Journal of the Nigerian Association of Mathematical Physics
Volume 63, (January – March, 2022 Issue), **pp39 – 44**

A SUBCLASS OF $\delta-$ VALENT FUNCTIONS FOR OPERATORS ON A HILBERT SPACE

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Abstract

In this article, the authors investigated a new subclass of analytic functions for operators on a Hilbert space relating to the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Interesting results on coefficient estimates, distortion theorem were pointed out for this subclasses.

Keywords: Univalent function, coefficient estimate, distortion theorem.

1. INTRODUCTION

Let A denote the usual class of analytic univalent functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

In the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let S denote the subclass of A, consisting of functions of the form (1) which are normalized in U.

A function $f\in A$ is said to be starlike of order $\eta\ (0\le\eta<1)$ if and only if

$$\Re\frac{zf'(z)}{f(z)}>\eta,\ z\in U \tag{2}$$

Similarly, a function $f\in A$ is said to be starlike of order $\eta\ (0\le\eta<1)$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\eta,\ \ z\in U \tag{3}$$

We represent by $S^*(\eta)$ and $k(\eta)$ respectively the classes of functions in S, which are starlike and convex of order η in U. The subclass $S^*(\eta)$ was introduced in [1] and was equally studied further in [2,3]. Also, certain classes of univalent functions with negative coefficients for properties such as coefficient estimates, distortion theorem, extreme points, and radius theorem with respect to symmetric and conjugate points and the radius of convexity was investigated in [4].

In [5], T represent the subclass of S consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \tag{4}$$

Now, let

$F_\lambda(z)=(1-\lambda)f(z)+\lambda zf(z),\ 0\le\lambda\le1,\ f\in T\cdot$

By applying line segment principle with (4) we have the form

$$F_\lambda(z)=z-\sum_{n=2}^{\infty}[1+\lambda(n-1)]a_nz^n \tag{5}$$

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The functions is in (4) and (5) were investigated in [5].
The function $f \in S$ is in the class $S_\lambda(\alpha,\beta,\mu)$ was investigated in [5] and it satisfies

$$\left|\frac{\frac{zF_\lambda'(z)}{F_\lambda(z)}-1}{\mu\frac{zF_\lambda'(z)}{F_\lambda(z)}+-(1+\mu)\alpha}\right|<\beta,\ z\in U, \tag{6}$$

Where $0\le\alpha<1, 0<\beta\le1$ and $0\le\mu\le1$. The investigation in [5], focused on the coefficient estimates, distortion theorem for certain subclass as well as application to operators based on fractional calculus for the class $S_\lambda(\alpha,\beta,\mu)$. On the other hand,in [5} the study of T the subclass of S was equally investigated to consist of functions of the form

$$S_\lambda^*(\alpha,\beta,\mu)=S_\lambda(\alpha,\beta,\mu)\cap T. \tag{7}$$

It is significant to emphasize that the study of various sub-classes of S and other related works are numerous in literature and few out of many are in [6,7,8] .
According to [5}, H is a complex Hilbert space and A is an operator on H. For an analytic function f defined on U, we denote by $f(A)$ the operator on H defined by the well-known Riese-Dunford integral

$$f(A)=\frac{1}{2\pi i}\int_C f(z)(zI-A)^{-1}dz, \tag{8}$$

where I is the identity operator on H, C is a positively oriented simple closed contour lying in U and containing the spectrum of A on the interior of the domain. The conjugate operator of A is denoted by A^*.
A function given by (8) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$ if it satisfies the condition

$$\|AF_\lambda'(A)-F_\lambda(A)\|<\beta\|\mu AF_\lambda'(A)+F_\lambda(A)-(1+\mu)\alpha F_\lambda(A)\|$$

With the same constraints as α, β and μ, given in (6) and for all A with $\|A\|<1$, $A\neq\theta$, where θ is the zero operator on H. This type of work was earlier studied in [9,10].
Also in this present study, we have established coefficient estimates, distortion theorem for $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$ and the application to a class of operator defined through fractional calculus were equally considered.
Let T^δ denote the subclass of S^δ consisting of functions of the form

$$f(z)^\delta=z^\delta-\sum_{n=2}^{\infty}a_n(\delta)z^{n+\delta-1}, a_n(\delta)\ge0\cdot \qquad \delta\ge1 \tag{9}$$

By setting

$$F_\lambda(z)^\delta=(1-\lambda)f(z)^\delta+\lambda zf'(z),\quad 0\le\lambda\le1, \qquad f\in T^\delta, \tag{10}$$

The function in (10) evidently takes the form given below:

$$F_\lambda(z)^\delta=(1-\lambda(1-\delta))z^\delta-\sum_{n=2}^{\infty}[1+\lambda(\delta+n-2)]a_n(\delta)z^{\delta+n-1}, \qquad \delta\ge1. \tag{11}$$

Definition 1: A function $f^\delta\in S^\delta$ is said to be in the class $S_\lambda(\delta,\sigma,\beta,\mu)$ if it satisfies

$$\left|\frac{\frac{z(F_\lambda(z)^\delta)'}{F_\lambda(z)}-1}{\mu\frac{z(F_\lambda(z)^\delta)'}{F_\lambda(z)}+1-(1+\mu)\alpha}\right|<\beta, \qquad z\in U, \tag{12}$$

where $\delta\ge1$, $0\le\sigma<1$, $0<\beta\le1$ and $0\le\mu\le1$.
Where γ is real, $0\le\alpha<1$, $0<\beta\le1$ and $0\le\mu\le1$.
Let us define

$$S_\lambda^*(\delta,\sigma,\beta,\mu)=S_\lambda(\delta,\sigma,\beta,\mu)\bigcap T^\delta \tag{13}$$

H is a complex Hilbert space and A is an operator on H, for an analytic function f defined on U, we denote by $f(A)^\delta$ the operator on H defined by the well-known Riese-Dunford integral

$$f(A)^{\delta} = \frac{1}{2\pi i}\int_C f(z)^{\delta}(zI - A)^{-1}dz, \tag{14}$$

where I is the identity operator on H, C is a positively oriented simple closed contour lying in U and containing the spectrum of A on the interior of the domain. The conjugate operator of A is denoted by A^*.

Definition 2: A function given by (9) is in the class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$ if it satisfies the condition

$$\left\|A\left(F_\lambda(A)^{\delta}\right)' - F_\lambda(A)^{\delta}\right\| < \beta\left\|\mu A\left(F_\lambda(A)^{\delta}\right)' + F_\lambda(A)^{\delta} - (1+\mu)\alpha F_\lambda(A)^{\delta}\right\| \tag{15}$$

With the same constraints as γ, α, β and μ, given in (12) and for all A with $\|A\| < 1$, $A \neq \theta$ where $A \neq \theta$ is the zero operator on A^*.

Definition 3: The fractional integral operator of order k associated with a function f^{δ} is defined by

$$D_\lambda^{-k} f(A)^{\delta} = \frac{1}{\Gamma(k)}\int A^k f(tA)^{\delta}(1-t)^{k-1}dt,$$

Where $\delta \geq 1$, $k > 0$ and f is an analytic function in a simply connected region of the complex plane containing the origin.

Definition 4: The fractional integral operator of order class k associated with a function f^{δ} is defined by

$$D_A^{-k} f(A)^{\delta} = \frac{1}{\Gamma(1-k)} g'(A)^{\delta} \quad , \tag{16}$$

Where $g(A)^{\delta} = \frac{1}{\Gamma(k)}\int_0^t A^k f(tA)^{\delta}(1-t)^{k-1}dt,\ \ 0 < k < 1,$

and f^{δ} is an analytic function in a simply connected region of the complex plane containing the origin.

This shall be used to investigate distortion theorem for functions belonging to the class $S_\lambda^*(\alpha,\beta,\mu;A)$ and it extends the study carried out in [5].

2. MAIN RESULTS

2.1 Coefficient estimates for class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$

Theorem 2.1: A function f^{δ} be given by (9) is in the class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$ for all proper contraction A with $A \neq \theta$ if and only if

$$\sum_{n=2}^{\infty}\{(n+\delta-2) + \beta[1+\mu n - \mu(1+\sigma-\delta) - \sigma]\}a_n(\delta) \leq \beta(1-\sigma(1+\mu)+\mu\delta) - (1-\delta) \tag{18}$$

for $\delta \geq 1$, $0 \leq \sigma < 1$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$.

The result is best possible for

$$f(z)^{\delta} = z^{\delta} - \frac{\beta(1-\sigma(1+\mu)+\mu\delta) - (1-\delta)}{(n+\delta-2) + \beta[1+\mu n - \mu(1+\sigma-\delta) - \sigma]} z^{n+\delta-1}, \quad n \in N \setminus \{1\} \tag{19}$$

Proof. Assuming (19) holds, we deduce that

$$\left\|A\left(F_\lambda(A)^{\delta}\right)' - F_\lambda(A)^{\delta}\right\| - \beta\left\|\mu A\left(F_\lambda(A)^{\delta}\right)' + F_\lambda(A)^{\delta} - (1+\mu)\alpha F_\lambda(A)^{\delta}\right\|$$

$$= \left\|(1-\delta)A^{\delta} - \sum_{n=2}^{\infty}(n+\delta-2)a_n(\delta)A^{n+\delta-1}\right\|$$

$$-\beta\left\|(1-\sigma(1+\mu)+\mu\delta)A^{n+\delta-1} - \sum_{n=2}^{\infty}\{1+\mu n - \mu(1+\sigma-\delta) - \sigma\}a_n A^{n+\delta-1}\right\|$$

$$\leq \sum_{n=2}^{\infty}\{(n+\delta-2) + \beta[1+\mu n - \mu(1+\sigma-\delta) - \sigma]\}a_n(\delta) - \beta(1-\sigma(1+\mu)+\mu\delta) + (1-\delta) \leq 0,$$

so, f^{δ} is in the class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$.

On contrary note, if we suppose that f^{δ} belongs to $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$, then

$$\left\|A\left(F_\lambda(A)^\delta\right)' - F_\lambda(A)^\delta\right\| < \beta\left\|\mu A\left(F_\lambda(A)^\delta\right)' + F_\lambda(A)^\delta - (1+\mu)\alpha F_\lambda(A)^\delta\right\|$$

$$= \left\|(1-\delta)A^\delta - \sum_{n=2}^{\infty}(n+\delta-2)a_n(\delta)A^{n+\delta-1}\right\|$$

$$\le \beta\left\|(1-\sigma(1+\mu)+\mu\delta)A^{n+\delta-1} - \sum_{n=2}^{\infty}\{1+\mu n-\mu(1+\sigma-\delta)-\sigma\}a_n A^{n+\delta-1}\right\|.$$

Choosing $A = eI$ $(0<e<1)$ in the above inequality, we get

$$\frac{(1-\delta)+\sum_{n=2}^{\infty}(n+\delta-2)a_n(\delta)e^{n+\delta-1}}{\le (1-\sigma(1+\mu)+\mu\delta) - \sum_{n=2}^{\infty}\{1+\mu n-\mu(1+\sigma-\delta)-\sigma]\}e^{n-1}} < \beta. \tag{20}$$

By simple simplification (20) and letting $e \to 1 (0<\sigma<1)$, then the following readily comes into existence:

$$\sum_{n=2}^{\infty}(n+\delta-2)a_n(\delta) \le \beta(1-\sigma(1+\mu)+\mu\delta) - \beta\sum_{n=2}^{\infty}\{\mu n+1-\mu(1+\sigma-\delta)-\sigma\}a_n(\delta) - (1-\delta)$$

This implies that a further simplification gives

$$\sum_{n=2}^{\infty}\{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]\}a_n(\delta) \le \beta(1-\sigma(1+\mu)+\mu\delta) - (1-\delta),$$

and this completes the proof.

Corollary 2.2 [4,5]**:**
Whenever given that function f^δ be given by (9) is in the class $S_\lambda^*(1,\sigma,\beta,\mu;A)$ then

$$a_n \le \frac{\beta(1+\mu)(1-\sigma)}{(n-1)+\beta[1+\mu n-(1+\mu)\sigma]}, \quad n = 2,3,4,...$$

2.2 Distortion theorem for class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$

Theorem 2.3. If the *f*unction f^δ given by (9) is in the class $S_\lambda^*(\delta,\sigma,\beta,\mu;A)$ for $\delta \ge 1$, $0 \le \sigma < 1$, $0 < \beta \le 1$, $0 \le \mu \le 1$, $\|A\| < 1$ and $\|A\| \ne \theta$, then

$$\|A\| - \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\delta]}\|A\|^2 \le \|f(A)\| \le \|A\| + \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\delta]}\|A\|^2. \tag{21}$$

The result is sharp *f*or the *f*unction

$$f(z)^\delta = z^\delta - \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]}z^{n+\delta-1}. \tag{22}$$

Proof. In view of Theorem 2.1, we have

$$(1-\delta)+\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]\sum_{n=2}^{\infty}a_n(\delta)$$

$$\le (1-\delta)+\sum_{n=2}^{\infty}\{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]\}a_n$$

$$\le \beta(1-\sigma(1+\mu)+2\mu\delta),$$

which gives us

$$\sum_{n=2}^{\infty}a_n \le \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]} \tag{23}$$

Hence, we have

$$\left\| f(A)^{\delta} \right\| \geq \|A\| - \|A\|^2 \sum_{n=2}^{\infty} a_n(\delta)$$
$$\geq \|A\| - \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]}\|A\|^2,$$
and
$$\left\| f(A)^{\delta} \right\| \leq \|A\| + \|A\|^2 \sum_{n=2}^{\infty} a_n(\delta) \leq$$
$$\|A\| + \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]}\|A\|^2,$$
which completes the proof.

2.3 Extreme points for class $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$

Theorem 2.4. *Let* $f_1(z)^{\delta} = z^{\delta}$, and
$$f_n(z)^{\delta} = z^{\delta} - \frac{\beta(1-\sigma(1+\mu)+\mu\delta)-(1-\delta)}{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]} z^{n+\delta-1}, n \geq 2. \tag{24}$$
Then, any function f^{δ} of the form (9) is in the class $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$ *if* and only *if* it can be expressed as,
$$f(z)^{\delta} = \sum_{n=1}^{\infty} \lambda_n f_n(z)^{\delta}, \text{ with } \lambda_n \geq 0, \ \sum_{n=1}^{\infty} \lambda_n = 1 \tag{25}$$
Proof. First, let us assume that
$$f(z)^{\delta} = \sum_{n=1}^{\infty} \lambda_n f_n(z)^{\delta} = z^{\delta} - \sum_{n=2}^{\infty} \frac{\beta(1-\sigma(1+\mu)+\mu\delta)-(1-\delta)}{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]} \lambda_n z^{n+\delta-1}.$$
Then, we have
$$\sum_{n=2}^{\infty} \frac{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]}{\beta(1-\sigma(1+\mu)+\mu\delta)-(1-\delta)} \lambda_n \frac{\beta(1-\sigma(1+\mu)+\mu\delta)-(1-\delta)}{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]}$$
$$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1,$$
Hence $f^{\delta} \in S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$.

Conversely, let us assume that the function f^{δ} f^{γ} given by (9) is in the class $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$. Applying Theorem 2.1 and specializing it we obtain
$$a_n(\delta) \leq \frac{\beta(1-\sigma(1+\mu)+\mu\delta)-(1-\delta)}{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]}.$$
Also, let
$$\lambda_n = \frac{(1-\delta)+(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]}{\beta(1-\sigma(1+\mu)+\mu\delta)} a_n(\delta),$$
and
$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,$$
Hence it is easy to check that f^{γ} can be expressed by (25), and this completes the proof of Theorem 2.4.

2.4 Distortion theorem involving fractional calculus for class $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$

Theorem 2.5. If the function f^{δ} given by (9) is in the class $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$ for γ is real, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$, then
$$\left\| D_A^{-k} f(A)^{\delta} \right\| \geq \frac{\|A\|^k}{\Gamma(k+2)} + \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]} \frac{\|A\|^{k+2}}{\Gamma(k+2)}$$
and
$$\left\| D_A^{-k} f(A)^{\delta} \right\| \leq \frac{\|A\|^k}{\Gamma(k+2)} - \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]} \frac{\|A\|^{k+2}}{\Gamma(k+2)}. \tag{26}$$
Proof. If we consider
$$F(A)^{\gamma} = \Gamma(k+2) A^{-k} D_A^{-k} f(A)^{\gamma}$$
$$= A^{\gamma} - \sum_{n=1}^{\infty} \frac{\Gamma(n+2)\Gamma(k+2)}{\Gamma(n+k+2)} a_{n+1}(\gamma) A^{n+\gamma} = A^{\gamma} - \sum_{n=2}^{\infty} B_n(\gamma) A^n,$$

$$F(A)^{\delta} = \Gamma(k+2)A^{-k}D_A^{-k}f(A)^{\delta}$$
$$= A^{\delta} - \sum_{n=1}^{\infty}\frac{\Gamma(n+2)\Gamma(k+2)}{\Gamma(n+k+2)}a_{n+1}(\delta)A^{n+\delta} = A^{\delta} - \sum_{n=2}^{\infty}B_n(\delta)A^n,$$

where $B_n(\delta) = \dfrac{\Gamma(n+1)\Gamma(k+2)}{\Gamma(n+k+1)}a_n(\delta)$, then we obtain

$$\sum_{n=2}^{\infty}\{(n+\gamma-2)+\beta[1+\mu n-\mu(1+\alpha-\gamma)-\alpha]\}B_n(\gamma)$$
$$\leq \sum_{n=2}^{\infty}\{(n+\gamma-2)+\beta[1+\mu n-\mu(1+\alpha-\gamma)-\alpha]\}a_n(\gamma) \leq \beta(1-\alpha(1+\mu)+\mu\gamma),$$
$$\sum_{n=2}^{\infty}\{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]\}B_n(\delta)$$
$$\leq \sum_{n=2}^{\infty}\{(n+\delta-2)+\beta[1+\mu n-\mu(1+\sigma-\delta)-\sigma]\}a_n(\delta) \leq \beta(1-\sigma(1+\mu)+\mu\delta),$$

as $0 < \dfrac{\Gamma(n+1)\Gamma(k+2)}{\Gamma(n+k+1)} < 1$, hence F^{δ} belongs to $S_{\lambda}^{*}(\delta,\sigma,\beta,\mu;A)$.

Therefore, by Theorem 2.3 we deduce that

$$\left\|D_A^{-k}f(A)^{\delta}\right\| \leq \frac{\|A\|^{k+2}}{\Gamma(k+2)} + \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]}\frac{\|A\|^{k+2}}{\Gamma(k+2)}$$

and

$$\left\|D_A^{-k}f(A)^{\delta}\right\| \geq \frac{\|A\|^{k+1}}{\Gamma(k+2)} - \frac{\beta(1-\sigma(1+\mu)+2\mu\delta)-(1-\delta)}{\delta+\beta[(1+2\mu)-2\mu(1+\sigma-\delta)-\sigma]}\frac{\|A\|^{k+2}}{\Gamma(k+2)}.$$

This complete the proof.

Note that $(A^{\frac{1}{q}})*A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q^*}});\ q \in N$ and by varying parameter in Corollary 3.8 in [12], we have $\|A^m\| = \|A\|^m$, where m is rational number and '*' is the Hadamard product or convolution product of two analytic functions. When S is any irrational number, we choose a single-valued branch of z^s and a single valued branch of z^{k_n} (K_n is a sequence of irrational numbers) such that $K_n \to s$ as $\|A^{k_n}\| = \|A\|^{k_n}$, and by following the study carried out in [13] we have $\|A^{k_n}\| \to \|A^s\|$, $\|A^{k_n}\| = \|A\|^{k_n} \to \|A^s\|$, $k_n \to s$.

That is $\|A^s\| = \|A\|^s$, hence $\|A^k\| = \|A\|^k$, $k > 0$.

Acknowledgment. The authors wish to express their sincere thanks to the referees of this paper for their meaningful contributions.

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