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δ −POLYNOMIAL BOUNDS FOR A SUBCLASS OF UNIVALENT FUNCTION REGARDING MODIFIED SIGMOID FUNCTION

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Abstract

In this article, the authors investigated a new subclass of analytic univalent function which relate to ameliorated sigmoid function and the classical special polynomial function known as the Chebyshev polynomials by employing the concept of subordination. This investigation produced new interesting coefficient bounds. The famous Fekete-Szego inequalities were also pointed out.

Keywords: Analytic function, sigmoid function, Chebyshev polynomials, Salagean operator

1. INTRODUCTION

The concept special functions is fast taking the focal point in the field of geometric function theory and are rapidly attracting the attention of several researchers owing to advancement in Science and Technology. A very good example of Special function in this investigation is the activation function. One of the most popular activation function in hardware implementation of Artificial Neural Network (ANN) is the sigmoid function. According to [1,2], the study of activation function, in particular, the sigmoid function happens to be a function that increases the size of the hypothesis space that represent the network can represent. Neural network can be used for complex learning tasks. It is therefore necessary to investigate the use of sigmoid function in geometric function theory.
The sigmoid function in [1,2] takes the form

$$h(s) = \frac{1}{1+e^{-s}} \quad s \geq 0,$$

is a bounded differentiable function and has the following properties:

1. It outputs real numbers between 0 and 1.
2. It maps a very large output domain to a small range of inputs.
3. It never losses information because it is a one – to – one function.
4. It increases monotonically.

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z: z \in \mathbb{C} \; and \; |z| < 1\}$.
In [3], the differential operator $D^n f, n \epsilon \mathbb{N}_0 = 0,1,2,...$ was applied on function $f(z)$ belonging to A class of analytic functions in the unit disk U and this takes the form :

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \qquad n \in \mathbb{N}_0$$

If $f(z)$ and $g(z)$ be analytic in $U = \{z: z \in \mathbb{C} \; and \; |z| < 1\}$, according to [4], we say that $f(z)$ is subordinate to g(z) when there exist a function $\omega(z)$ and $|\omega(z)| < 1$ such that

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$f(z) = g(\omega(z)) \qquad (|z| < 1)$.
In [5], the modified sigmoid function
$G(z) = \frac{2}{1 - e^{-z}}$
was studied and investigated in order to obtained another series of the modified sigmoid function which takes the form as
$$G(z) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\frac{(-1)^m}{n!} z^n\right)^m\right)$$
$$= 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 + \cdots$$
According to [1], the Chebyshev polynomials are a sequence of orthogonal polynomials which are related to De'Moivres formula and which are defined recursively. The use of Chebyshev polynomials in numerical analysis is on the increase in both theoretical and practical perspective. There are four kinds of Chebyshev polynomials. The dominant types are Chebyshev polynomials of first and second kinds which are $T_n(t)$ and $U_n(t)$ respectively and their numerous uses in different applications abound. Details are in [6,7]
Following [1], the Chebyshev polynomials of the first and second kind are defined respectively in the form:
$$T_n(t) = Cosn\alpha \qquad t \in (-1,1),$$
$$U_n(t) = \frac{Sin(n+1)\alpha}{Sin\alpha} \qquad t \in (-1,1),$$
Where n denotes the degree of the polynomial and $t = Cos\alpha$.
The Chebyshev polynomials of the first kind $T_n(t),\ t \in [-1,1]$ have the generating function of the form
$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \qquad (z \in D)$$
and that of the second kind is:
$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{Sin(n+1)\alpha}{Sin\alpha} z^n \qquad (z \in D)$$
for $|t| < 1$.
Note that if $t = Cos\alpha,\quad \alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then
$$H(z,t) = \frac{1}{1 - 2Cos\alpha z + z^2}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{Sin(n+1)\alpha}{Sin\alpha} z^n$$
Thus,
$$H(z,t) = 1 + 2Cos\alpha z + (3Cos^2\alpha - Sin^2\alpha)z^2 + \cdots$$
The investigation carried out in [1] has some basic interplay in the study done in [8]. According to the investigation in [8], we have the function given below:
$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \qquad \big(z \in D, \qquad t \in (-1,1)\big),$$
Where
$$U_{n-1} = \frac{Sin(n\ arcCost)}{\sqrt{1-t^2}} \qquad (n \in N).$$
This gives the Chybeshev polynomial of the second kind and It has the following general form as
$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$
Where $U_1(t) = 2t,\quad U_2(t) = 4t^2 - 1,\quad U_3(t) = 8t^3 - 4t, \ldots$
Lemma 1-1 [1,9]: If $\omega(z) = b_1 z + b_2 z^2 + \cdots, b_1 \neq 0$ is analytic and satisfy $|\omega(z)| < 1$ in the unit disk U, then for each $0 < r < 1$, $|\omega'(z)| < 1$ and $\left|\omega(re^{i\theta})\right| < 1$ unless $\omega(z) = e^{i\sigma}z$ for some real σ.
Lemma1.2[1,10]: Let$\omega \epsilon \Omega = \{\omega \epsilon A: |\omega(z)| \leq |z|, z \epsilon U\}$.If$\omega \epsilon \Omega, \omega(z) = \sum_{n=1}^{\infty} c_n z^n\ (z \epsilon U),\ then\ |c_n| \leq 1, n = 1,2, \ldots, |c_2| \leq 1 - |c_1|^2$ (1.1)
and $|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\} \quad (\mu \epsilon \mathbb{C})$ (1.2)
The result is sharp.
The functions
$$\omega(z) = z, \omega_a(z) = z\frac{z + a}{1 + \bar{a}z}\ (z \epsilon U, |a| < 1)$$
are extremal functions.

Also in [1], Salagean differential operator was combined with modified sigmoid function and this gives the form:
$f_\gamma(z) = Z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k,$ (1.3)
where
$\gamma(s) = \frac{2}{1+e^{-s}}, s \geq 0$, notice s has been used to replace z.
function of the form (1.3) belong to the class A_γ, where $A_1 \equiv A$.
Here, $D^n f_\gamma(z);\ \ n \in N_0$ denote the Salagean differential operator involving modified sigmoid function with the usual form as:

$$D^0 f_\gamma(z) = f_\gamma(z)$$
$$D^1 f_\gamma(z) = \gamma(s) z f^1{}_\gamma(z)$$
$$\vdots$$
$$D^n f_\gamma(z) = D\left(D^{n-1} f_\gamma(z)\right) = \gamma(s) z \left(D^{n-1} f_\gamma(z)\right)^1 \qquad (1.4)$$

When $\gamma(s) = 1$, we have the Salagean differential operator that was introduced in [3]. Further details in [1].
In [12], coefficient estimates for a Spirallike functions in the space of Sigmoid function was investigated. Further more, in [13] the Fekete – Szego functional for a subclass of analytic functions related to Sigmoid function was studied.
In [7], the definition given below inspired by investigation carried out in [6] was considered:
Definition 1.3[1**]**:A function $f_\gamma(z) \in A_\gamma$ is said to be in the class

$$H_\gamma(n, \mu, \lambda), \quad 0 \leq \lambda \leq 1, \qquad \mu \geq 0, \qquad \gamma(s) = \frac{2}{1+e^{-2}} \qquad s \geq 0, \quad n \in N_0$$

If the following subordination principle holds

$$(1-\lambda)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu} + \lambda f^1{}_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} \prec H(z,t), \qquad (1.5)$$

Where, $D^n f_\gamma(z)$ is the Salagean differential operator involving Modified Sigmoid Function defined as follows:

$$D^0 f_\gamma(z) = f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k$$
$$D^1 f_\gamma(z) = D f_\gamma(z) = \gamma(s) z + \sum_{k=2}^{\infty} k \gamma^2(s) a_k z^k,$$
$$\vdots$$
$$D^n f_\gamma(z) = D\left(D^{n-1} f_\gamma(z)\right) = \gamma^n(s) \sum_{k=2}^{\infty} k^n \gamma^{n-1}(s) a_k z^k \qquad (1.6)$$

We can re-express (1.3) in the form

$$f_\gamma(z)^\delta = z^\delta + \sum_{k=2}^{\infty} \gamma(s) a(\delta)_k z^{\delta+k-1} \qquad (1.7)$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$ $s \geq 0$ and δ is real $(\delta \geq 1)$.
Observe that function in (1.7) belongs to the subclass $A_\gamma^\delta \epsilon A_\gamma$, where $A_1^1 \equiv A$.
Applying the Salegean differential operator in (1.7) gives the form

$$D^n f_\gamma(z)^\delta = D\left(D^{n-1} f_\gamma(z)^\delta\right) = \gamma(s) z \left(D^{n-1} f_\gamma(z)^\delta\right)' = \gamma^n(s) z^\delta + \sum_{k=2}^{\infty} k^n \gamma^{n+1}(s) a_k(\delta) z^{\delta+k-1} \qquad (1.8)$$

where, $D^n f_\gamma(z)^\delta$ is the Salagean differential operator involving δ −valent function for Modified Sigmoid Function.
Remark: When $\gamma(s) = 1$ and $\delta = 1$, we have the usual Salagean differential operator [10].
Lemma 1.4. [14]. Let $P(z) = 1 + c_1 z z^2 + c_1 z^2 + \cdots$ is analytic function with positive real part in U, then

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2, if\ v \leq 0; \\ 2, if\ 0 \leq v \leq 1; \\ 4v - 2, if\ v \geq 1. \end{cases}$$

Lemma 1.5 [15]. If a function p ϵ P is given by $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ then $|p_k| \leq 2$, $k \in \mathbb{N}$, where p is the family of all functions analytic in U for which $p(0) = 1$ and $\mathbb{R}e(p(z)) > 0, (z \in U)$.
Definition 1.6: A function $f_\gamma(z)^\delta \epsilon A_\gamma^\delta$ is said to be in the class $S_\gamma(\delta, n, \rho; \psi)$, $0 \leq \psi \leq 1$, $\rho \geq 0$, $\delta \geq 1$, $\gamma(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, $n \epsilon N_0$, if the following subordination holds

$$(1-\psi)\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^{\rho} + \psi \frac{f_\gamma''(z)^\delta}{\delta}\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^{\rho-1} \prec H(z,t) \qquad (1.9)$$

where D^n is the Salagean differential operator[10].

2. MAIN RESULTS

Theorem 2.1. If $f_\gamma(z)^\delta$ belongs to the class $S_\gamma(\delta,n,\rho,\psi)$, and, $\delta\geq 1,\rho\geq 0, 0\leq\psi\leq 1, n\epsilon N_o$ then

$$|c_2(\delta)|\leq\frac{2t}{A_N+B_N}, \qquad (1.10)$$

$$|c_3(\delta)|\leq\frac{2t+(4t^2-1)}{(A'_N+B'_N)}+\frac{4t^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)}, \qquad (1.11)$$

$$|c_4(\delta)|\leq\frac{8t^3+8t^2-2t-2}{(D'_N+E'_N)}+\frac{(F_N+G_N)(4t^2+2t(4t^2-1)}{(A_N+B_N)(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(D_N+E_N)(F_N+G_N)}{(A_N+B_N)^3(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(F'_N+G'_N)}{(A_N+B_N)^3(D'_N+E'_N)} \qquad (1.12)$$

$$|c_5(\delta)|\leq\frac{16t^4+24t^3-10t-4}{(H_N+I_N)}+\frac{(H'_N+I'_N)(16t^4+16t^3-4t^2-4t)}{(A_N+B_N)(D'_N+E'_N)(H_N+I_N)}+\frac{(F_N+G_N)(H'_N+I'_N)(16t^4+8t^3-4t^2)}{(A_N+B_N)^2(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4(D_N+E_N)(F_N+G_N)(H'_N+I'_N)}{(A_N+B_N)^4(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4(F'_N+G'_N)(H'_N+I'_N)}{(A_N+B_N)^4(D'_N+E'_N)(H_N+I_N)}+\frac{(16t^4+16t^3-4t^2-4t+1)(J_N+K_N)}{(A'+B')^2(H_N+I_N)}+\frac{(D_N+E_N)(J_N+K_N)(32t^4+16t^3-8t^2)}{(A_N+B_N)^2(A'_N+B'_N)^2(H_N+I_N)}+\frac{(D_N+E_N)^2(J_N+K_N)16t^4}{(A_N+B_N)^4(A'+B')^2(H_N+I_N)}+\frac{(J'_N+K'_N)(16t^4+8t^3-4t^2)}{(A_N+B_N)^2(A'_N+B'_N)(H_N+I_N)}+\frac{16t^4(D_N+E_N)(J'_N+K'_N)}{(A_N+B_N)^4(A'_N+B'_N)(H_N+I_N)}+\frac{(L_N+M_N)16t^4}{(A_N+B_N)^4(H_N+I_N)} \qquad (1.13)$$

Where

$A_N=2^n\rho(1-\psi)\gamma^{n\rho+1}(s)$,

$B_N=\left(2^n(\rho-1)+\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)$,

$A'_N=3^n\rho(1-\psi)\gamma^{n\rho+1}(s)$,

$B'_N=\left(3^n(\rho-1)+\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)$,

$D_N=-(1-\psi)2^{2n}\frac{\rho(\rho-1)}{2!}\gamma^{n\rho+2}(s)$,

$E_N=-\left(2^{2n}\frac{(\rho-1)(\rho-2)}{2!}+2^n(\rho-1)\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)$,

$D'_N=4^n\rho(1-\psi)\gamma^{n\rho+1}(s)$,

$E'_N=\left(4^n(\rho-1)+\left(\frac{\delta+3}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)$,

$F_N=-(1-\psi)2.6^n\frac{\rho(\rho-1)}{2!}\gamma^{n\rho+2}(s)$,

$G_N=-\left(2.6^n\frac{(\rho-1)(\rho-2)}{2!}+(\rho-1)\left(3^n\left(\frac{\delta+1}{\delta}\right)+2^n\left(\frac{\delta+2}{\delta}\right)\right)\right)\psi\gamma^{n(\rho-1)+2}(s)$,

$F'_N=-(1-\psi)2^{3n}\frac{\rho(\rho-1)(\rho-2)}{3!}\gamma^{n\rho+3}(s)$,

$G'_N=-\left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n}+\frac{(\rho-1)(\rho-2)}{2!}2^{2n}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+3}(s)$,

$H_N=5^n\rho(1-\psi)\gamma^{n\rho+1}(s)$,

$I_N=\left(5^n(\rho-1)+\left(\frac{\delta+4}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)$,

$H'_N=-(1-\psi)2.8^n\frac{\rho(\rho-1)}{2!}\gamma^{n\rho+2}(s)$,

$I'_N=-\left(2.8^n\frac{(\rho-1)(\rho-2)}{2!}+(\rho-1)\left(4^n\left(\frac{\delta+1}{\delta}\right)+2^n\left(\frac{\delta+3}{\delta}\right)\right)\right)\psi\gamma^{n(\rho-1)+2}(s)$,

$J_N=-(1-\psi)3^{2n}\frac{\rho(\rho-1)}{2!}\gamma^{n\rho+2}(s)$,

$K_N=-\left(3^{2n}\frac{(\rho-1)(\rho-2)}{2!}+3^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)$,

$J'_N=-(1-\psi)3.12^n\frac{\rho(\rho-1)(\rho-2)}{3!}\gamma^{n\rho+3}(s)$,

$K'_N=-\left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}3.12^n+\frac{(\rho-1)(\rho-2)}{2!}\left(2.6^n\left(\frac{\delta+1}{\delta}\right)+2^{2n}\left(\frac{\delta+2}{\delta}\right)\right)\right)\psi\gamma^{n(\rho-1)+3}(s)$,

$L_N=-(1-\psi)2^{4n}\frac{\rho(\rho-1)(\rho-2)(\rho-3)}{4!}\gamma^{n\rho+4}(s)$,

$M_N=-\left(2^{4n}\frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}+\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+4}(s)$.

Proof:

If $f_\gamma(z)^\delta\epsilon S_\gamma(\delta,n,\rho;\psi)$, then from (1.9) we have

$$(1-\psi)\left(\frac{D^nf_\gamma(z)^\delta}{z^\delta}\right)^\rho+\psi\frac{f''_\gamma(z)^\delta}{\delta}\left(\frac{D^nf_\gamma(z)^\delta}{z^\delta}\right)^{\rho-1}\prec H(z,t)$$

Where $\gamma(s)$ and δ as given in 1.7).

From (1.8) we have

$$\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^\rho = \gamma^{n\rho}(s) + 2^n\rho\gamma^{n\rho+1}(s)c_2(\delta)z + \left(3^n\rho\gamma^{n\rho+1}(s)c_3(\delta) + \frac{\rho(\rho-1)}{2!}2^{2n}\gamma^{n\rho+2}(s)c_2^2(\delta)\right)z^2 + \left(4^n\rho\gamma^{n\rho+1}(s)c_4(\delta) + \frac{\rho(\rho-1)}{2!}2.6^n\gamma^{n\rho+2}(s)c_2(\delta)c_3(\delta) + \frac{\rho(\rho-1)(\rho-2)}{3!}2^{3n}\gamma^{n\rho+3}(s)c_2^3(\delta)\right)z^3 + \left(5^n\rho\gamma^{n\rho+1}(s)c_5(\delta) + \frac{\rho(\rho-1)}{2!}9^n\gamma^{n\rho+2}(s)c_3^2 + \frac{\rho(\rho-1)}{2!}2.8^n\gamma^{n\rho+2}(s)c_2(\delta)c_4(\delta) + \frac{\rho(\rho-1)(\rho-2)}{3!}3.12^n\gamma^{n\rho+3}(s)a_2^2(\delta)a_3(\delta) + \frac{\rho(\rho-1)(\rho-2)(\rho-3)}{4!}2^{4n}\gamma^{n\rho+4}(s)c_2^4(\delta)\right)z^4 + \cdots \qquad (1.14)$$

Also,

$$\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^{\rho-1} = \gamma^{n(\rho-1)} + 2^n(\rho-1)\gamma^{n(\rho-1)+1}(s)c_2(\delta)z + \left(3^n(\rho-1)\gamma^{n(\rho-1)+1}(s)c_3(\delta) + \frac{(\rho-1)(\rho-2)}{2!}2^{2n}\gamma^{n(\rho-1)+2}(s)c_2^2(\delta)\right)z^2 + \left(4^n(\rho-1)\gamma^{n(\rho-1)+1}(s)c_4(\delta) + \frac{(\rho-1)(\rho-2)}{2!}2.6^n\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_3(\delta) + \frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n}\gamma^{n(\rho-1)+3}(s)c_2^3(\delta)\right)z^3 + \left(5^n(\rho-1)\gamma^{n(\rho-1)+1}(s)c_5(\delta) + \frac{(\rho-1)(\rho-2)}{2!}9^n\gamma^{n(\rho-1)+2}(s)c_3^2 + \frac{(\rho-1)(\rho-2)}{2!}2.8^n\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_4(\delta) + \frac{(\rho-1)(\rho-2)(\rho-3)}{3!}3.12^n\gamma^{n(\rho-1)+3}(s)c_2^2(\delta)c_3(\delta) + \frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}2^{4n}\gamma^{n(\rho-1)+4}(s)c_2^4(\delta)\right)z^4 + \cdots \qquad (1.15)$$

Multiplying (1.14) via by $(1-\psi)$, we have

$$(1-\psi)\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^\rho = (1-\psi)\gamma^{n\rho}(s) + (1-\psi)2^n\rho\gamma^{n\rho+1}(s)c_2(\delta)z + \left((1-\psi)3^n\rho\gamma^{n\rho+1}(s)c_3(\delta) + \frac{(1-\psi)\rho(\rho-1)}{2!}2^{2n}\gamma^{n\rho+2}(s)c_2^2(\delta)\right)z^2 + \left((1-\psi)4^n\rho\gamma^{n\rho+1}(s)c_4(\delta) + \frac{(1-\psi)\rho(\rho-1)}{2!}2.6^n\gamma^{n\rho+2}(s)c_2(\delta)c_3(\delta) + \frac{(1-\psi)\rho(\rho-1)(\rho-2)}{3!}2^{3n}\gamma^{n\rho+3}(s)c_2^3(\delta)\right)z^3 + \left((1-\psi)5^n\rho\gamma^{n\rho+1}(s)c_5(\delta) + \frac{(1-\psi)\rho(\rho-1)}{2!}9^n\gamma^{n\rho+2}(s)c_3^2 + \frac{(1-\psi)\rho(\rho-1)}{2!}2.8^n\gamma^{n\rho+2}(s)c_2(\delta)c_4(\delta) + \frac{(1-\psi)\rho(\rho-1)(\rho-2)}{3!}3.12^n\gamma^{n\rho+3}(s)c_2^2(\delta)c_3(\delta) + \frac{(1-\psi)\rho(\rho-1)(\rho-2)(\rho-3)}{4!}2^{4n}\gamma^{n\rho+4}(s)c_2^4(\delta)\right)z^4 + \cdots \qquad (1.16)$$

Differentiating (1.7) with respect to z, multiply both sides by ψ and further divide via by $\delta z^{\delta-1}$ we have

$$\psi\frac{z^{1-\delta}f_\gamma'(z)^\delta}{\delta} = \psi + \psi\left(\frac{\delta+1}{\delta}\right)\gamma(s)c_2(\delta)z + \psi\left(\frac{\delta+2}{\delta}\right)\gamma(s)c_3(\delta)z^2 + \psi\left(\frac{\delta+3}{\delta}\right)\gamma(s)c_4(\delta)z^3 + \psi\left(\frac{\delta+4}{\delta}\right)\gamma(s)c_5(\delta)z^4 + \cdots \qquad (1.17)$$

Multiplying (1.16) and (1.17), we have

$$\psi\frac{z^{1-\delta}f_\gamma'(z)^\delta}{\delta}\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^{\rho-1} = \psi\gamma^{n(\rho-1)}(s) + \left(2^n(\rho-1) + \frac{\delta+1}{\delta}\right)\psi\gamma^{n(\rho-1)+1}(s)c_2(\delta)z + \left(\left(3^n(\rho-1) + \left(\frac{\delta+2}{\delta}\right)\right)\gamma^{n(\rho-1)+1}(s)c_3(\delta) + \left(\frac{(\rho-1)(\rho-2)}{2!}2^{2n} + 2^n(\rho-1)\left(\frac{\delta+1}{\delta}\right)\right)\gamma^{n(\rho-1)+2}(s)c_2^2(\delta)\right)\psi z^2 + \left(\left(4^n(\rho-1) + \left(\frac{\delta+3}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)c_4(\delta) + \left(\frac{(\rho-1)(\rho-2)}{2!}2.6^n + 3^n(\rho-1)\left(\frac{\delta+1}{\delta}\right) + 2^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_3(\delta) + \left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n} + \frac{(\rho-1)(\rho-2)}{2!}2^{2n}\frac{\delta+1}{\delta}\right)\psi\gamma^{n(\rho-1)+3}(s)c_2^3(\delta)\right)z^3 + \left(\left(5^n(\rho-1) + \left(\frac{\delta+4}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)c_5(\delta) + \left(\frac{(\rho-1)(\rho-2)}{2!}9^n + 3^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_3^2 + \left(\frac{(\rho-1)(\rho-2)}{2!}2.8^n + 4^n(\rho-1)\left(\frac{\delta+1}{\delta}\right) + 2^n(\rho-1)\left(\frac{\delta+3}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_4(\delta) + \left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}3.12^n + \frac{(\rho-1)(\rho-2)}{2!}2.6^n\left(\frac{\delta+1}{\delta}\right) + \frac{(\rho-1)(\rho-2)}{2!}2^{2n}\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+3}(s)c_2^2(\delta)c_3(\delta) + \left(\frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}2^{4n} + \frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+4}(s)c_2^4(\delta)\right)z^4 + \cdots \qquad (1.18)$$

Furthermore, we now have the following:

$$(1-\psi)\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^\rho + \psi\frac{z^{1-\delta}f_\gamma'(z)^\delta}{\delta}\left(\frac{D^n f_\gamma(z)^\delta}{z^\delta}\right)^{\rho-1} = (1-\psi)\gamma^{n\rho}(s) + \psi\gamma^{n(\rho-1)}(s) + \left(2^n\rho(1-\psi)\gamma^{n\rho+1}(s) + 2^n(\rho-1)\psi\gamma^{n(\rho-1)+1}(s) + \psi\left(\frac{\delta+1}{\delta}\right)\gamma^{n(\rho-1)+1}(s)\right)c_2(\delta)z + \left(3^n\rho(1-\psi)\gamma^{n\rho+1}(s)c_3(\delta) + \frac{(1-\psi)\rho(\rho-1)}{2!}2^{2n}\gamma^{n\rho+2}(s)c_2^2(\delta) + \left(3^n(\rho-1)\psi\gamma^{n(\rho-1)+1}(s) + \psi\left(\frac{\delta+2}{\delta}\right)\gamma^{n(\rho-1)+1}(s)\right)c_3(\delta) + \left(\frac{(\rho-1)(\rho-2)}{2!}2^{2n}\psi\gamma^{n(\rho-1)+2}(s) + 2^n(\rho-1)\psi\left(\frac{\delta+1}{\delta}\right)\gamma^{n(\rho-1)+2}(s)\right)c_2^2(\delta)\right)z^2 + \left(4^n(1-\psi)\rho\gamma^{n\rho+1}(s)a_4(\delta) + \frac{(1-\psi)\rho(\rho-1)}{2!}2.6^n\gamma^{n\rho+2}(s)c_2(\delta)c_3(\delta) + \frac{(1-\psi)\rho(\rho-1)(\rho-2)}{3!}2^{3n}\gamma^{n\rho+3}(s)c_2^3(\delta) + \left(4^n(\rho-1)\psi + \psi\left(\frac{\delta+3}{\delta}\right)\right)\gamma^{n(\rho-1)+1}(s)c_4(\delta) + \left(\frac{(\rho-1)(\rho-2)}{2!}2.6^n + 3^n(\rho-1)\left(\frac{\delta+1}{\delta}\right) + 2^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_3(\delta) + \left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n} + \frac{(\rho-1)(\rho-2)}{2!}2^{2n}\left(\frac{\delta+1}{\delta}\right)\right)\gamma^{n(\rho-1)+3}(s)\psi c_2^3(\delta)\right)z^3 +$$

$\Big(5^n\rho(1-\psi)\gamma^{n\rho+1}(s)c_5(\delta)+\frac{(1-\psi)\rho(\rho-1)}{2!}9^n\gamma^{n\rho+2}(s)c_3^2(\delta)+\frac{(1-\psi)\rho(\rho-1)}{2!}2.8^n\gamma^{n\rho+2}(s)c_2(\delta)c_4(\delta)+$
$\frac{(1-\psi)\rho(\rho-1)(\rho-2)}{3!}3.12^n\gamma^{n\rho+3}(s)c_2^2(\delta)c_3(\delta)+\frac{(1-\psi)\rho(\rho-1)(\rho-2)(\rho-3)}{4!}2^{4n}\gamma^{n\rho+4}(s)c_2^4(\delta)+\left(5^n(\rho-1)+\left(\frac{\delta+4}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)c_5(\delta)+$
$\left(\frac{(\rho-1)(\rho-2)}{2!}9^n+3^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_3^2(\delta)+\left(\frac{(\rho-1)(\rho-2)}{2!}2.8^n+4^n(\rho-1)\left(\frac{\delta+1}{\delta}\right)+2^n(\rho-$
$1)\left(\frac{\delta+3}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)c_2(\delta)c_4(\delta)+\left(\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}3.12^n+\frac{(\rho-1)(\rho-2)}{2!}2.6^n\left(\frac{\delta+1}{\delta}\right)+\right.$
$\left.\frac{(\rho-1)(\rho-2)}{2!}2^{2n}\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+3}(s)c_2^2(\delta)a_3(\delta)+\left(\frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}2^{4n}+\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}2^{3n}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+4}(s)c_2^4(\delta)\Big)z^4$
(1.19)

More so, notice that

$H(z,t)=1+U_1(t)z+U_2(t)z^2+U_3(t)z^3+U_4(t)z^4+\cdots$ (1.20)

$H(\omega(z,t))=1+U_!(t)\omega(z)+U_2(t)\omega(z)^2+U_3(t)\omega(z)^3+U_4(t)\omega(z)^4+\cdots$ (1.21)

Also, observe that $\omega(z)=d_!z+d_2z^2+d_3z^3+d_4z^4+d_5z^5+\cdots$ (1.22)

$\omega(z)^2=d_1^2z^2+2d_1d_2z^3+(d_2^2+2d_1d_3)z^4+(2d_1d_4+2d_2d_3)z^5+\cdots$ (1.23)

$\omega(z)^2=d_!z^3+3d_1^2d_2z^4+(3d_1^2d_3+3d_1d_2^2)z^5+\cdots$ (1.24)

$\omega(z)^4=d_1^4z^4+4d_1^3d_2z^5+\cdots,\quad \omega(z)^5=d_1^5z^5$ (1.25)

Substituting (1.22), (1.23), (1.24), and (1.25) into (1.21), we have

$H(\omega(z,t))=1+U_!(t)(d_1z+d_2z^2+d_3z^3+d_4z^4+d_5z^5+\cdots)+U_2(t)(d_1^2z^2+2d_1d_2z^3+(d_2^2+2d_1d_2)z^3+$
$(2d_1d_4+2d_2d_3)z^5+\cdots)+U_3(t)(d_1^3z^3+3d_1^2d_2z^4+(3d_1^2d_3+3d_1d_2^2)z^5)+U_4(t)(d_1^4z^4+4d_1^3d_2z^5+\cdots)+\cdots$

$H(\omega(z,t))=1+U_!(t)d_1z+U_!(t)d_2z^2+U_!(t)d_3z^3+U_!(t)d_4z^4+U_!(t)d_5z^5+\cdots$
$U_2(t)d_1^2z^2+2U_2(t)d_1d_2z^3+(d_2^2+2d_1d_3)U_2(t)z^4+(2d_1d_4+2d_2d_3)U_2(t)z^5+\cdots$
$U_3(t)d_1^3z^3+3d_1^2d_2U_3(t)z^4+(3d_1^2d_3+3d_1d_2^2)U_3(t)z^5+\cdots$
$U_4(t)(d_1^4z^4+4d_1^3d_2z^5)+\cdots$

Further simplification, gives

$H(\omega(z,t))=1+U_1(t)d_1z+(U_1(t)d_2+U_2(t)d_1^2)z^2+(U_1(t)d_3+2d_1d_2U_2(t)+U_3(t)d_1^3)z^3+(U_1(t)d_4+(d_2^2+$
$2d_1d_3)U_2(t)+3d_1^2d_2U_3(t)+U_4(t)d_1^4)z^4+\cdots$ (1.26)

Equating (1.19) and (1.26) and comparing the coefficients, we have

$U_1(t)d_1=\left(2^n\rho(1-\psi)\gamma^{n\rho+1}(s)+\left(2^n(\rho-1)+\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)\right)c_2(\delta)$

$U_1(t)d_2+U_2(t)d_1^2=\left(3^n\rho(1-\psi)\gamma^{n\rho+1}(s)+\left(3^n(\rho-1)+\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)\right)c_3(\delta)+\Big(2^{2n}\frac{\rho(\rho-1)}{2!}(1-\psi)\gamma^{n\rho+2}(s)+$
$\left(2^{2n}\frac{(\rho-1)(\rho-2)}{2!}+2^n(\rho-1)\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)\Big)c_2^2(\delta)$

$U_1(t)d_3+2U_2(t)d_1d_2+U_3(t)d_1^3=\left(4^n\rho(1-\psi)\gamma^{n\rho+1}(s)+\left(4^n(\rho-1)+\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)\right)c_4(\delta)+\Big(2.6^n\frac{\rho(\rho-1)}{2!}(1-$
$\psi)\gamma^{n\rho+2}(s)+\left(2.6^n\frac{(\rho-1)(\rho-2)}{2!}+\left(3^n\left(\frac{\delta+1}{\delta}\right)+2^n\left(\frac{\delta+2}{\delta}\right)\right)(\rho-1)\right)\psi\gamma^{n(\rho-1)+2}(s)\Big)c_2(\delta)c_3(\delta)+\Big(2^{3n}\frac{\rho(\rho-1)(\rho-2)}{3!}(1-\psi)\gamma^{n\rho+3}(s)+$
$\left(2^{3n}\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}+2^{2n}\frac{(\rho-1)(\rho-2)}{2!}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+3}(s)\Big)c_2^3(\delta)$

$U_1(t)d_4+U_2(t)(d_2^2+2d_1d_3)+U_3(t)3d_1^2d_2+U_4(t)d_1^4=\left(5^n\rho(1-\psi)\gamma^{n\rho+1}(s)+\left(5^n(\rho-1)+\left(\frac{\delta+4}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s)\right)c_5(\delta)+$
$\Big(2.8^n\frac{\rho(\rho-1)}{2!}(1-\psi)\gamma^{n\rho+2}(s)+\left(2.8^n\frac{(\rho-1)(\rho-2)}{2!}+\left(4^n\left(\frac{\delta+1}{\delta}\right)+2^n\left(\frac{\delta+3}{\delta}\right)\right)(\rho-1)\right)\psi\gamma^{n(\rho-1)+2}(s)\Big)c_2(\delta)c_4(\delta)+\Big(9^n\frac{\rho(\rho-1)}{2!}(1-$
$\psi)\gamma^{n\rho+2}(s)+\left(9^n\frac{(\rho-1)(\rho-2)}{2!}+3^n(\rho-1)\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s)\Big)c_3^2(\delta)+\Big(3.12^n\frac{\rho(\rho-1)(\rho-2)}{3!}(1-\psi)\gamma^{n\rho+3}(s)+$
$\left(3.12^n\frac{(\rho-1)(\rho-2)(\rho-3)}{3!}+\frac{(\rho-1)(\rho-2)}{2!}\left(2.6^n\left(\frac{\delta+1}{\delta}\right)+2^n\left(\frac{\delta+2}{\delta}\right)\right)\right)\psi\gamma^{n(\rho-1)+3}(s)\Big)c_2^2(\delta)c_3(\delta)+\Big(2^{4n}\frac{\rho(\rho-1)(\rho-2)(\rho-3)}{4!}(1-\psi)\gamma^{n\rho+4}(s)+$
$\left(2^{4n}\frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}+2^{4n}\frac{(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{4!}\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+4}(s)\Big)c_2^4(\delta)$ (1.27)

Simplifying (1.27), we have

$$c_2(\delta)=\frac{2tc_1}{A_N+B_N},$$

$$c_3(\delta)=\frac{2tc_2+c_1^2(4t^2-1)}{A'_N+B'_N}+\frac{4t^2c_1^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)},$$

$$c_4(\delta)=\frac{2tc_3+2(4t^2-1)c_1c_2+(8t^3-4t)c_1^3}{D'_N+E'_N}+\frac{(F_N+G_N)(4t^2c_1c_2+2tc_1^3(4t^2-1))}{(A_N+B_N)(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3c_1^3(D_N+E_N)(F_N+G_N)}{(A_N+B_N)^3(A'_N+B'_N)(D'_N+E'_N)}+\frac{8c_1^3t^3(F'_N+G'_N)}{(A_N+B_N)^3(D'_N+E'_N)},$$

$$c_5(\delta)=\frac{2tc_4+(c_2^2+2c_1c_3)(4t^2-1)+3c_1^2c_2(8t^3-4t)+c_1^4(16t^4-12t^2-1)}{H_N+I_N}+\frac{(4t^2c_1c_3+(16t^3-4t)c_1^2c_2+(16t^4-8t^2)c_1^4)(H'_N+I'_N)}{(A_N+B_N)(D'_N+E'_N)(H_N+I_N)}+$$
$$\frac{(8t^3c_1^2c_2+4t^2c_1^4(4t^2-1)(F_N+G_N))(H'_N+I'_N)}{(A_N+B_N)^2(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4c_1^4(D_N+E_N)(F_N+G_N)(H'_N+I'_N)}{(A_N+B_N)^4(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4c_1^4(F'_N+G'_N)(H'_N+I'_N)}{(A_N+B_N)^4(D'_N+E'_N)(H_N+I_N)}+$$
$$\frac{(4t^2c_2^2+4tc_1^2c_2(4t^2-1)+c_1^4(4t^2-1)^2)(J_N+K_N)}{(A'_N+B'_N)^2(H_N+I_N)}+\frac{(16t^3c_1^2c_2+8t^2c_1^4(4t^2-1))(D_N+E_N)(J_N+K_N)}{(A'_N+B'_N)^2(A_N+B_N)^2(H_N+I_N)}+\frac{16t^4c_1^4(D_N+E_N)^2(J_N+K_N)}{(A_N+B_N)^4(A'_N+B'_N)^2(H_N+I_N)}+$$
$$\frac{(8t^2c_1^2c_2+(16t^4-4t^2)c_1^4)(J'_N+K'_N)}{(A_N+B_N)^2(A'_N+B'_N)(H_N+I_N)}+\frac{16t^4c_1^4(D_N+E_N)(J'_N+K'_N)}{(A_N+B_N)^4(A'_N+B'_N)(H_N+I_N)}+\frac{16t^4c_1^4(L_N+M_N)}{(A_N+B_N)^4(H_N+I_N)}$$

where $U_1(t)=2t, U_2(t)=4t^2-1, U_3(t)=4t^3-4t, U_4(t)=16t^4-12t^2+1.$

Then, applying Lemma 1.5, we have

$$|c_2(\delta)|\le\frac{2t}{A_N+B_N},$$

$$|c_3(\delta)|\le\frac{2t+4t^2-1}{A'_N+B'_N}+\frac{4t^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)},$$

$$|c_4(\delta)|\le\frac{8t^3+8t^2-2t-2}{D'_N+E'_N}+\frac{(F_N+G_N)(4t^2+2t(4t^2-1))}{(A_N+B_N)(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(D_N+E_N)(F_N+G_N)}{(A_N+B_N)^3(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(F'_N+G'_N)}{(A_N+B_N)^3(D'_N+E'_N)},$$

$$|c_5(\delta)|\le\frac{16t^4+24t^3-10t-4}{H_N+I_N}+\frac{(16t^4+16t^3-4t^2-4t)(H'_N+I'_N)}{(A_N+B_N)(D'_N+E'_N)(H_N+I_N)}+\frac{(8t^3+4t^2(4t^2-1))(F_N+G_N)(H'_N+I'_N)}{(A_N+B_N)^2(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}$$
$$+\frac{16t^4(D_N+E_N)(F_N+G_N)(H'_N+I'_N)}{(A_N+B_N)^4(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4(F'_N+G'_N)(H'_N+I'_N)}{(A_N+B_N)^4(D'_N+E'_N)(H_N+I_N)}$$
$$+\frac{(4t^2+4t(4t^2-1)+(4t^2-1)^2)(J_N+K_N)}{(A'_N+B'_N)^2(H_N+I_N)}+\frac{(16t^3+8t(4t^2-1))(D_N+E_N)(J_N+K_N)}{(A'_N+B'_N)^2(A_N+B_N)^2(H_N+I_N)}$$
$$+\frac{16t^4(D_N+E_N)^2(J_N+K_N)}{(A_N+B_N)^4(A'_N+B'_N)^2(H_N+I_N)}+\frac{(16t^4+8t^3-4t^2)(J'_N+K'_N)}{(A_N+B_N)^2(A'_N+B'_N)(H_N+I_N)}$$
$$+\frac{16t^4(D_N+E_N)(J'_N+K'_N)}{(A_N+B_N)^4(A'_N+B'_N)(H_N+I_N)}+\frac{16t^4(L_N+M_N)}{(A_N+B_N)^4(H_N+I_N)}$$

This complete the proof.

By specializing some parameters, the following deduced corollaries.

Corollary 2.2[1]: If $f_\gamma(z)^\delta$ belongs to the class $S_\gamma(\delta,n,\rho,\psi)$**,** and, $\delta=1,\rho\ge 0, 0\le\psi\le 1, n\epsilon N_o$ then

$$|c_2(1)|\le\frac{2t}{A+B},$$

$$|c_3(1)|\le\frac{2t+(4t^2-1)}{(A'_N+B'_N)}+\frac{4t^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)},$$

$$|c_4(1)|\le\frac{8t^3+8t^2-2t-2}{(D'_N+E'_N)}+\frac{(F_N+G_N)(4t^2+2t(4t^2-1)}{(A_N+B_N)(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(D_N+E_N)(F_N+G_N)}{(A_N+B_N)^3(A'_N+B'_N)(D'_N+E'_N)}+\frac{8t^3(F'_N+G'_N)}{(A_N+B_N)^3(D'_N+E'_N)}$$

$$|c_5(1)|\le\frac{16t^4+24t^3-10t-4}{(H_N+I_N)}+\frac{(H'_N+I'_N)(16t^4+16t^3-4t^2-4t)}{(A_N+B_N)(D'_N+E'_N)(H_N+I_N)}+\frac{(F_N+G_N)(H'_N+I'_N)(16t^4+8t^3-4t^2)}{(A_N+B_N)^2(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+$$
$$\frac{16t^4(D_N+E_N)(F_N+G_N)(H'_N+I'_N)}{(A_N+B_N)^4(A'_N+B'_N)(D'_N+E'_N)(H_N+I_N)}+\frac{16t^4(F'_N+G'_N)(H'_N+I'_N)}{(A_N+B_N)^4(D'_N+E'_N)(H_N+I_N)}+\frac{(16t^4+16t^3-4t^2-4t+1)(J_N+K_N)}{(A'+B')^2(H_N+I_N)}+$$
$$\frac{(D_N+E_N)(J_N+K_N)(32t^4+16t^3-8t^2)}{(A_N+B_N)^2(A'_N+B'_N)^2(H_N+I_N)}+\frac{(D_N+E_N)^2(J_N+K_N)16t^4}{(A_N+B_N)^4(A'+B')^2(H_N+I_N)}+\frac{(J'_N+K'_N)(16t^4+8t^3-4t^2)}{(A_N+B_N)^2(A'_N+B'_N)(H_N+I_N)}+\frac{16t^4(D_N+E_N)(J'_N+K'_N)}{(A_N+B_N)^4(A'_N+B'_N)(H_N+I_N)}+$$
$$\frac{(L_N+M_N)16t^4}{(A_N+B_N)^4(H_N+I_N)}$$

Corollary 2.3: if $f(z)^\delta{}_\gamma$ belongs to the class $S_\gamma(1,0,1,0)$, then

$$|c_2(1)|\le\frac{2t}{\gamma(s)},$$

$$|c_3(1)|\le\frac{2t+4t^2-1}{\gamma(s)},$$

$$|c_4(1)| \leq \frac{8t^3+8t^2-2t-2}{\gamma(s)},$$
$$|c_5(1)| \leq \frac{16t^4+24t^3-10t-4}{\gamma(s)}$$

Proof: Setting $\delta = 1, n = 0, \rho = 1$ and $\psi = 0$ in (1.13), the results follows.

Corollary 2.4: if $f(z)^{\delta}{}_{\gamma}$ belongs to the class $S_1(1,0,1,0)$, then

$$|c_2(1)| \leq 2t,$$
$$|c_3(1)| \leq 2t + 4t^2 - 1,$$
$$|c_4(1)| \leq 8t^3 + 8t^2 - 2t - 2,$$
$$|c_5(1)| \leq 16t^4 + 24t^3 - 10t - 4$$

Proof: Setting $\gamma = 1, \delta = 1, n = 0, \rho = 1$ and $\psi = 0$ in (1.13), the results follows.

Corollary 2.5: if $f(z)^{\delta}{}_{\gamma}$ belongs to the class $S_\gamma(1,0,1,1)$, then

$$|c_2(1)| \leq \frac{2t}{2\gamma(s)},$$
$$|c_3(1)| \leq \frac{2t+4t^2-1}{3\gamma(s)},$$
$$|c_4(1)| \leq \frac{8t^3+8t^2-2t-2}{4\gamma(s)},$$
$$|c_5(1)| \leq \frac{16t^4+24t^3-10t-4}{5\gamma(s)}$$

Proof: Setting $\delta = 1, n = 0, \rho = 1$ and $\psi = 1$ in (1.13), the results follows.

Corollary 2.6: if $f(z)^{\delta}{}_{\gamma}$ belongs to the class $S_\gamma(1,1,1,0)$, then

$$|c_2(1)| \leq \frac{2t}{2\gamma^2(s)},$$
$$|c_3(1)| \leq \frac{2t+4t^2-1}{3\gamma^2(s)},$$
$$|c_4(1)| \leq \frac{8t^3+8t^2-2t-2}{4\gamma^2(s)},$$
$$|c_5(1)| \leq \frac{16t^4+24t^3-10t-4}{5\gamma^2(s)}$$

Proof: Setting $\delta = 1, n = 1, \rho = 1$ and $\psi = 0$ in (1.13), the results follows.

Corollary 2.7: if $f(z)^{\delta}{}_{\gamma}$ belongs to the class $S_\gamma(1,1,1,1)$, then

$$|c_2(1)| \leq \frac{2t}{2\gamma(s)},$$
$$|c_3(1)| \leq \frac{2t+4t^2-1}{3\gamma(s)},$$
$$|c_4(1)| \leq \frac{8t^3+8t^2-2t-2}{4\gamma(s)},$$
$$|c_5(1)| \leq \frac{16t^4 + 24t^3 - 10t - 4}{5\gamma(s)}$$

Proof: Setting $\delta = 1, n = 1, \rho = 1$ and $\psi = 1$ in (1.13), the results follows.

Corollary 2.8: if $f(z)^{\delta}{}_{\gamma}$ belongs to the class $S_\gamma(1,0,\rho,\psi)$, then

$$|c_2(1)| \leq \frac{2t}{A+B},$$
$$|c_3(1)| \leq \frac{2t+4t^2-1}{A'+B'} + \frac{2t^2(A'+B')(1-\rho)}{(A+B)^2(A'+B')},$$
$$|c_4(1)| \leq \frac{8t^3+8t^2-2t-2}{\rho+3\psi} + \frac{(1-\rho)(4t^2+2t(4t^2-1))}{(\rho+\psi)(\rho+2\psi)} + \frac{4t^3(\rho-1)^2}{(\rho+\psi)^3} - \frac{4t^3(\rho-1)(\rho-2)}{3(\rho+\psi)^3},$$
$$|c_5(1)| \leq \frac{16t^4+24t^3-10t-4}{\rho+4\psi} + \frac{(1-\rho)(16t^4+16t^3-4t^2-4t)}{(\rho+\psi)(\rho+3\psi)} + \frac{(1-\rho)^2\left(8t^3+4t^2(4t^2-1)\right)}{\rho+\psi^2(\rho+2\psi)} + \frac{(1-\rho)^2 8t^4}{(\rho+\psi)^4} + \frac{8t^4(\rho-1)^2(\rho-2)}{3(\rho+\psi)^4} +$$
$$\frac{32t^3+8t^2-8t+(4t^2-1)^2(1-\rho)}{2(\rho+2\psi)^2} + \frac{(1-\rho)^2\left(4t^3+2t^2(4t^2-1)\right)}{(\rho+2\psi)(\rho+\psi)^2} + \frac{2t^4(1-\rho)^3}{(\rho+\psi)^4} - \frac{(\rho-1)(\rho-2)(8t^4+4t^3-2t^2)}{(\rho+\psi)^2(\rho+2\psi)} + \frac{4t^2(1-\rho)^2}{(\rho+\psi)^4} - \frac{(\rho-1)(\rho-2)(\rho-3)2t^4}{3(\rho+\psi)^4}$$

Proof: Setting $\delta = 1, n = 0, \rho = 1$ and $\gamma = 1$ in (1.13), the results follows.

Theorem 2.9

If $\boldsymbol{f(z)}$ belongs to the class $S_\gamma(m,\mu,\lambda,\sigma)$ and $\mu \geq 1, \sigma > 0, 0 \leq \lambda \leq 1, m \in N_0$ then for any real number $\boldsymbol{\mu}$**:**

$$\left|c_3-\mu c_2^2\right|\le 2k\begin{cases}1-\dfrac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-\mu 4t^2(A'_N+B'_N)}{t(A'_N+B'_N)} & ,if\ \mu\le k_1\\ 1 & ,if\ k_1\le\mu\le k_2\\ \dfrac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-\mu 4t^2(A'_N+B'_N)}{t(A'_N+B'_N)}-1 & ,if\ \mu\ge k_2\end{cases} \quad (1.28)$$

where

$$k=\frac{2t}{(A_N+B_N)^2},$$

$$k_1=\frac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)}{4t^2(A'_N+B'_N)},$$

$$k_2=\frac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-2t(A'_N+B'_N)}{4t^2(A'_N+B'_N)},$$

$$A_N=2^n\rho(1-\psi)\gamma^{n\rho+1}(s),$$

$$B_N=\left(2^n(\rho-1)+\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s),$$

$$A'_N=3^n\rho(1-\psi)\gamma^{n\rho+1}(s),$$

$$B'_N=\left(3^n(\rho-1)+\left(\frac{\delta+2}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+1}(s),$$

$$D_N=-(1-\psi)2^{2n}\frac{\rho(\rho-1)}{2!}\gamma^{n\rho+2}(s),$$

$$E_N=-\left(\frac{(\rho-1)(\rho-2)}{2!}2^{2n}+2^n(\rho-1)\left(\frac{\delta+1}{\delta}\right)\right)\psi\gamma^{n(\rho-1)+2}(s).$$

Proof:
Using the coefficient bounds from theorem 2.1, which gives

$$a_2(\delta)=\frac{2tc_1}{A_N+B_N},$$

The square of the coefficient bounds gives

$$c_2^2(\delta)=\frac{4t^2d_1^2}{(A_N+B_N)^2}.$$

$$c_3(\delta)=\frac{2td_2+d_1^2(4t^2-1)}{A'_N+B'_N}+\frac{4t^2d_1^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)}.$$

$$c_3(\delta)-\mu c_2^2(\delta)=\frac{2td_2+d_1^2(4t^2-1)}{A'_N+B'_N}+\frac{4t^2d_1^2(D_N+E_N)}{(A_N+B_N)^2(A'_N+B'_N)}-\frac{\mu 4t^2d_1^2}{(A_N+B_N)^2}$$

$$c_3(\delta)-\mu c_2^2(\delta)=\frac{2td_2(A'_N+B'_N)+d_1^2(4t^2-1)(A'_N+B'_N)+4t^2d_1^2(D_N+E_N)-\mu 4t^2d_1^2(A'_N+B'_N)}{(A_N+B_N)^2(A'_N+B'_N)}$$

$$c_3(\delta)-\mu c_2^2(\delta)=\frac{2t(A'_N+B'_N)}{(A_N+B_N)^2(A'_N+B'_N)}\left(d_2+\frac{((4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-\mu 4t^2(A'_N+B'_N))d_1^2}{2t(A'_N+B'_N)}\right)$$

Notice $v=\dfrac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-\mu 4t^2(A'_N+B'_N)}{2t(A'_N+B'_N)}$

Case1: when $v\le 0$

$$-(4t^2-1)(A'_N+B'_N)-4t^2(D_N+E_N)+\mu 4t^2(A'_N+B'_N)\le 0,$$

$$\mu\le\frac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)}{4t^2(A'_N+B'_N)},$$

$$\mu\le k_1.$$

Case 2; when $v\ge 1$, we have

$$\frac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-\mu 4t^2(A'_N+B'_N)}{2t(A'_N+B'_N)}\leq 1,$$

$$-(4t^2-1)(A'_N+B'_N)-4t^2(D_N+E_N)+\mu 4t^2(A'_N+B'_N)\geq -2t(A'_N+B'_N),$$

$$\mu\geq\frac{(4t^2-1)(A'_N+B'_N)+4t^2(D_N+E_N)-2t(A'_N+B'_N)}{4t^2(A'_N+B'_N)},$$

$$\mu\geq k_2$$

By applying lemma 1.4 with simple substitution of the various variables involves completes the desire result of the proof of the **theorem 2.1.**

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