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A NOTE ON THE CONNECTIONS AND CONTRASTS OF SOME SELECTED RESULTS IN GROUPS AND MULTIGROUPS

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Abstract

This study undertakes a comparative examination of some selected properties (results) of groups and multigroups with a view to discover their similarities and differences. We compare the nature of the order of elements, behaviour of elements with respect to Lagrange's theorem and homomorphisms in both structures.

***Keywords*: groups, multigroups, order of elements, Lagrange's theorem, homomorphisms.**

1. Introduction

Set theory as an aspect of mathematical logic that studies collection of well-defined object has over the years and up till today form a building block upon which mathematics is done. In group theory for instance, it is the first issue that comes to mind because the group structure itself is built with a non-empty set as its underlying foundation. This however translates to saying without sets, mathematicians wouldn't have been gifted the honor of enjoying the aspect of group theory.

Further, it is known that in classical sets, repetition of elements is not allowed. But when the notion of sets is viewed in light of real-life situations, we cannot dispute that repetition of objects cannot be ignored. This fact however brought about the idea of multisets. Multiset is an unordered collection of objects in which unlike a standard (cantorian) set, duplicates or multiples of objects are admitted [1]. Multiset is an aspect of non-classical set theory.

The multigroup is a structure developed from groups as its root set, but whose elements are multisets. The concept multigroups has been studied by several authors but from various stand point. These studies were carried out first without the idea of multisets in view. The study began with [2] who first studied what is known as hypergroup. Hypergroup can be obtained if the binary operation in the case of a group is taken to be multivalued. The structure turns out to be a generalisations of groups. A different perspective to the hypergroup was considered by [3]. He stated the definition of hypergroup in a way quite different from that in [2]. He saw it as a system in which the product ab of any two elements a, b forms a complex n not necessarily distinct elements of the system, where n is a fixed integer ($n \geq 1$). In a further development, a most general definition to hypergroup was proposed by [4] and called it a multigroup. They did this by considering only the existence of solutions to linear relations while neglecting the existence of two-sided units and inverses. A multigroup is an algebraic system with the operation of multiplication which satisfies the classical group axioms but product is not unique [4].

In recent times, what appear to be in use as a definition to multigroup is with respect to multisets and this was introduced in [5]. A multigroup is an extension or a generalisation of the classical group due to the fact that it is defined with reference to multisets which takes cognizance of multiplicities as count functions and it equally follows the trend of other non-classical groups [5]. Further studies on multigroups can be found in [6-9].

Juxtaposing the properties of some mathematical structures brings us to seeing some similarities or differences in terms of their structural properties. In this paper, we study some existing properties that are peculiar to both the classical group and the multigroup with a view to establish similarities or differences between them. This paper is subdivided into three sections. In the second section, we present preliminary definitions and existing results relating to both groups and multigroups, and the last section presents the findings of this paper.

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2. Preliminaries

In this section, we present definitions and preliminary results in groups and multigroups. Some basic concepts in classical group theory are omitted since they can be found handy in any group theory text like [10].

Definition 2.1 Group [10]

An algebraic structure $(G,*)$ is said to be a group if the binary operation $*$ satisfies the following axioms:

i. Closure property i.e, $a\,b \in G \;\forall\, a, b \in G$
ii. Associativity i.e $(ab)c = a(bc), \forall\, a, b, c \in G$
iii. Existence of identity: There exists an element $e \in G$ such that $ea = ae \;\forall a \in G$ The element e is called the identity element of the group.
iv. Existence of inverse: for each $a \in$ G, there exists an element $b \in G$ such that $ba = e = ab$. The element b is then called inverse of a.

Definition 2.2 Count Function of a Multiset [9]

Let P be any multiset drawn from a set X. The count function C_P is defined by

$C_P : X \to K = \{0,1,2,3,...\}$

$C_P(x)$ denotes the number of times the element $x \in X$ appears in the multiset P.

Definition 2.3 Multigroup [5]

Let X be a group. A multiset G over X is said to be a multigroup over X if the count function C_G satisfies the following two conditions:

i. $C_G(xy) \geq C_G(x) \wedge C_G(y) \;\forall x, y \in X$
ii. $C_G(x^{-1}) \geq C_G(x) \;\forall x \in X$.

The set of all multigroups over X is denoted by $MG(X)$. Note that "∧" in the definition denotes minimum. The example of multigroup below is adopted from Nazmul et. al, 2013.

Let $X = \{e, a, b, c\}$ be the Klein's 4-group and $G = \{e, e, e, a, a, b, b, b, c, c\}$ be a multiset over X: Now

1. $C_G(ea) = C_G(a) = 2 \geq [C_G(e) \wedge C_G(a)]$, $C_G(eb) = C_G(b) = 3 \geq [C_G(e) \wedge C_G(b)]$
$C_G(ec) = C_G(c) = 2 \geq [C_G(e) \wedge C_G(c)]$, $C_G(ab) = C_G(c) = 2 \geq [C_G(a) \wedge C_G(b)]$
$C_G(bc) = C_G(a) = 2 \geq [C_G(b) \wedge C_G(c)]$, $C_G(ca) = C_G(b) = 3 \geq [C_G(c) \wedge C_G(a)]$
$C_G(a^2) = C_G(e) = 3 \geq [C_G(a) \wedge C_G(a)]$, $C_G(b^2) = C_G(e) = 3 \geq [C_G(b) \wedge C_G(b)]$
$C_G(c^2) = C_G(e) = 3 \geq [C_G(c) \wedge C_G(c)]$, $C_G(e^2) = C_G(e) = 3 \geq [C_G(e) \wedge C_G(e)]$
2. $C_G(a^{-1}) = C_G(a) = 2$, $C_G(b^{-1}) = C_G(b) = 3$, $C_G(c^{-1}) = C_G(c) = 3$, $C_G(e^{-1}) = C_G(e) = 3$.

Therefore G is a multigroup over X.

Definition 2.4 Submultigroup [5]

Let $A, B \in MG(X)$. Then A is said to be a submultigroup of B if A ⊆B.

Definition 2.5 Regular Multigroup [9]

Let $A \in MG(X)$. Then A is called regular if the count function of A occurs with the same multiplicity. The set of all regular multigroups over X is denoted by $RMG(X)$.

Definition 2.6 Order of an element of a Multigroup [9]

Let $A \in MG(X)$ and $x \in X$. If there exists a positive integer n such that $C_A(x^n) = C_A(e)$, then the least such n is called the order of an element x with respect to A. If no such n exists, x is said to be of infinite order with respect to A. The order of an element x with respect to A is denoted
by $O_A(x)$.

Definition 2.7 Homomorphism in Multigroup [5]

Let X and Y be groups and let $f: X \to Y$ be a homomorphism from X to Y. Let A and B be multisets over X and Y respectively. Then

i. the image of A under f denoted by $f(A)$ is a multigroup of Y defined by

$$C_{f(A)}(y) = \begin{cases} \vee_{x \in f^{-1}(y)} C_A(x) & f^{-1}(y) \neq \emptyset \\ 0 & otherwise. \end{cases}$$

For each $y \in Y$.

ii. the inverse image of B under f denoted by $f^{-1}(B)$ is a multigroup of X defined by,
$C_{f^{-1}(B)}(x) = C_B(f(x))$ for all $x \in X$.

Definition 2.8 Multigroup Restricted to a Subgroup [9]

Let H be a subgroup of a group G and $A \in MG(X)$. The restriction of the multigroup A to the subgroup H is defined as the multigroup consisting of only the element of H. This is denoted by $A|H$.

In what follows, we present results in classical group setting and their corresponding multigroup analogue. These results will be resourceful in driving home our idea.

Theorem 2.1 [10]

The order of an element x in a group G is the same as the order of its inverse, x^{-1}

Theorem 2.2 [9]

Let $A \in MG(X)$. Then $O_A(x) = O_A(x^{-1})$.

The above results translates to saying that the order of an element x in a multigroup A is the same as the order of its inverse.

Theorem 2.3 [10] - Lagrange's Theorem

The order of each subgroup of a finite group is a divisor of the order of the group

Proposition 2.4 [9]

Let H be a subgroup of the group X. If $H \leq X$ and $A \in MG(X)$, then $o(A|H) \leq o(A)$.

Theorem 2.5 [9]

Let H be a subgroup of the group X and $A \in MG(X)$, $A|H \in RMG(X)$. Then $o(A\big|H)|o(A)$

Theorem 2.6 [11]

Let G be a finite multigroup of a group X and let H be a complete submultigroup of G wherein the count of every element in H is a factor of the count of the corresponding element in G. Then, the order of H divides the order of G.

The reader can refer to the references against Theorems 2.1 – 2.6 for proofs and concretizations.

3. Results

This section achieves the result of comparing and contrasting some properties of groups and multigroups. We devote three sub-sections to these findings. These sub-sections discuss the results for order of elements, Lagrange's theorem and homomorphisms in groups and multigroups respectively. Throughout these sections we let $A \in MG(X)$, where X is a group. To begin, we observe from definitions 2.1 and 2.6 that

i. any multigroup A over a group X inherits the binary operation of the group X. Should there be any need for an operation in A, the operation in X does it.
ii. in the classical group setting, emphasis is not on the number of element but the idea of the definition of a multigroup is built around the number of times $x \in X$ appear in A.
iii. while the condition for a structure to be a group must be satisfied for all $x \in X$, all the elements from the group are needed to meet the conditions for the multigroup.
iv. the multigroup A will always depend on the group X from which it is drawn to establish its validity.

3.1 Order Relations in Groups and Multigroups.

Considering definitions 2.9 and 2.11,

i. It is known from elementary group theory that the order of a group X is defined as $|X| = n$ $(n \in \mathbb{N})$, the cardinality of X. The order in multigroup is the sum of the multiplicities of a multigroup $A \in MG(X)$. This technically makes the idea of $o(A)$ and the $o(X)$ the same.
ii. From classical groups, it is true that $o(x) = n$ if $x^n = e$ where, $x \in X$ and $(n \in \mathbb{N})$, where $o(x)$ denotes the order of $x \in X$. It is clear from definition 2.11 that the idea of the definition of the order of an element in a multigroup A is built from the definition of order of an element in the classical group X.
iii. We can visualize from theorems 2.1 and 2.2, that while in group theoretical setting, the order of an element and its inverse always coincide, this property equally holds good in the multigroup setting.

3.2 Lagrange's Theorem in Groups and its Analogues in Multigroups

From theorems 2.3, 2.5 and 2.6, we observe thus:

i. while aware that in group setting the Lagrange's theorem asserts that the order of any subgroup Y of a group X divides the order of the group, this argument doesn't immediately follow in multigroup case. For the analogue of Lagrange's theorem to hold, certain conditions need to be attached to the behaviour of the corresponding submultigroup of a multigroup as seen from theorems 2.5 and 2.6. It is evident from [9] in theorem 2.5, that to achieve what appears to be an analogue to the Lagrange's Theorem, the extra idea of restricting a multigroup to a subgroup (definition 2.13) was introduced. The modification of theorem 2.5 made by [11] requires the idea of complete submultigroup.

3.4. Homomorphism in Groups and Multigroups

From definition 2.12, we observe that

i. while the operation preserving property ($\psi(g_1 g_2) = \psi(g_1).\psi(g_2), \forall g_1, g_2 \in X$, where ψ is a group morphism) is the very heart of homomorphism in the classical group setting we see it play a role only as number of images appearing in the multigroups.
ii. the homomorphism in multigroups inherits the binary operation in the group homomorphism.
iii. While the method of determining the image set appears easy based on the given function, determining the image set in multigroup setting appears to be somewhat cumbersome and it follows a somewhat rigorous process.

4.0 Conclusion

We have in this study considered some properties associated to both groups and Multigroups and have spelt clearly the similarities they share and how they differ. Basically we conclude on the note that although the Multigroup structure is drawn from Groups as its root set, it still possesses some properties different from that of the group from which it is drawn.

References

[1] Ibrahim, A. M., Singh, D., and Singh, J. N. (2011). *An Outline of Multiset Space Algebra.* International Journal of Algebra, 5(31):1515–1525.
[2] Marty, F. (1934). *Sur une Generalisation de la Notion de Groupe*. Huitieme Congres des Mathematiciens Scandinaves, Stockholm.
[3] Wall, H. S. (1937). *Hypergroups*. American Journal of Mathematics, 59(1):77–98.
[4] Dresher, M. and Ore, O. (1938). *Theory of Multigroups*. American Journal of Mathematics, 60:705–733.
[5] Nazmul, S. K., Majumdar, P., and Samanta, S. K. (2013). *On Multisets and Multigroups.* Annals of Fuzzy Mathematics and Informatics, 6(3):643–656.
[6] Tella, Y. and Daniel, S. (2013a). *A Study of Group Theory in the Context of Multiset Theory*. International Journal of Science and Technology, 2(8):609–615.
[7] Tella, Y. and Daniel, S. (2013b). *Symmetric Groups Under Multiset Perspective*. IOSR Journal of Mathematics, 7(5):47–52.
[8] Awolola, J. A. and Ibrahim, A. M. (2016). *Some Results on Multigroups*. Quasigroups and Related System, 24:169–177.
[9] Awolola, J. A. and Ejegwa, P. A. (2017). *On Some Algebraic Properties of Order of an Element of a Multigroup*. Quasigroups and Related Systems, 25:21–26.
[10] Vasishtha, A. R. (2009). Modern Algebra. *First Edition. Krishna Prakashan, Mandir,Meerut India*
[11] Ejegwa, P. A and Ibrahim, M. A. (2017). *Some Homomorphic Properties of Multigroups.* Buletinul Academiei de Stiinte. A Republicii Moldova Matematica. Vol 1(83), pp. 67-76.