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COMPUTATIONAL ANALYSIS OF HIGHER- ORDER INTEGRO-DIFFERENTIAL EQUATIONS BY CANONICAL POLYNOMIAL BASIS FUNCTIONS

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Abstract

This paper presents the computational analysis approach to the solution of higher order Integro- Differential Equations (IDES) via the canonical polynomial basis function. The canonical polynomial basis function generated by re-defining the differential part in operator form. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. Two numerical examples were considered with the use of Mathematical Software (MATLAB 2009b) to illustrate the performance, efficiency and implementation of the method. Hence, the results showing the performance and effectiveness of the technique were presented in tabular form. The technique has approachable better performance than variational iteration method when compared.

Keywords**:** Integro Differential Equations (IDEs), canonical, integral equations.

Introduction

In recent years, there has been a growing interest in the Integro-Differential Equations (IDEs). IDEs play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics. The higher-order integro differential equations arises in mathematical, applied and engineering sciences, astrophysics, solid state physics, astronomy, fluid dynamics, fiber optics and chemical reaction diffusion models; [1-6] and references therein. The mentioned integro-differential equations are usually difficult to solve analytically; so a numerical method is required.

The developments of the collocation tau method (see Taiwo et al[7] and Ortiz and Samara [8]) with bases functions (Canonical and Chebyshev) for the numerical solution of linear integro-differential equations (IDEs). Also, variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [9-13] which successfully applied to many situations. For example He [9] solved the classical Blasius' equation using VIM. He [10] used VIM to obtained approximate solutions for some well-known nonlinear problems. He [4] used VIM on autonomous ordinary differential systems. He [11] combined iteration method with the perturbation method to solve the well-known Blasius equation. He [13] solved nonlinear equations by discretizing the problem using VIM. Soliman [14] applied the VIM to solve the KDV-Burger's and lax's seventh-order KDV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa *et al*. [15]. The VIM applied in solving nonlinear differential equations of fractional order by Odibat *et al*.[16]. Bildik *et al*. [17] used VIM for solving different types of nonlinear partial differential equations. Dehghan and Tatari [18] used VIM to solve a Fokker-Planck equation. Wazwaz [19] also worked on comparative study between the variational iteration method and Adomian decomposition method. Tamer *et al*.[20] introduced a modification of VIM. Abbasandy [21] solved one example of the quadratic Riccati differential equation by He's VIM by using Adomian's polynomials. Moreover, application of the Chebyshev and canonical polynomials and their numerical merits in solving ODEs and PDEs numerically have been discussed in a series of papers [4,13, 16, 17, 22-26].

In the paper, we are, therefore, motivated to work in this direction of extending the collocation proposed in the literatures to

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handle IDEs numerically especially higher-order equations. As variational iteration method were developed [4, 10, 26-30] to solve effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions so also Chebyshev Tau method and Legendry polynomial shows similar with small error. Hence, higher-order integro differential equations are equivalent to the system of integral equations which can be solved efficiently using the canonical polynomial with the given problems..

Problem to be considered

Let D be a linear differential operator of order v with polynomial coefficients defined by

$$D := \sum_{n=0}^{v} g_n(x)\frac{d^n}{dx^{-n}}, \tag{1}$$

We shall write for $g_n(x)$

$$g_n(x) := \sum_{j=0}^{\alpha r} g_{nj}x^j, \tag{2}$$

Where α_n is the degree of $g_n(x)$ and $\underline{\underline{g}}_n = (g_{n0},\ldots g_{n\alpha},0,0,\ldots), \underline{X} = (1,x,x^2,\ldots)^T$.

Unless otherwise stated, x will always be the independent variable of the functions which appear throughout this paper and will be defined in a finite interval.

Let $y(x)$ be the exact solution of the integro-differential equation,

$$Dy(x) - \lambda\int_a^b m(x,t)y(t)dt = f(x), \qquad x \in [a,b] \tag{3}$$

With

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}y^{(m-1)}(a) + c_{jm}^{(2)}y^{(m-1)}(b)\right] = d_j, \qquad j = 1,\ldots,v, \tag{4}$$

Where $f(x)$ and $m(x,t)$ are given continuous functions $\lambda, a, b, c_{jm}^1, c_{jm}^2$ and d_j some given constants.

Matrix representation for the different parts

Let $\underline{V} := \{v_0(x), v_1(x),\ldots\}$ be a polynomial basis by $\underline{V} := V\underline{X}$, where V is a non-singular lower triangular matrix and degree $(v_i(x)) \le i$, for $i = 0,1,2,\ldots.$. Also for any matrix P, $P_v = VPV^{-1}$.

Now we convert the Eq. (3) and (4) to the corresponding linear algebraic equations in three parts; (a), (b) and (c).

(a) Matrix representation for $Dy(x)$:

Ortiz and Samara proposed in an alternative for the Tau technique which they called the operational approach as it reduces differential problems to linear algebraic problems. The effect of differentiation, shifting and integration on the coefficients vector

$\underline{\underline{\tilde{a}}}_n := (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_n, 0,0,\ldots)$

Of a polynomial $u_n(x) = \underline{\underline{\tilde{a}}}_n\underline{X}$ is the same as that of post-multiplication of $\underline{\underline{\tilde{a}}}_n$ by the matrices η, μ and i respectively,

$$\frac{du_n(x)}{dx} = \underline{\underline{\tilde{a}}}_n\eta\underline{X}, \quad u_n(x) = \underline{\underline{\tilde{a}}}_n\mu\underline{X} \qquad \text{and} \qquad \int_0^x u_n(t)dt = \underline{\underline{\tilde{a}}}_n i\underline{X} \tag{5}$$

Where

$$\eta = \begin{bmatrix} 0 & 0 & \ldots & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 2 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \mu = \begin{bmatrix} 0 & 1 & \ldots\ldots \\ 0 & 0 & 1\ldots \\ \ldots & \ldots & \ldots\ldots \end{bmatrix}, i = \begin{bmatrix} 0 & 1 & \ldots\ldots \\ 0 & 0 & \tfrac{1}{2}\ldots \\ \ldots & \ldots & \ldots\ldots \end{bmatrix}.$$

We recall now the following theorem given by Ortiz and Samara.

(b) Matrix representation for the integral term:

Let us assume that

$$m(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} m_{ij}v_i(x)v_j(t),$$

And,

$$y(x)=\sum_{i=0}^{\infty}a_iv_i(x)=\underline{\underline{a}}\underline{V}.$$

Then, we can write

$$\int_a^b m(x,t)y(t)dt=\sum_{1=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}m_{ij}a_iv_i(x)\int_a^b v_j(t)v_i(t)dt=\underline{\underline{aM}}\underline{V}, \tag{6}$$

Where,

$$\underline{\underline{M}}\begin{bmatrix}\sum_{j=0}^{n}m_{0j}\alpha_{jo} & \cdots & \sum_{j=0}^{n}m_{nj}\alpha_{jo} & 0 & 0 & \\ . & & . & . & . & \cdots \\ \sum_{j=0}^{n}m_{0j}\alpha_{jn} & \cdots & \sum_{j=0}^{n}m_{nj}\alpha_{jn} & 0 & 0 & \cdots \\ . & & . & . & . & \cdots\end{bmatrix}$$

With,

$$\alpha_{ji}=\int_a^b v_j(t)v_1(t), \qquad \text{for} \qquad \text{j, } l=0,\dots,n.$$

(c) Matrix representation for the supplementary conditions:

Replacing $y(x)=\sum_{i=0}^{\infty}a_iv_i(x)$ in the left hand side of (4), it can be written as

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}y^{(m-1)}(a)+c_{jm}^{(2)}y^{(m-1)}(b)\right]=\sum_{i=0}^{\infty}\sum_{m=1}^{v}\left[c_{jm}^{(1)}v_i^{m-1}(a)+c_{jm}^{(2)}v_i^{(m-1)}(b)\right]=\underline{\underline{a}}B_j, \tag{7}$$

Where for j = 1, …,v,

$$B_j=\begin{bmatrix}c_{jm}^{(1)}v_0(a)+c_{jm}^{(2)}v_0(b)\\ \sum_{m=1}^{2}\left[c_{jm}^{(1)}v_1^{m-1}(a)+c_{jm}^{(2)}v_1^{(m-1)}(b)\right]\\ \vdots\\ \sum_{m=1}^{v}\left[c_{jm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right]\\ \vdots\end{bmatrix} \tag{8}$$

We refer to B as the matrix representation of the supplementary conditions and B_j as its jth column. The following relations for computing the elements of the matrix B can be deduced from (7):

$$b_{ij}=\sum_{m=1}^{v}\left[c_{jm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right], \text{ for } i,j=1,2,\dots,v, \tag{9}$$

And,

$$b_{ij}=\sum_{m=1}^{v}\left[c_{jkm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right], \qquad \text{for } i=v+1,v+2,\dots,j=1,2,\dots,v. \tag{10}$$

We introduce $\underline{\underline{d}}=(d_1,d_2,\dots,d_v)$, the vector that contains right hand sides of conditions. Then the supplementary conditions take the form

$$\underline{\underline{a}}B=\underline{\underline{d}}. \tag{11}$$

It follows from (5) and (6) that

$$Dy(x)-\lambda\int_a^b m(x,t)y(t)d=\underline{\underline{a}}(\prod{}_v-\lambda M)\underline{V}. \tag{12}$$

Let $M_v:=\prod{}_v-\lambda\underline{\underline{M}}$ and M_{vi} stands for its ith column and let $f(x)=\sum_{i=0}^{n}f_iv_i(x)=f\underline{V}$ with $f=(f_0,\dots,f_n,0,0,\dots)$.. Then the coefficient of exact solution $y=\underline{\underline{a}}\underline{V}$ of problem (3) and (4) satisfies the following infinite algebraic system:

$$\begin{cases}\underline{\underline{a}}M_{vi}=f_i; & i=0,\dots,n,\\ \underline{\underline{a}}M_{vi}=0; & i\geq n+1,\\ \underline{\underline{a}}B_j=d_i; & j=1,2,\dots,v.\end{cases} \tag{13}$$

Setting,

$$G=(B_1,\dots,B_v,M_{v0},M_{v1},\dots),$$

And,

$$g=(d_1,\dots,d_v,f_0,f_1,\dots),$$

We can write instead of (13)

$$\underline{\underline{a}}G = g. \qquad (14)$$

Remark:

For $v = 0$ and $G_0(x) = 1$, Eq. (3) is transformed into a Fredholm integral equation of second kind and for $\lambda = 0$, it is transformed into a differential equation.

Approximation by Canonical polynomial basis

For the purpose of our discussion, we illustrate the basic concept of the technique consider the following general differential equation

$$Lu + Nu = g(x) \qquad (15)$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to canonical polynomial method, we can construct a correct functional as follows:

Assume an approximate solution of the form

$$y_N(x) = \sum_{i=0}^{N} a_i Q_i(x) \qquad (16)$$

Where a_r are constants to be determined and $Q_i(x)$ are the canonical polynomials generated below:

Let D be a linear differential operator of order v with polynomial coefficients defined by

$$DQ_i(x) = x^i \qquad (17)$$

From (1)

$$D \equiv \sum_{i=0}^{N} g_n(x) \tfrac{d^n}{dx^n} = g_0 + g_1 \tfrac{d}{dx} + . \quad . \quad . + g_n \tfrac{d^n}{dx^n}$$

$$Dx^i = g_0 x^i + g_1 i x^{i-1} + g_2 i(i-1)x^{i-2} + \ldots + g_n i(i-1)(1-2)\ldots(i-n)x^n \qquad (18)$$

$$D\{DQ_i(x)\} = Dx^i \qquad (19)$$

On Solving, we have

$$DQ_i(x) = i(i-1)(i-2)\ldots(i-v)Q_{i-v}(x) + g_0 Q_i(x) \qquad (20)$$

Hence,

$$Q_i(x) = \frac{1}{g_0}\left\{x^i - i(i-1)(i-2)\ldots(i-v)Q_{i-v}(x)\right\}, \qquad i \geq 0 \qquad (21)$$

The canonical polynomial used to solve the form includes:

$$Q_i(x) = x^i - i(i-1)Q_{i-2}(x);$$

$$Q_0(x) = 1, \quad Q_1(x) = x, \quad Q_2(x) = x^2 - 2, \quad Q_3(x) = x^3 - 6x$$

Solution of Integro Differential Equations:

We consider the linear boundary value problem for the higher-order integro differential equation

$$y^{(n)}(x) = g(x) + f(x)y(x) + \lambda \int_0^x k(x,t)y(t)dt, \qquad (22)$$

With initial conditions

$$y(0) = \alpha_0, \; y^1(0) = \alpha_1, \ldots, y^{(n-1)}(0) = \alpha_{n-1} \qquad (23)$$

We transformation the system as

$$y(x) = y_1(x), \; \frac{dy}{dx} = y_2(x), \; \frac{d^2y}{dx^2} = y_3(x), \ldots, \frac{d^{(n-1)}y}{dx^{(n-1)}} = y_n(x) \qquad (24)$$

We rewrite the above higher-order boundary value problem as a system of differential equations:

$$\frac{dy_1}{dx} = y_2(x),$$
$$\frac{dy_2}{dx} = y_3(x)$$
$$\vdots$$
$$\frac{dy_n}{dx} = g(x) + f(x)y_1(x) + \lambda\int_0^x K(x,t)y_1(t)dt$$

with initial conditions:

$$y_1^0(x) = \alpha_0,\ y_2^0(x) = \alpha_1, \ldots, y_n^0(x) = \alpha_{n-1}$$

The above system of differential equations can be written as a system of integral equations and substituting the canonical polynomial generated from the above construction as

$$y_1^{(k+1)}(x) = y_1^0(x) + \int_0^x y_2^k(s)ds,$$
$$y_2^{(k+1)}(x) = y_2^0(x) + \int_0^x y_3^k(s)ds,$$
$$y_3^{(k+1)}(x) = y_3^0(x) + \int_0^x y_4^k(s)ds,$$
$$\vdots$$
$$y_n^{(k+1)}(x) = y_1^0(x) + \int_0^x \left[g(s) + f(s)y_1^k(s) + \lambda\int_0^s k(s,t)y_1^k(t)dt \right] ds$$

For example with k = 0 we obtain

$$y_1^{(1)}(x) = \alpha_0 + \int_0^x \alpha_1 ds = \alpha_0 + \alpha_1 x,$$
$$y_2^{(1)}(x) = \alpha_1 + \int_0^x \alpha_2 ds = \alpha_1 + \alpha_2 x,$$
$$\vdots$$
$$y_n^{(1)}(x) = \alpha_{n-1} + \int_0^x \left[g(s) + f(s)\alpha_0 + \lambda\int_0^s k(s,t)\alpha_0 dt \right] ds$$

Error Estimate

In this section, an error estimator for the approximate solution of (16) is obtained. We defined $e_N(x) = y(x) - y_N(x)$ as the error function of the approximate solution $y_N(x)$ to $y(x)$, where, $y(x)$ is the exact solution and $y_N(x)$ is the approximate solution computed for various values of N.

Numerical Examples: Given below are numerical examples to illustrate the simplicity and the applicability of the discussed method. The experiments were carried out by the Mathematical software (MATLAB 2009b) and the results were presented.

Example1: Consider the following integro differential equation

$$y^{iv}(x) = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt$$

With boundary conditions:

y(0) = 1, y'(0) = 1, y(1) = 1 + e , y'(1) = 2e

The exact solution for this problem is: y(x) = 1 + xe^x.

Using the transformation $y_1 = y,\ y_2 = y',\ y_3 = y'',\ y_4 = y'''$

We rewrite the above problem as a system of differential equations:

$$\begin{cases} \dfrac{dy_1}{dx} = y_2(x) \\ \dfrac{dy_2}{dx} = y_3(x) \\ \dfrac{dy_3}{dx} = y_4(x) \\ \dfrac{dy_4}{dx} = x(1+e^x) + 3e^x + y_1(x) - \int_0^x y_1(t)dt. \end{cases}$$

The above system of differential equations can be written as a system of canonical polynomial $\lambda_i = 1$ i = 1,2,…,n

$$y_1^{(k+1)}(x) = y_1^{(0)}(x) + \int_0^x y_2^{(k)}(s)ds$$

$$y_2^{(k+1)}(x) = y_2^{(0)}(x) + \int_0^x y_3^{(k)}(s)ds$$

$$y_3^{(k+1)}(x) = y_3^{(0)}(x) + \int_0^x y_4^{(k)}(s)ds$$

$$y_4^{(k+1)}(x) = y_4^{(0)}(x) + \int_0^x \left[s(1+e^s + 3e^s + y_1^{(k)}(s) - \int_0^s y_1^{(k)}(t)dt \right] ds$$

With $y_1^{(0)} = 1,\ y_2^{(0)} = 1,\ y_3^{(0)} = C,\ y_4^{(0)} = D$

Consequently, we obtain the following approximations:

$y_1^1 = 1 + x$, $y_2^1 = 1 + Cx$, $y_3^1 = C + Dx,$ $y_4^1 = D + xe^x + 2e^x + x + 1$

$y_1^2 = 1 + x + \dfrac{Cx^2}{2}$, $y_2^2 = 1 + Cx + \dfrac{Dx^2}{2}$, $y_3^2 = C + Dx + xe^x + 3e^x + \dfrac{x^2}{2} + x,$

$y_4^2 = D + xe^x + 4e^x + x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

.

Using the boundary conditions at x = 1, we have

C = 1.9999999 D = 3.000002

Table 1: Numerical comparison of the exact solution, existing method, canonical method and its error

X	Exact Solution	VIM	Canonical Method	Error of VIM	Error of Canonical Method
0	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.1	1.11105170	1.11105170	1.11105172	2.0E-09	2.0E-08
0.2	1.24428055	1.24428054	1.24428057	1.5E-09	2.02E-08
0.3	1.40495764	1.40495760	1.40495769	4.0E-08	5.15E-08
0.4	1.59672987	1.59672985	1.59672990	2.1E-08	3.1E-08
0.5	1.82436063	1.82436060	1.82436069	3.2E-08	6.5E-08
0.6	2.09327128	2.09327006	2.09327120	1.2E-06	8.3E-08
0.7	2.40962689	2.40962585	2.40962588	1.4E-06	1.01E-06
0.8	2.78043274	2.78043070	2.78043175	2.0E-06	9.90E-07
0.9	3.21364280	3.21364261	3.21364354	1.9E-07	7.45E-07
1.0	3.71828182	3.71828180	3.71828193	2.0E-09	1.13E-07

Example2: Consider the following integro differential equation

$$y^{iv}(x) = 1 + \int_0^x e^{(-x)} y^2(t)dt$$

With boundary conditions:

$y(0) = 1, y'(0) = 1,\ y(1) = e,\ y'(1) = e$

The exact solution for this problem is: $y(x) = e^x$.

Transform into system of equation to have $y_1 = y,\ y_2 = y',\ y_3 = y'',\ y_4 = y'''$

We rewrite the above problem as a system of differential equations:

$$\begin{cases} \dfrac{dy_1}{dx} = y_2(x) \\ \dfrac{dy_2}{dx} = y_3(x) \\ \dfrac{dy_3}{dx} = y_4(x) \\ \dfrac{dy_4}{dx} = 1 + \int_0^x e^{-x} y_1^2(t)dt. \end{cases}$$

The above system of differential equations can be written as a system of integral equations with canonical polynomial generated above $\lambda_i = 1$, i= 1,2,…,n

$$y_1^{(k+1)}(x) = y_1^{(0)}(x) + \int_0^x y_2^{(k)}(s)ds$$

$$y_2^{(k+1)}(x) = y_2^{(0)}(x) + \int_0^x y_3^{(k)}(s)ds$$

$$y_3^{(k+1)}(x) = y_3^{(0)}(x) + \int_0^x y_4^{(k)}(s)ds$$

$$y_4^{(k+1)}(x) = y_4^{(0)}(x) + \int_0^x \left[1 + \int_0^s e^{-s}(y_1^{(k)}(t))^2 dt\right] ds$$

With $y_1^{(0)} = 1,\ y_2^{(0)} = 1,\ y_3^{(0)} = M,\ y_4^{(0)} = N$

Consequently, we obtain the following approximations:

$$y_1^1 = 1 + x\ ,\quad y_2^1 = 1 + Mx\ ,\quad y_3^1 = M + Nx,\quad y_4^1 = N + e^{-x} + 2x - 1$$

$$y_1^2 = 1 + x + \frac{Mx^2}{2}\ ,\quad y_2^2 = 1 + Mx + \frac{Nx^2}{2}\ ,\quad y_3^2 = M + Nx - e^{-x} + x^2 + 1,$$

$$y_4^2 = N + x^2 e^{-x} + 6e^{-x} + 11e^{-x} + 6x + 6$$

$\vdots$

Using the boundary conditions at x = 1, we have

M = 0.99708595, N = 1.0109940.

Table 2: Numerical comparison of the exact solution versus the approximation method

X	Exact	VIM	CANONICAL	Error VIM	Error of Canonical
0	1.00000000	1.00000000	1.00000000	0.00E+00	0.00E+00
0.1	1.10515818	1.10515817	1.10515815	1.00E-08	3.00E-08
0.2	1.22140227	1.22140225	1.22140221	2.00E-08	6.00E-08
0.3	1.34985880	1.34985778	1.34985868	1.02E-06	1.20E-07
0.4	1.49182469	1.49182437	1.49182395	3.20E-07	7.40E-07
0.5	1.64872127	1.64872034	1.64871992	9.30E-07	1.35E-06
0.6	1.82211880	1.82211777	1.82211689	1.03E-06	1.91E-06
0.7	2.01375270	2.01375078	2.01374995	1.92E-06	2.75E-06
0.8	2.22554092	2.22554083	2.22553894	9.00E-08	1.98E-06
0.9	2.45960311	2.4596031	2.45959874	1.00E-08	4.37E-06
1.0	2.71828182	2.71828153	2.71827964	2.90E-07	2.18E-06

Conclusion:

In this paper, canonical polynomial was successfully employed for solving higher-order integro-differential equations. The numerical results in the tables compared the new method of canonical polynomial with the existing variational iterative method where both methods show that they have highly accurate numerical solutions for solving Integro Differential equations.

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