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# STRONG CONVERGENCE ANALYSIS FOR FIXED POINT, VARIATIONAL INEQUALITY AND EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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***Abstract***

***In this paper, we introduce and study strong convergence analysis for finding a common element of the set of fixed points of asymptotically nonexpansive mapping, the set of solutions of generalized mixed equilibrium problem and the set of solutions of variational inequality problem. We prove that the sequence generated converges strongly to the common element of the three aforementioned problems. Furthermore, an optimization problem is solved using the theorems in real Hilbert spaces.***

***Keyword:*** Inverse strongly monotone mapping, asymptotically nonexpansive mapping, equilibrium problem, Hilbert space.

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## 1. INTRODUCTION

Let $K$ be a nonempty subset of a real Hilbert space H. A mapping A: K → H is called;
(i) Monotone, if
$\langle Ax - Ay, x - y\rangle \geq 0 \ \forall x, y \in K$ (1.1)
(ii) Inverse-strongly monotone if there exists a positive real number λ such that
$\langle Ax - Ay, x - y\rangle \geq \lambda \parallel Ax - Ay \parallel^2 \ \forall x, y \in K.$
(iii) Relaxed (λ, γ)-cocoercive, if there exist λ, γ > 0 such that
$\langle Ax - Ay, x - y\rangle \geq -\lambda \parallel Ax - Ay \parallel^2 + \gamma \parallel x - y \parallel \ \forall x, y \in K.$
(iv) μ-Lipschitzian, if there exists μ > 0 such that
$\parallel Ax - Ay \parallel \leq \mu \parallel x - y \parallel \ \forall x, y \in K.$
(v) Nonexpansive, if $\parallel Tx - Ty \parallel \leq \parallel x - y \parallel \ \forall x, y \in K.$
Let A: K →H be a nonlinear mapping. The variational inequality problem is to find an $x^* \in K$ such that
$\langle Ax^*, y - x^*\rangle \geq 0 \ \forall x, y \in K$ (1.2)
We shall denote the set of solutions of the variational inequality problem (1.2) by VI(K; A) and the set of fixed points of T by F(T). One important generalization of the class of nonexpansive mappings that has appeared in the literature is the class of asymptotically nonexpansive mappings introduced in [1].
Let $K$ be a nonempty subset of a real normed linear space E. A mapping T: K → E is called asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$; $k_n > 1$, such that $\lim_{n=1} k_n$=1, and
$\parallel Tx - Ty \parallel \leq k_n \parallel x - y \parallel \ \forall x, y \in K.$
Many authors have studied the approximation of fixed points of asymptotically nonexpansive maps [1, 2, 3, 4, 5, and 6].
A monotone mapping A: H → H is said to be maximal if the graph G (A) is not properly contained in the graph of any other monotone map, where G(A) := $\{(x, y) \in H \times H: y \in Ax\}$ -for a multi-valued mapping A. It is also known that A is

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maximal if and only if for $(x,f) \epsilon H \times H, \langle x-y, f-g \rangle \geq 0$ for every $(x,y) \epsilon G(A)$ which implies that $f \epsilon Ax$.
Let A be a monotone mapping defined from K into H and $N_k q$ be a normal cone to K at $q \epsilon K$; i.e., $N_k q = \{p \epsilon H: \langle q - u, p \rangle \geq 0, \forall u \epsilon K\}$ Define a mapping M by

$$Mq := \begin{cases} Aq + N_k q, & q \in K, \\ \emptyset, & if\ q \notin K. \end{cases}$$

Then, M is maximal monotone. Furthermore, $x^* \epsilon M^{-1}(0) \Leftrightarrow x^* \in VI(K,A)$ [7].
The computation of fixed points is important in the study of many problems including inverse problems.
In particular, construction of fixed points of nonexpansive mappings is applied in image recovery, signal processing and in transition operators for initial value problems of differential
Inclusions [8]. Finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem has been studied extensively in the literature [9, 10, 11, 12, 13, 14] and the references contained therein.
Similarly, the strong convergence of the set of solutions of equilibrium problem and fixed point problem has also been obtained by many researchers recently [15, 16] and references therein).
Let $\varphi: K \to \mathbb{R} \cup \{+\infty\}$ be a real-valued, proper, lower semi-continuous and convex function and A: K → H be a nonlinear mapping. Suppose $F: K \times K \to \mathbb{R} \cup \{+\infty\}$ is an equilibrium bifunction, that is, $F(x,x) = 0, \forall x \in K$.
The generalized mixed equilibrium problem is to find $x \in K$, [17] such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x \rangle \geq 0, \forall y \in K \quad (1.3)$$

We shall denote the set of solutions of this generalized mixed equilibrium problem by GMEP.
Thus $\{x^* \in K: F(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \forall y \in K\}$
The generalised mixed equilibrium problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems as special cases [17] and the references therein. Some methods have been proposed to solve the generalised mixed equilibrium problem [17] and the references therein. For solving the generalised mixed equilibrium problem for a bifunction $F: K \times K \to \mathbb{R} \cup \{+\infty\}$.
Let us assume that $F, \varphi$ and K satisfy the following conditions:
(A1) : $F(x,x) = 0\ for\ all\ x \in K$;
(A2) : $F$ is monotone; i.e; $F(x,y) + F(y,x) \geq 0, \forall x,y \in K$.
(A3) : For each $x,y \in K, \lim_{t\to 0} F(tz + (1-t), y) \leq F(x,y)$;
(A4) : For each $x \in K, y \to F(x,y)$ is a convex and lower semicontinuous.
(B1) : For each $x \in K$ and r > 0 there exists a bounded subset $D_x$ of K and $y_x \in K$ such that for any

$$z \in K, F(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0; \quad (1.4)$$

(B2): K is a bounded set.

Several weak convergence results have been proved for finding a common element of the set of fixed points of nonexpansive mappings and either the set of solutions of equilibrium problem or the set of solutions of generalised mixed equilibrium problem in certain Banach spaces [18,19, 20] and the references therein). In order to obtain strong convergence theorems for finding a common element of the set of solutions of equilibrium problem (or generalised mixed equilibrium problem), variational inequality problem and fixed point problem, many authors have obtained their results using the hybrid method of CQ algorithm and viscosity approximation methods [21, 22] and the references therein. The CQ method involves the computation, at each step of the iteration process, two convex subsets C and Q of H, computation of CQ and projecting the initial vector onto CQ. This is certainly not convenient to implement in application.

In [23], the author introduced an algorithm which does not involve either the CQ algorithm or the viscosity approximation method and proved strong convergence of the scheme to a common element of the fixed points set of a nonexpansive mapping, the set of solutions of a variational inequality problem for a lipschitzian, relaxed (λ, γ) cocoercive mapping and the set of solutions of a GM EP in the framework of Hilbert spaces. He proved the following theorem.

**Theorem [23]:** Let K be a closed convex subset of a real Hilbert space H. Let F be a bifunction from K × K satisfying (A1) - (A4), $\varphi: K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (B1) or (B2) , let A be a μ-Lipschitzian, relaxed ( λ,γ )-cocoercive mapping of K into H and be an α-inverse, strongly monotone mapping of K into H. Let T be a nonexpansive mapping of K into H such that $\mathcal{F}$ = F(T ) ∩V I(A, K) ∩ GMEP ≠ ∅. Let $\{x_n\}_{n=1,}^{\infty}$ $\{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,

$$\begin{cases} y_n = P_K(1-\alpha_n)x_n \\ u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T P_K(u_n - s_n A u_n) \end{cases} \quad (1.5)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}$, $\{r_n\}_{n=1,}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$

(ii) $0 < \lim_{n\to\infty} inf \beta_n \leq \lim_{n\to\infty} sup \beta_n < 1$

(iii) $0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$

(iv) $0 < \alpha \leq s_n \leq b < \frac{2(\gamma - \lambda\mu^2)}{\mu^2}, \lim_{n\to\infty} |s_{n+1} - s_n| = 0.$

Then $\{x_n\}_{n=1,}^{\infty}$ converges strongly to $z \in \mathcal{F}$.

We observe that under the hypothesis of theorem [23], the map $I - s_n A$ is a strict contraction map.

Consequently, the map $P_K(I - s_n A)$ is also a strict contraction map. By the Banach contraction mapping principle, $P_K(I - s_n A)$ has a unique fixed point. Furthermore, it can be shown that A is $\delta$- strongly monotone, for some $\delta > 0$; using the assumptions on A: Hence A is $\mu$- Lipschitzian and $\delta$- strongly monotone. It is well known that with such a map, $x^* \in VI(K,A)$, $x^* = P_K(I - s_n A)x^*$. Hence, the solution of VI(K, A); under the setting of theorem [23] is unique. So, VI(K, A) is a singleton which implies that $\mathcal{F}$ is a singleton and there are simpler algorithms than the one studied in theorem [23] for approximating such a solution.

But we also observe that;

(i) The assumption that A is inverse strongly monotone is weaker than the assumption that A is k- lipschitz imposed in [23],

(ii) The class of asymptotically nonexpansive map is more general than the class of nonexpansive map considered in [23].

In this paper, we introduce a new iterative scheme for the class of asymptotically nonexpansive maps and prove strong convergence theorems for approximating a common element of the set of fixed points of asymptotically nonexpansive maps, the set of solutions of variational inequality problem when A is assumed to be an inverse strongly monotone map, and the set of solution of generalised mixed equilibrium problem in a real Hilbert space. Our theorems improve significantly the results of [20, 23], and a host of other authors from the class of nonexpansive maps to the more general class of asymptotically nonexpansive maps. Moreover, the condition $\lim_{n\to\infty} | s_{n+1-} s_n| = 0$ imposed in [23] is dispensed with. Furthermore, our iterative scheme does not involve the CQ method and the conditions imposed on the operator A do not make the set VI(K; A) a singleton, which is the case in [23].

## 2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and let K be a nonempty closed convex subset of H. It is known that for any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$\|u - P_K u\| \leq \| u - y \| \ \forall\, y \in K$ (2.1)

$P_K$ is called the metric projection of H onto K. It is also know that $P_K$ is a nonexpansive mapping of H onto K and satisfies the following inequality:

$\langle x - y, P_K x - P_K y \rangle \geq \| P_K x - P_K y \|^2$ (2.2)

for all $x, y \in H$: Furthermore, $P_K x$ is characterised by the properties $P_K x \in K$ and

$\langle x - P_K x, y - P_K y \rangle \geq 0$ (2.3)

For all $k \in K$.

In the context of the variational inequality problem,

$x^* \in VI(K,A) \Leftrightarrow x^* = P_K(x^* - s_n A x^*) \forall\, \lambda > 0$ (2.4)

In what follows, we shall make use of the following lemmas.

**Lemma 2.1**. [24] Let $\{x_n\}_{n=1,}^{\infty}$ and $\{y_n\}_{n=1,}^{\infty}$ be bounded sequences in a Banach space E and let $\{\delta_n\}_{n=1,}^{\infty}$ be a sequence in [0,1] with $0 < \lim_{n\to\infty} inf \delta_n \leq \lim_{n\to\infty} suo \delta_n < 1$. Suppose $x_{n+1} = (1-\delta_n)y + \delta_n x_n$ for all integers $n \geq 0$ $\lim_{n\to\infty} \sup(\| x_{n+1} - x_n \| - \| y_{n+1} - y_n \|) \leq 0$. Then, $\lim_{n\to\infty} \| y_n - x_n \| = 0$.

**Lemma 2.2.** [25] Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \delta_n \sigma_n + \gamma_n, n \in \mathbb{N}$$

Where

(i) $\{\delta_n\} \subset (0,1), \lim_{n\to\infty} \delta_n = 0, \sum_{n=0}^{\infty} \delta_n = \infty$

(ii) $\lim_{n\to\infty} sup_n \sigma_n \leq 0$;

(iii) $\gamma_n \geq 0, n \geq 1, \sum \gamma_n + \infty$. Then, $\lim_{n\to\infty} \alpha_n = 0$

**Lemma 2.3.** [2,3] Let E be a uniformly convex Banach space, K be a nonempty closed con-vex subset of E and T : K → K be an asymptotically nonexpansive mapping. Then, $(I - T)$ is demi-closed at zero.

**Lemma 2.4.** [26] Suppose E is a Banach space with uniform normal structure, K a nonempty bounded subset of E and T: K → K is uniformly L- Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$.

Suppose also there exists a nonempty closed convex subset A of K with the following property

$x \in A$ implies $w_w(x) \in A$,

Where $w_w(x)$ is the weak-limit set of T at $x$, that is the set $\{y \in E: y = weak - limT^{n_j}x \ for \ some \ n_{j\to\infty}\}$. Then, T has a fixed point in A.

**Lemma 2.5.** Let H be real Hilbert space. Then

$\| \boldsymbol{x} + \boldsymbol{y} \|^2 \leq \| \boldsymbol{x} \|^2 + 2\langle y, x+y \rangle, \forall x, y \in K$

**Lemma 2.6.** [11] Let $(a_1, a_2, \dots,) \in l^{\infty}$ be such that $\mu_n(a_n) \leq 0$ for all Banach limit $\mu$ and let $\lim_{n\to\infty} \sup(a_{n+1} - a_n) \leq 0$. Then, $\lim_{n\to\infty} \sup a_n \leq 0$.

**Lemma 2.6.** [27] Let K be a nonempty closed convex subset of H and let F be a bifunction of K × K into R satisfying (A1)- (A4) and $\varphi: K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either B1 or B2 holds. Let $r > 0$. Define a mapping $T_r^{(F,\varphi)}: H \to 2^K$ as follows;

$$T_r^{(F,\varphi)}(x) := \left\{z \in K: F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, y \in K\right\}$$

For all $x \in H$. Then, the following hold:

(1) For each $x \in H, T_r^{(F,\varphi)}(x) \neq \emptyset$

(2) $T_r^{(F,\varphi)}$ is single valued

(3) $T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e., for each $x, y \in H$

$\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \|^2 \leq \langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\rangle$;

(4) $T_r^{(F,\varphi)} = GMEP(F)$

(5) GMEP(F) is closed and convex.

## 3. MAIN RESULT

**Theorem 3.1.** Let K be a closed convex subset of a real Hilbert space H. Let F be a bifunction from K× K into ℝ satisfying the following: (A1)-(A4), let $\varphi: K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (B1) or (B2) , let A be a μ-inverse strongly monotone mapping of K into H and $\psi$ be an α-inverse, strongly monotone mapping of K into H. Suppose T is an asymptotically nonexpansive mapping of K into K such that

Γ := F(T ) ∩V I(A, K) ∩ GMEP ≠ ∅. Let $\{x_n\}_{n=1,}^{\infty}$ $\{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,

$$\begin{cases} y_n = P_K(1-\alpha_n)x_n \\ u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T^n P_K(u_n - s_n A u_n) \end{cases} \quad (3.1)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}$, $\{r_n\}_{n=1,}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(ii) $0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$

(iii) $0 < \alpha \leq s_n \leq b < 2\mu$;

(iv) $0 < \lim_{n\to\infty} inf\beta_n \leq \lim_{n\to\infty} sup\beta_n < 1$;

(v) $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$.

Then $\{x_n\}_{n=1,}^{\infty}$ converges strongly to $u \in \Gamma$.

Proof. We divide our proof into six (6) steps;

**Step 1.** We prove that $\{x_n\}_{n=1,}^{\infty}$ is bounded.

We remark that with restriction on the sequences $\{s_n\}_{n=1,}^{\infty}$ and $\{r_n\}_{n=1,}^{\infty}$, $(I - s_nA)$ and $(I - r_nA)$ are nonexpansive maps for all $n \geq 1$.

Furthermore, $x^* \in \Gamma$, using (3.1), we obtain:

$$\|x_{n+1} - x^*\| = \|(1-\beta_n)(x_n - x^*) + \beta_n[T^nP_K(I - s_nA)u_n - T^nP_K(I - s_nA)x^*]\|$$
$$\leq (I-\beta_n)\|x_n - x^*\| + \beta_n k_n \|u_n - x^*\|$$
$$\leq (I-\beta_n)\|x_n - x^*\| + \beta_n k_n((1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x^*\|)$$
$$\leq \left(1-\beta_n + \beta_n k_n(1-\alpha_n)\right)\|x_n - x^*\| + \beta_n\alpha_n k_n\|x^*\|$$
$$\leq [1-\beta_n + \beta_n k_n(1-\alpha_n) + \beta_n\alpha_n k_n] max\{\|x_n - x^*\|, \|x^*\|\}$$
$$\leq [1 + \beta_n(k_n - 1)] max\{\|x_n - x^*\|, \|x^*\|\}$$
$$\leq \prod_{j=1}^{n}(1 + (k_j - 1)) max\{\|x_1 - x^*\|, \|x^*\|\}$$
$$\leq e^{\sum_{j=1}^{\infty}(k_j - 1)} max\{\|x_1 - x^*\|, \|x^*\|\} < \infty.$$

Hence, $\{x_n\}$ is bounded. Consequently $\{y_n\}$, $\{u_n\}$ and $\{Au_n\}$ are bounded.

**Step 2.** We prove that $\lim_{n\to\infty}\|u_n - y_n\| = 0$. Let $\rho_n := P_K(u_n - s_nAu_n)$. Clearly,
$\|\rho_{n+1} - \rho_n\| \leq \|u_{n+1} - u_n\|$.

From $u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n)$ and $u_{n+1} = T_{r_{n+1}}^{(F,\varphi)}(y_{n+1} - r_{n+1}\psi y_{n+1})$ and using Lemma (2.7), we obtain:

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \psi y_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \forall y \in K. \quad (3.3)$$

and

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \langle \psi y_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - y_{n+1}\rangle \geq 0, \forall y \in K. \quad (3.4)$$

Substituting $y = u_{n+1}$ in (3.3) and $y = u_n$ in (3.4), we have;

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \langle \psi y_n, u_{n+1} - u_n\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - y_n\rangle \geq 0. \quad (3.5)$$

and

$$F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \langle \psi y_{n+1}, u_n - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - y_{n+1}\rangle \geq 0. \quad (3.6)$$

Adding inequalities (3.5) and (3.6) and using (A2), we have

$$\langle \psi y_{n+1} - \psi y_n, u_n - u_{n+1}\rangle + \langle u_{n+1} - u_n, \frac{u_n - y_n}{r_n} - \frac{u_{n+1} - y_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence, $0 \leq \langle u_n - u_{n+1}, r_n(\psi y_{n+1} - \psi y_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - y_{n+1}) - (u_n - y_n)\rangle$

$$= \langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - y_{n+1}) + (y_{n+1} - r_n\psi y_{n+1}) - (y_n - r_n\psi y_n)\rangle$$

It then follows, using the nonexpansiveness of $(I - r_n\psi)$ that

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\|\right\}$$

and so we have,

$$\|u_{n+1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\|,$$

and using condition (ii) of Theorem (3.1), we get

$$\|u_{n+1} - u_n\| \leq \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\|$$
$$\leq M_1c^{-1}|r_{n+1} - r_n| + \|y_{n+1} - y_n\| \quad (3.7)$$

$where\ M_1 := sup_{n\geq 1}\|u_n - y_n\|$

For $m \geq 1$, set $z_n := T^m\rho_n, n \geq 1$. Then using boundedness of $\{z_n\}$ and inequalities (3.2) and (3.7), we get:

$$\|z_{n+1} - z_n\| = \|T^m\rho_{n+1} - T^m\rho_n\|$$
$$\leq k_m\|u_{n+1} - u_n\|$$
$$\leq k_m\|y_{n+1} - y_n\| + k_mM_1c^{-1}|r_{n+1} - r_n|$$
$$\leq k_m\|(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n\| + k_mM_1c^{-1}|r_{n+1} - r_n|$$
$$\leq k_m\|x_{n+1} - x_n\| + k_m\alpha_{n+1}\|x_{n+1}\| + k_m\alpha_n\|x_n\| + k_mM_1c^{-1}|r_{n+1} - r_n| \quad (3.8)$$

So,
$\| z_{n+1} - z_n \| \leq k_m \| x_{n+1} - x_n \| + k_m \alpha_{n+1} \| x_{n+1} \| + k_m \alpha_n \| x_n \| + k_m M_1 c^{-1} |r_{n+1} - r_n|$.
Hence, $\lim_{n\to\infty} sup\, (\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \leq 0$. Using this and Lemma (2.1), we get $\lim_{n\to\infty} \| z_n - x_n \| = 0$.
Thus, $\lim_{n\to\infty} \| x_{n+1} - x_n \| = \lim_{n\to\infty} \beta_n \| x_n - z_n \| = 0$.
Using (3.8), we have that $\| y_{n+1} - y_n \| \leq \| (1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n \| \to 0$ as $n \to \infty$, and
$\lim_{n\to\infty} \| u_{n+1} - u_n \| = 0$. (3.9)
Using (3.9) and the definition of $\rho_n$, we obtain that
$\lim_{n\to\infty} \| \rho_{n+1} - \rho_n \| = 0$ (3.10)
Let $\{T_{r_n}\}_{n=1}^{\infty}$ be a sequence of mappings defined as in Lemma (2.7). Then we have $x^* = P_K(x^* - s_n A x^*) = T_{r_n}^{(F,\varphi)}(x^* - r_n \psi x^*)\ \forall\, x^* \in F$. For each $n \geq 1$, using the fact that $\psi$ and A are inverse strongly monotone, we obtain
$\| u_n - x^* \|^2 \leq \| T_{r_n}^{(F,\phi)}(y_n - r_n \psi y_n) - T_{r_n}^{(F,\phi)}(x^* - r_n \psi x^*) \|^2$
$\leq \| y_n - x^* \|^2 - r_n(2\alpha - r_n) \| \psi y_n - \psi x^* \|^2$,
and $\| \rho_n - x^* \|^2 \leq \| (I - s_n A)u_n - (I - s_n A)x^* \|^2$
$= \| u_n - x^* \|^2 - 2 s_n \langle u_n - x^*, A u_n - A x^* \rangle + {s_n}^2 \| A u_n - A x^* \|^2$
$\leq \| y_n - x^* \|^2 - s_n(2\mu - s_n) \| A u_n - A x^* \|^2$.
Furthermore, using the convexity of $\|.\|^2$,
we obtain $\| x_{n+1} - x^* \|^2 = \| (1-\beta_n)(x_n - x^*) + \beta_n (T^n \rho_n - x^*) \|^2$
$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 \| \rho_n - x^* \|^2$.
$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 \| x_n - x^* - \alpha_n x_n \|^2 + \beta_n {k_n}^2 s_n ((s_n - 2\mu) \| A u_n - A x^* \|^2)$

$\leq \| x_n - x^* \|^2 + \beta_n ({k_n}^2 - 1) \| x_n - x^* \|^2 + 2 \alpha_n \beta_n {k_n}^2 \| x_n - x^* \| \| x_n \| + {\alpha_n}^2 \beta_n {k_n}^2 \| x_n \|^2 - \beta_n {k_n}^2 s_n (2\mu - s_n) \| A u_n - A x^* \|^2$.
This implies $\beta_n {k_n}^2 s_n (2\mu - s_n) \| A u_n - A x^* \|^2 \leq \| x_n - x_{n+1} \| M_2 + \left({k_n}^2 - 1\right) M_3 + \alpha_n M_4$
Where $M_2 := sup_n \{ \| x_n - x^* \| + \| x_{n+1} - x^* \| \}, M_3 := sup_n \{ \| x_n - x^* \|^2 \}\ and\ M_4 := sup_n \{ 2 \beta_n {k_n}^2 \| x_n - x^* \| \| x_n \| + {\alpha_n}^2 \beta_n {k_n}^2 \| x_n \|^2 \}$
Using condition (iii), we have

$\beta_n \alpha \left((2\mu - b) \| A u_n - A x^* \|^2\right) \leq \| x_n - x_{n+1} \| M_2 + \left({k_n}^2 - 1\right) M_3 + \alpha_n M_4$. (3.11)

Taking $limsup\ as\ n \to \infty$ in (3.11) and using the fact that $\alpha_n \to 0, as\ n \to \infty, \| x_{n+1} - x_n \| \to 0\ as\ n \to \infty,\ {k_n}^2 - 1 \to 0\ as\ n \to \infty$ and condition (iv), we get $\| A u_n - A x^* \| \to 0$, as $n \to \infty$.

Moreover, $\| x_{n+1} - x^* \|^2 \leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 \| \rho_n - x^* \|^2$

$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 (\| y_n - x^* \|^2 - r_n (2\alpha - r_n) \| \psi y_n - \psi x^* \|^2)$
$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 (\| x_n - x^* \|^2 + 2\alpha \| x_n - x^* \| \| x_n \| + {\alpha_n}^2 \| x_n \|^2 - \beta_n {k_n}^2 r_n (2\mu - r_n) \| \psi y_n - \psi x^* \|^2)$

$\leq \left[1 + \beta_n \left({k_n}^2 - 1\right)\right] \| x_n - x^* \|^2 + \alpha_n M_4 - (2\alpha - r_n) \beta_n {k_n}^2 r_n \| \psi y_n - \psi x^* \|^2$.
Hence, $\beta_n {k_n}^2 r_n (2\alpha - r_n) \| \psi y_n - \psi x^* \|^2 \leq \| x_{n+1} - x_n \| M_2 + \left({k_n}^2 - 1\right) M_3 + \alpha_n M_4$.
This implies, using condition (ii) that
$\beta_n c (2\alpha - d) \| \psi y_n - \psi x^* \|^2 \leq \| x_{n+1} - x_n \| M_2 + \left({k_n}^2 - 1\right) M_3 + \alpha_n M_4$. (3.12)
Taking $limsup\ as\ n \to \infty$ in (3.12) and using the fact that $\alpha_n \to 0, as\ n \to \infty, \| x_{n+1} - x_n \| \to 0\ as\ n \to \infty,\ {k_n}^2 - 1 \to 0\ as\ n \to \infty$ and condition (iv), we get $\| \lim_{n\to\infty} \| \psi y_n - \psi x^* \| \| = 0$. Furthermore, using Lemma (2.7)(3) and the nonexpansiveness of $(I - r_n \psi)$, we have: $\leq \| T_{r_n}^{(F,\varphi)}(y_n - r_n \psi y_n) - T_{r_n}^{(F,\varphi)}(x^* - r_n \psi x^*) \|^2 \leq \langle (y_n - r_n \psi y_n) - (x^* - r_n \psi x^*), u_n - x^* \rangle$
$= \frac{1}{2} (\| (y_n - r_n \psi y_n) - (x^* - r_n \psi x^*) \|^2) + \| u_n - x^* \|^2 - (\| (y_n - r_n \psi y_n) - (x^* - r_n \psi x^*) - u_n - x^* \|^2) \leq$
$\frac{1}{2} (\| y_n - x^* \|^2) + \| u_n - x^* \|^2 - (\| (y_n - r_n \psi y_n) - (x^* - r_n \psi x^*) - (u_n - x^*) \|^2)$

$= \frac{1}{2}(\|\| y_n - x^* \|^2 + \| u_n - x^* \|^2) - \frac{1}{2}(\| (y_n - u_n) - r_n(\psi y_n - \psi x^*)) \|^2)$
$= \frac{1}{2}(\|\| y_n - x^* \|^2 + \| u_n - x^* \|^2) - (\frac{1}{2} \| u_n - y_n \|^2 + r_n\langle y_n - u_n, \psi y_n - \psi x^* \rangle - \frac{1}{2} r_n^2 \| \psi y_n - \psi x^* \|^2)$ and hence,

$\| u_n - x^* \|^2 \leq \| y_n - x^* \|^2 - \| u_n - y_n \|^2 + 2r_n \| y_n - u_n \|^2 \| \psi y_n - \psi x^* \|$ (3.13)
By convexity of $\|.\|^2$ and using inequality (3.13), we have

$\| x_{n+1} - x^* \|^2 \leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 \| u_n - x^* \|^2$
$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 (\| y_n - x^* \|^2) - \| u_n - y_n \|^2 + 2r_n \| y_n - u_n \|\|\| \psi y_n - \psi x^* \|$
$\leq \| x_n - x^* \|^2 + ({k_n}^2 - 1)M_3 + \alpha_n M_4 - \beta_n {k_n}^2 \| u_n - y_n \|^2 + 2r_n \| y_n - u_n \|\|\| \psi y_n - \psi x^* \|$
Consequently,
$\beta_n {k_m}^2 \| u_n - y_n \|^2 \leq \| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2 + ({k_n}^2 - 1)M_3 + \alpha_n M_4 + 2d \| y_n - u_n \|\|\| \psi y_n - \psi x^* \|$ (3.14)
Taking $limsup \ as \ n \to \infty$ in (3.14), we have $\lim_{n\to\infty} \|u_n - y_n\| = 0$, completing step 2.

**Step 3.** We show that $\lim_{n\to\infty} \|u_n - \rho_n\| = 0$, using the nonexpansiveness of $(I - s_n A)$ and inequality (3.14), we have
$\| \rho_n - x^* \|^2 = \| P_K(u_n - s_n A u_n) - P_K(x^* - s_n A x^*) \|^2$
$\leq \langle (u_n - s_n A u_n) - (x^* - s_n A x^*), P_K(u_n - s_n A u_n) - P_K(u_n - s_n A u_n) - x^* \rangle,$
$= \frac{1}{2} (\| u_n - s_n A u_n - (x^* - s_n A x^*) \|^2 + \| P_K(u_n - s_n A u_n) - x^* \|^2$
$\quad - \| (u_n - s_n A u_n) - (x^* - s_n A x^*) - P_K(u_n - s_n A u_n) - x^* \|^2)$
$\leq \frac{1}{2}(\| u_n - x^* \|^2 + \| P_K(u_n - s_n A u_n) - x^* \|^2 - \| (u_n - P_K(u_n - s_n A u_n)) - s_n (A u_n - A x^*) \|^2$

$= \frac{1}{2}(\| u_n - x^* \|^2 + \| P_K(u_n - s_n A u_n) - x^* \|^2$
$\quad - \| (u_n - P_K(u_n - s_n A u_n)) + 2s_n \langle u_n - P_K(u_n - s_n A u_n), A u_n - A x^* \rangle \|^2)$

Therefore,
$\| \rho_n - x^* \|^2 \leq \| u_n - x^* \|^2 - \| u_n - \rho_n \|^2 + 2s_n \langle (u_n - P_K(u_n - s_n A u_n)) - s_n (A u_n - A x^*) \rangle$
$\quad - {s_n}^2 \| A u_n - A x^* \|^2$ (3.15)

Hence, we have using inequality (3.15) and the fact that
$\| u_n - x^* \|^2 \leq \| y_n - x^* \|^2 = \| x_n - x^* - \alpha_n x_n \|^2$ that
$\| x_{n+1} - x^* \|^2 \leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 \| \rho_n - x^* \|^2$
$\leq (1-\beta_n) \| x_n - x^* \|^2 + \beta_n {k_n}^2 [\| u_n - x^* \|^2 - \| u_n - \rho_n \|^2 + 2s_n \langle u_n - \rho_n, A u_n - A x^* \rangle - {s_n}^2 \| A u_n - A x^* \|^2]$
$\| x_n - x^* \|^2 + \beta_n({k_n}^2 - 1) \| x_n - x^* \|^2 + 2\alpha_n \beta_n {k_n}^2 \| x_n - x^* \|\|x_n\| + {\alpha_n}^2 \beta_n {k_n}^2 \| x_n \|^2 - \beta_n {k_n}^2 \| u_n - \rho_n \|^2 +$
$2s_n \beta_n {k_n}^2 \langle u_n - \rho_n, A u_n - A x^* \rangle - \beta_n {k_n}^2 {s_n}^2 \| A u_n - A x^* \|^2$
$\leq \| x_n - x^* \|^2 + \beta_n({k_n}^2 - 1)M_3 + \alpha_n {k_n}^2 M_4 - \beta_n {k_n}^2 \| u_n - \rho_n \|^2 + 2s_n \beta_n {k_n}^2 \| x_n - x^* \| \| A u_n - A x^* \|.$
Since $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$, $\lim_{n\to\infty} ({k_n}^2 - 1) = 0$, $\lim_{n\to\infty} \| A u_n - A x^* \| = 0$
and condition (iv), we obtain $\lim_{n\to\infty} \| u_n - \rho_n \| = 0$, completing step 3.

**Step 4.** We show that $\lim_{n\to\infty} \| x_n - T^n x_n \| = 0$.
$\| x_n - T^n x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T^n x_n \|$
$\leq \| x_n - x_{n+1} \| + (1-\beta_n) \| x_n - T^n x_n \| + \beta_n \| T^n \rho_n - T^n x_n \|$
$\leq \| x_n - x_{n+1} \| + (1-\beta_n) \| x_n - T^n x_n \| + \beta_n k_n \| \rho_n - y_n \| + \beta_n k_n \| y_n - x_n \|$
$\leq \| x_n - x_{n+1} \| + (1-\beta_n) \| x_n - T^n x_n \| + \beta_n k_n \| \rho_n - y_n \| + \beta_n \alpha_n k_n \| x_n \|,$
So that
$\| x_n - T^n x_n \| \leq \frac{1}{\beta_n} \{\| x_n - x_{n+1} \| + \beta_n k_n \| \rho_n - y_n \| + \beta_n \alpha_n k_n \| x_n \|\}.$
Since from step 2 and step 3, $\lim_{n\to\infty} \| \rho_n - u_n \| = 0$, $\lim_{n\to\infty} \| u_n - y_n \| = 0$, we have that $\lim_{n\to\infty} \| \rho_n - y_n \| = 0$. Thus, $\lim_{n\to\infty} \| x_n - T^n x_n \| = 0$ completing step 4.

**Step 5.** We prove that $\lim_{n\to\infty} \| x_n - T^n x_n \| = 0$.

$\| x_{n+1} - T^n x_n \| \leq \| x_{n+1} - x_n \| + \| x_n - T^n x_n \| \to 0, n \to \infty.$

$\| x_{n+1} - T x_n \| \leq \| x_{n+1} - T^n x_n \| + \| T^n x_n - T x_n \|$

$\leq \| x_{n+1} - T^n x_n \| + \| T^{n-1} x_n - x_n \|$

$\leq \| x_{n+1} - T^n x_n \| + [\| T^{n-1} x_n - T^{n-1} x_{n-1} \| + \| T^{n-1} x_{n-1} - x_n \| + \| x_{n-1} - x_n \|]$

$\leq \| x_{n+1} - T^n x_n \| + k_n - 1 \| x_n - x_{n-1} \| + \| T^{n-1} x_{n-1} \| + \| x_{n-1} - x_n \|.$

Thus, $\lim_{n\to\infty} \| x_{n+1} - T x_n \| = 0$.

Hence, $\| x_n - T x_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - T x_n \| \to 0,\ as\ n \to \infty$, completing step 5.

As $\{x_n\}_{n=1,}^{\infty}$ is bounded, there exists a subsequence $\left\{x_{n_j}\right\}_{j=1}^{\infty} of\ \{x_n\}_{n=1,}^{\infty}$ such that $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ converges weakly to some $u \in K$.

**Step 6.** We show that $u \in \Gamma$.

(i) We first show that $u \in GMEP$.

Since $u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n \psi y_n), n \geq 1$, Applying lemma 2.7 (3), we have for any $y \in K$ that $F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \psi y_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0$.

Furthermore, replacing n by $n_j$ in the inequality and using (A2), we obtain:

$$\varphi(y) - \varphi\left(u_{n_j}\right) + \langle \psi y_{n_j}, y - u_{n_j} \rangle + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - y_{n_j} \rangle \geq F\left(y, u_{n_j}\right) \quad (3.16)$$

Let $z_t := ty + (1-t)u$ for all $t \in (0,1]$ and $y \in K$. This implies that $z_t \in K$.

Then, by inequality (3.16), we have

$$\langle z_t - u_{n_j}, \psi z_t \rangle \geq \varphi\left(u_{n_j}\right) - \varphi(z_t) + \langle z_t - u_{n_j}, \psi z_t \rangle - \langle z_t - u_{n_j}, \psi y_{n_j} \rangle$$
$$- \langle z_t - u_{n_j}, \frac{u_{n_j - y_{n_j}}}{r_{n_j}} \rangle + F(z_t, u_{n_j})$$
$$= \varphi\left(u_{n_j}\right) - \varphi(z_t) + \langle z_t - u_n, \psi z_t - \psi u_{n_j} \rangle + \langle z_t - u_{n_j}, \psi u_{n_j} - \psi y_{n_j} \rangle$$
$$- \langle z_t - u_{n_j}, \frac{u_{n_j - y_{n_j}}}{r_{n_j}} \rangle + F(z_t, u_{n_j}).$$

Since $\| y_{n_j} - u_{n_j} \| \to 0, j \to \infty$ by step 2, we obtain $\| \psi y_{n_j} - \psi u_{n_j} \| \to 0, j \to \infty$.

Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j} \rangle \geq 0$. Also,

$\| x_n - u_n \| \leq \| u_n - y_n \| + \| y_n - x_n \| \to 0$

Implies that $\left\{u_{n_j}\right\}_{j=1,}^{\infty}$ converges weakly to u. Then, by (A4) we obtain as $j \to \infty$,

$$\langle z_t - u, \psi z_t \rangle \geq \varphi(u) - \varphi(z_t) + F(z_t, u) \qquad (3.17)$$

Using (A1), (A4) and inequality (3.17) we also obtain

$0 = F(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \leq tF(z_t, y) + (1-t)F(z_t, u) + t\varphi(y) + (1-t)\varphi(u) - \varphi(z_t) + t\varphi z_t - t\varphi z_t \leq t[F(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)\langle z_t - u, \psi z_t \rangle = t[F(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)\langle y - u, \psi z_t \rangle$

and hence $F(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle z_t - u, \psi z_t \rangle$. (3.18)

Letting $t \to 0$, we obtain, for each $y \in K, 0 \leq F(u, y) + \varphi(y) - \varphi(u)\langle y - u, \psi u \rangle$.

This implies that $u \in GMEP$.

(ii) Next, we show $u \in VI(K, A)$. Put

$$Mw := \begin{cases} Aw + N_k w, & w \in K, \\ \emptyset, & if\ w \notin K. \end{cases}$$

Since A is $\mu$ −inverse strongly monotone, it is monotone. Thus, M is maximal monotone [7].

Let $(w_1, w_2) \in G(M)$. Since $w_2 - Aw_1 \in N_k w_1$ and $\rho_n \in K$, we have

$$\langle w_1 - \rho_n, w_2 - Aw_1 \rangle \geq 0. \qquad (3.19)$$

On the other hand, from $\rho_n = P_K(I - s_n A)u_n$ and inequality (3.19), we have $\langle w_1 - \rho_n, \rho_n - (I - s_n A)u_n \rangle \geq 0$ and hence $\langle w_1 - \rho_n, \frac{\rho_n - u_n}{s_n} + Au_n \rangle \geq 0$. It follows from inequality (3.19) with n replaced with $n_j$ and the monotonicity of A that

$\langle w_1-\rho_{n_j},w_2\rangle \geq \langle w_1-\rho_{n_j},Aw_1\rangle \geq \langle w_1-\rho_{n_j},Aw_1\rangle - \langle w_1-\rho_{n_j},\frac{\rho_{n_j}-u_{n_j}}{s_{n_j}}+Au_{n_j}\rangle$

$= \langle w_1-\rho_{n_j},Aw_1-A\rho_{n_j}\rangle + \langle w_1-\rho_{n_j},A\rho_{n_j}-Au_{n_j}\rangle - \langle w_1-\rho_{n_j},\frac{\rho_{n_j}-u_{n_j}}{s_{n_j}}\rangle$

$\geq \langle w_1-\rho_{n_j},A\rho_{n_j}-Au_{n_j}\rangle - \langle w_1-\rho_{n_j},\frac{\rho_{n_j}-u_{n_j}}{s_{n_j}}\rangle$

Which implies by step 3 and $u_{n_j}\rightharpoonup u(j\to\infty)$ that $\langle w_1-u,w_2\rangle \geq 0$. So, we have $u\in M^{-1}0$ and hence $u\in VI(K,A)$.

(iii) We now show that $u\in F(T)$. Using Lemma (2.4), the fact that T is asymptotically nonexpansive, $x_{n_j}$ converges weakly to u and $\| x_{n_j}-Tx_{n_j}\| \to 0\ as\ n\to\infty$ we obtain that $u\in F(T)$.

(iv) Now we prove that $\lim_{n\to\infty}\langle -u,x_n-u\rangle \leq 0$.

Define a map : $H\to\mathbb{R}$ by $\phi(x) := \mu_n \| x_n - x\|^2\ \forall\ x\in H$.

Then, $\phi(x)\to\infty$ as $\| x\| \to\infty,\phi$ is continuous and convex, so there exists $y^*\in H$ such that $\phi(y^*)=\min_{w\in H}\phi(w)$

Hence, the set $K^* := \left\{x\in H : \phi(x)=\min_{w\in H}\phi(w)\right\} \neq \emptyset$.

We shall make use of Lemma (2.4). if $x\in K^*$ and $y^* := w - T^{n_j}x$, for some $m_j, j\to\infty$ , then using the weak lower semi-continuity of $\phi$ and $\lim_{n\to\infty}\| x_n - Tx_n\| = 0$, we have (since $\lim_{n\to\infty}\| x_n - Tx_n\| = 0, implies\ that\ \lim_{m\to\infty}\| x_n - T^m x_n\| = 0$, m≥ 1 (by induction)):

$\phi(y^*) \leq \lim_{j\to\infty}\inf\phi(T^{m_j}x) \leq \lim_{m\to\infty}\sup\phi(T^{m_j}x) = \lim_{m\to\infty} sup(\mu \| x_n - T^m x\|^2)$

$= \lim_{m\to\infty} sup(\mu \| x_n - T^m x_n + T^m x_n - T^m x\|^2)$

$\leq \lim_{m\to\infty} sup(\mu \| T^m x_n - T^m x\|^2) \leq \lim_{n\to\infty} sup(\mu {k_m}^2 \| x_n - x\|^2)$

$\leq \lim_{n\to\infty} sup(\mu \| x_n - x\|^2) = \phi(x) = \min_{w\in H}\phi(w)$.

By Lemma (2.7), $K^*\cap F(T)\neq\emptyset$. Assume that $y^* = u\in K^*\cap F(T)$. Let $t\in(0,1)$. Then, it follows that $\phi(u)\leq \phi(u-tu)$ and using Lemma (2.5), we obtain that

$\| x_n - u + tu\|^2 \leq \| x_n - u\|^2 + 2t\langle u,x_n-u+tu\rangle$ which implies that

$\mu_n\langle -u,x_n-u+tu\rangle \leq 0$.

Furthermore, we obtain, as $t\to\infty, \langle -u,x_n-u\rangle - \langle -u,x_n-u+tu\rangle \to 0$.

Hence, for $\epsilon>0$, there exists $\delta>0$ such that $\forall\ t\in(0,\delta)$ and for all $n\geq 1$,

$\langle -u,x_n-u\rangle < \langle -u,x_n-u+tu\rangle + \epsilon \leq \epsilon$.

Consequently, $\mu_n\langle -u,x_n-u\rangle < \mu_n\langle -u,x_n-u+tu\rangle + \epsilon \leq \epsilon$.

Since $\epsilon$ is arbitrary, we have $\mu_n\langle -u,x_n-u+tu\rangle \leq 0$.

Furthermore, since $\lim_{n\to\infty}\| x_{n+1}-x_n\| = 0$, we also have:

$\lim_{n\to\infty}\sup(\langle -u,x_n-u\rangle < \langle -u,x_{n+1}-u+u\rangle) \leq 0$. And so we obtain by Lemma (2.6) that

$\lim_{n\to\infty}\sup\langle -u,x_n-u\rangle \leq 0$.

From the recursion formula (1.5), we have:

$\| x_{n+1}-u\|^2 \leq (1-\beta_n)\| x_n-u\|^2 + \beta_n {k_n}^2 \leq (1-\beta_n)\| x_n-u\|^2 + \beta_n {k_n}^2 \| y_n-u\|^2$

$\leq (1-\beta_n)\| x_n-u\|^2 + \beta_n {k_n}^2[(1-\alpha_n)\| x_n-u\|^2 + 2\alpha_n\langle u,u-x_n\rangle + {\alpha_n}^2 \| x_n\|^2]$

$\leq \left(1-\alpha_n\beta_n {k_n}^2\right)\| x_n-u\|^2 + \alpha_n\beta_n {k_n}^2[2\langle -u,x_n-u\rangle] + \left[{\alpha_n}^2 + \left({k_n}^2-1\right)\right]M,$

Where $M := sup_{n\geq 1}\left\{{k_n}^2 \| x_n\|^2 + \beta_n \| x_n-u\|^2\right\}$.

Using Lemma (2.1), we get that $\{x_n\}$ converges strongly to $u\in\Gamma$.

This completes the proof.

**3.2 Corollary:** Let K be a closed convex subset of a real Hilbert space H. Let F be a bifunction from K× K into ℝ satisfying the following: (A1)-(A4), let $\varphi: K\to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (B1) or (B2) , let A be a μ-inverse strongly monotone mapping of K into H and $\psi$ be an α-inverse, strongly monotone mapping of K into H. Let T be a nonexpansive mapping of K into itself such that Γ := F(T ) ∩V I(A, K) ∩ GMEP ≠ ∅. Let $\{x_n\}_{n=1,}^{\infty}$ $\{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1\in K$,

$$\begin{cases} y_n = P_K[(1-\alpha)x_n] \\ u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T P_K(u_n - s_n A u_n) \end{cases} \quad (3.20)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}$, $\{r_n\}_{n=1,}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$

(iii) $0 < a \leq s_n \leq b < 2\mu$;

(iv) $0 < \lim_{n\to\infty} inf \beta_n \leq \lim_{n\to\infty} sup \beta_n < 1$.

Then $\{x_n\}_{n=1,}^{\infty}$ converges strongly to $u \in \Gamma$.

## 4. APPLICATION

We now study the following optimization problem:

$$\min_{u\in K} \varphi(u), \quad (4.1)$$

Where K is a nonempty closed convex subset of a real Hilbert space H and $\varphi: K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. We denote the set of solutions of problem (4.1) by $\mathcal{F}$. Let $F(x,y) = 0, \forall x,y \in K, T \equiv I$ and $\psi \equiv 0$ in Theorem (3.1). Then,GMEP $= \mathcal{F}$. It also follows from Theorem (3.1) that the iterative sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = P_K[(1-\alpha)x_n] \\ u_n = arg \min_{u\in K}[\varphi(u) + \frac{1}{2r_n} \parallel u - y_n \parallel^2] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n P_K(u_n - s_n A u_n) \end{cases} \quad (4.2)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}$, $\{r_n\}_{n=1,}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$

(iii) $0 < a \leq s_n \leq b < 2\mu$;

(iv) $0 < \lim_{n\to\infty} inf \beta_n \leq \lim_{n\to\infty} sup \beta_n < 1$;

converges strongly to an element of $VI(K,A) \cap \mathcal{F}$.

Furthermore, let $F(x,y) = 0, \forall x,y \in K, T \equiv I$ and $\psi \equiv 0$ in Theorem (3.1).
Then, GMEP $= \mathcal{F}$. It also follows from Theorem (3.1) that the iterative sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = P_K[(1-\alpha)x_n] \\ u_n = arg \min_{u\in K}[\varphi(u) + \frac{1}{2r_n} \parallel u - y_n \parallel^2] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n u_n \end{cases} \quad (4.3)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}$, $\{r_n\}_{n=1,}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iii) $0 < \lim_{n\to\infty} inf \beta_n \leq \lim_{n\to\infty} sup \beta_n < 1$;

converges strongly to an element of $VI(K,A) \cap \mathcal{F}$.

Remark: Let A be a $\mu$ −Lipschitzian and relaxed $(\lambda,\gamma)$ −cocoercive map with $\lambda \leq \frac{\gamma}{\mu^2}$.

Then A is α-inverse, strongly monotone with $\sigma \coloneqq (\frac{\gamma}{\mu^2} - \lambda)$. In this case, the assumption that A is α-inverse, strongly monotone is weaker than the assumptions that A be a $\mu$ −Lipschitzian and relaxed $(\lambda,\gamma)$ −cocoercive imposed in [23]. Consequently, our result is a significant improvement on [23] and host of other important results in this direction of research.

**Prototypes:** The prototypes of our iteration parameters are:

$$\alpha_n := \frac{1}{n}, n \leq 1;\ \beta_n := \frac{1}{2}\left(\frac{n}{n+1}\right), n \geq 1; r_n := \frac{d_n}{n+2},$$

$$and\ s_n := \frac{b_n}{n+1}, n \geq 1, a = \frac{\mu}{4}, b = \mu, c = \frac{\alpha}{4}, d = \alpha$$

## 5. CONCLUSION

In this research paper, we studied convergence analysis for fixed point theorem of an asymptotically nonexpansive mapping, variational inequality and equilibrium problems. We constructed a new iterative algorithm that is devoid of the inherent problem in CQ and Krasnosel'skii mann type iterations methods. Our algorithm approximates the common element of the set of solution of the above three aforementioned problems. The strong convergence result of our result has been established under a suitable set of control conditions. Moreover, the applicability of the result is also shown in the paper.

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