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AMPLITUDE DISTRIBUTIONS DESCRIBING OCEAN WAVE FIELD EVENTS

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Abstract

Observed physical manifestations associated with marine wave fields are analysed. The essential interest in this study is on the wave crest elevations and the distribution of wave amplitude. Concerning amplitude fluctuations in a random wave field, the significant wave height appears distinct. This is so because of its role as the reference wave height in marine research activities. Finally, an important and interesting conclusion emanating from this study is the significant improvement in our deductions when the wave height is very close to the crest elevation which stands, in this case, as reference wave parameter in the study. This suggests that the probability density in this consideration is good for analysing events in the wave field and the related physical manifestations.

Keywords: marine wave field, wave crest elevations, wave amplitude, wave height, wave parameter, probability density
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1 Introduction

Intense wave activities induce chaotic state in ocean surfaces. The physical phenomena had been ongoing for long and had been treated adequately by researchers with interest in marine physics. The issue is the activities of ocean wave groups with related joint amplitude wave length distribution for wave train modulating as surface waves. Detail study of the kind of marine activities had been associated with a number of scientists. Among those very prominent in the study are [1, 2, 3] and several others.

Knowledge of wavelength/height joint distribution in locality is frequently asked for before any marine engineering construction is commenced. Again, studies concerning wave breaking in deep or shallow water zones is based on data for the height/wave length distribution not only in the locality but in the far field. Freak wave evaluations (numerical) are based on the identical wave parameters. Hence, this kind of studies form the background in most areas in ocean wave research.

Thus, we shall be concerned with some of the manifestations related with wave crest elevation. This forms an interesting topic of investigation in the family of ocean wave physics. Further, consideration of wave amplitude maximum in this study may reveal further manifestations related to rogue wave events in Gaussian seas.

2. Specifications

Previous studies had been based on time, t, at a fixed point (x_o, y_o, z_o). Our approach will be space dependent in this study. This provides a better picture of the event and its physical manifestations. Thus, *x-axis* is directed perpendicular to the wave front with the origin on the undisturbed sea level. A modulating wave train is considered. This is made up of N component parts with each component identified by a wave number, k.

3. Physical behaviour of two sinusoids

Consider two sine waves in spatial domain. They are defined as:

$$\eta(x) = a_0 \cos k_1 x + a_0 \cos k_2 x \ldots \qquad (3.1)$$

This takes the form:

$$\eta(x) = 2a_0 \cos\tfrac{1}{2}[(k_1 - k_2)x] \cos\tfrac{1}{2}[(k_1 + k_2)x] \qquad (3.2a)$$

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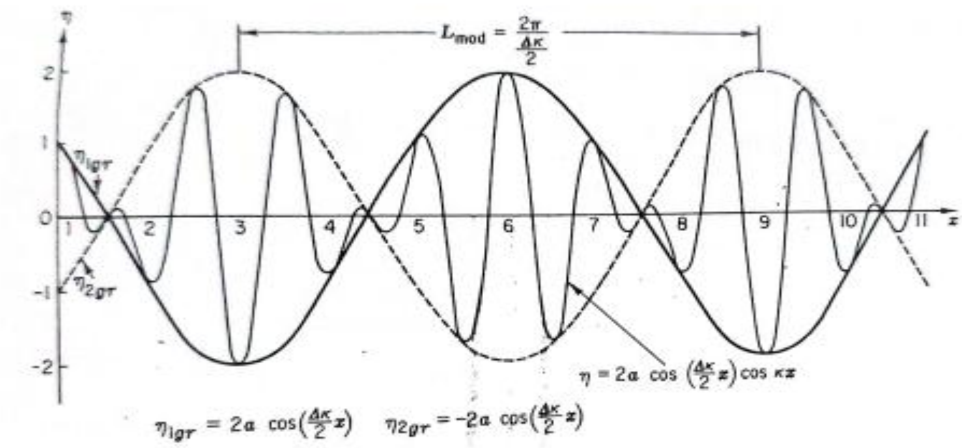

Fig. 1: *The group behaviour of an amplitude modulated wave*

This models an amplitude modulated wave event with k_1 and k_2 as the wave number components and with $\eta(x)$ as wave crest elevation. $\cos\frac{1}{2}[(k_1+k_2)x]$ is the carrier wave with wave length $\frac{2\pi}{k_1+k_2}$. This is modulated by an envelope $2a_0\cos\frac{1}{2}[(k_1-k_2)x]$ with wave length $\frac{2\pi}{k_1-k_2}$. In real situation, k_1 is quite near to k_2. Hence $\frac{2\pi}{k_1-k_2}$ contains a large number of $\frac{2\pi}{k_1+k_2}$ (i.e., $k_1 + k_2 >> k_1 - k_2$). In other words, $\frac{2\pi}{k_1-k_2} = \frac{2\pi N}{k_1+k_2}$, where N is the number of component waves that are independent but form the wave group. The corresponding wave frequency components σ_1 and σ_2 are calculated using the dispersion equation which is of the form [4] (see **Fig. I)**:

$$\sigma^2 = kg\tanh kh \tag{3.2b}$$

A modulated wave train is made of a large N number of component wave heights $H_1, H_2, \ldots, H_N$ for which $H_1 = 2a_0$, a_0 being the corresponding wave amplitude.

Assume that $H_1, H_2, \ldots, H_N$ are arranged in descending order of magnitude for N wave components:

i. The mean wave height, $\bar{H}$, is defined as

$$\bar{H} = \frac{1}{N}(H_1 + H_2 + \cdots + H_N) \tag{3.2c}$$

ii. The root mean square (rms) wave height, H_{rms} is defined as

$$H_{rms}^2 = \frac{1}{N}\sum_{s=1}^{N} H_s^2 \tag{3.2d}$$

iii. If the fraction, q, is $(0 < q \le 1)$ such that qN is an integer greater than zero, H_q is defined by

$$\bar{H}_q = \frac{1}{qN}\sum_{s=1}^{qN} H_s,\ \ 1 < s < N,\ \ qN \le N,\ \ 0 < q < 1 \tag{3.3}$$

From (3.2c) and (3.3), $\bar{H} = \bar{H}_N,\ \ \bar{H}_q \ge \bar{H},\ \ H_{rms} > H_q \ge \bar{H}$.

A useful wave parameter is deduced from (3.3) in the form

$$\bar{H}_{\frac{1}{3}} = \frac{3}{N}\sum_{s=1}^{N/3} H_s \qquad H_i < H_{i-1}, i = 1,2,\ldots,N \tag{3.4}$$

In other words, $H_{1/3}$ is the mean of the highest one-third of the wave heights for N-component waves arranged in descending order of magnitude.

The wave parameter $H_{1/3}$ is called Significant wave height, a name associated with some famous marine physicists [5]. It is regarded as a reference wave parameter in calculations involved in marine constructions, for example, wave crest distributions for rogue waves events in particular.

Further of interest in marine research is the parameter, H_{max}, called the maximum wave height. It is the parameter that describes the highest wave crest elevation in the wave field activities.

If the H_{rms} is known, then [4]

$$H_{max} = \left(R + \frac{0.2886}{R}\right) H_{rms} \qquad R = (\ln N)^{1/2} \qquad 3.4$$

N = number of wave component parts in the wave group or wave train as previously mentioned.

4. Relationship among amplitude parameters H_q, H_{rms}, H_{max}

Let $k_1 - k_2 = \Delta k$, then, integrating over the range $0 < x < {}^{q\pi}/_{\Delta k}$,

$$H_q = \frac{\Delta k}{\pi q}\int_0^{\frac{\pi q}{\Delta k}} 2H_0 \cos\left[\left(\frac{\Delta k}{2}\right)x\right]dx \qquad (4.1)$$

$$= \frac{\Delta k}{\pi q}.2H_0.\frac{2}{\Delta k}\sin(\frac{\Delta k}{2})x\Big|_0^{\pi q/\Delta k}$$

$$= \frac{4}{\pi q}H_0 \sin(\frac{\pi q}{2})$$

For $H_0 q = {}^{Hq}/_{2}$, $H_0 = 2a_0$ and thus,

$$Hq = \frac{4H_0}{\pi q}\sin\left(\frac{\pi q}{2}\right) \qquad (4.2)$$

$$H_{1/3} = \frac{12H_0}{\pi}\sin\left(\frac{\pi}{6}\right) = \frac{12H_0}{22/7}\sin(30^o) = \frac{42}{22}H_0 = \frac{21}{11}H_0 \qquad (4.3)$$

Thus,

$$H_{1/3} = \frac{21}{11}H_0 \qquad (4.4)$$

$$H_{rms}^2 = \frac{\Delta k}{\pi}\int_0^{\pi/\Delta k} 4H_0^2\cos^2\left[\frac{\Delta k}{2}x\right]dx$$

$$= \frac{\Delta k}{\pi}4H_0^2\int_0^{\pi/\Delta k}\frac{1}{2}[1+\cos(x\Delta k)]dx$$

$$= 4H_0^2\left[\frac{1}{2}(x + \frac{1}{\Delta k}\sin(x\Delta k))\right]\Bigg|_0^{\frac{\pi}{\Delta k}}\frac{\Delta k}{\pi}$$

$$= 2H_0^2$$

Thus,

$$H_{rms} = \sqrt{2}H_0 \qquad (4.5a)$$

$$H_{1/3} = \frac{21}{11}H_0 = \frac{21}{11\sqrt{2}}H_{rms} \qquad (4.5b)$$

$$H_{max} = \sqrt{2}H_{rms} \qquad (4.5c)$$

Generally,

$$Hq = \frac{2\sqrt{2}}{\pi q}\sin\left(\frac{\pi q}{2}\right)H_{rms} \qquad 0 < q \le 1 \qquad (4.6)$$

H_0 (meters)	0.5	1.0	1.5	2.0	3.5	4.3	5.0
$H_{1/3}$ (meters)	0.95	1.9	2.9	3.8	6.65	8.17	9.5
H_{rms} (meters)	0.71	1.41	2.13	2.83	4.74	6.10	7.80
H_{max} (meters)	1.0	2.0	3.0	4.0	7.0	8.60	10.0

Table 1: *Distribution of wave heights (in metres)*

Surprisingly, the above calculations follow nicely with observations (Marine Science Laboratory, Menai Bridge, Anglesey, North Wales). Comparing marine measurements with Table 1, it is suggested that

i. 70% of ocean waves are less than 1.1m high
ii. 60% less than 1m
iii. 4m and above are regarded as rare events in oceanography.

5. The Probability Density Function, $P(\eta)$ for Sinusoidal waves

We construct the probability function, $P(\eta)$, for an event described by a cosine function. $P(\eta)$ is regarded as a probability density function for wave crest elevation, $\eta(x)$. The probability of wave crest event between $\eta(x)$ and $\eta(x) + d\eta(x)$ is $P(\eta)d\eta$. The probability of the crest event in region kdx of the interval $0 < kx < \pi/2$ is $\frac{2kdx}{\pi}$; consequently,

$$P(\eta)|d\eta| = \frac{2kdx}{\pi} \tag{5.1}$$

Consider the crest elevation described simply by

$$\eta(x) = H_0 \cos kx \tag{5.2}$$

$0 < kx < \pi/2, \quad k = \frac{2\pi}{L},$ L is the wave length of the wave train. $H = 2a_0$. Thus, from (5.1) and (5.2),

$$p(\eta) = \frac{2k}{\pi}\frac{|dx|}{|d\eta|} = \frac{2}{\pi H_0 \sin kx} \tag{5.3}$$

$$\eta^2 = H_0^2 \cos^2 kx = H_0^2(1 - \sin^2 kx) \tag{5.4}$$

$$\sin^2 kx = 1 - \frac{\eta^2}{H_0^2} = \frac{1}{H_0^2}(H_0^2 - \eta^2) \tag{5.5}$$

$$\sin kx = \frac{1}{H_0}(H_0^2 - \eta^2)^{1/2} \tag{5.7}$$

$$P(\eta) = \begin{cases} \frac{2}{\pi}\frac{1}{(H_0^2 - \eta^2)^{1/2}}, & |\eta| \le H_0 \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

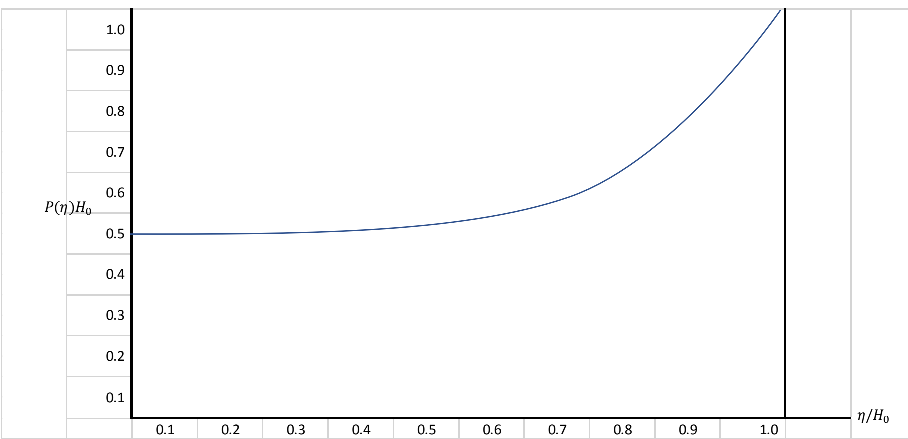

Fig. II: *Modulated wave train event*

Fig. II suggests that $\eta = 2a_0$ defines H_{max} (maximum wave height) associated with this simple model (eqn. 5.2)

6.0 Narrow band spectrum

Consider a wave group consisting of a large number of component waves with different wave numbers and phases. These phases are randomly distributed, and the energy of the group is concentrated within a narrow band of wave numbers; signifying that the spectrum of the event is realistically regarded as narrow. As before, let $a_1, a_2, a_3, \ldots, a_N$ be the amplitude respectively of component waves and that of the envelope be A, then as $N \to \infty$,

$$A = \sum_{s=1}^{N} a_s \tag{6.1}$$

Because the spectrum is narrow, $A = a_0$ [1, 6 and 3].

In this event, the probability of existence for wave crest height, A, between $\eta(x)$ and $\eta(x) + \delta\eta(x)$ is as before $P(\eta)d\eta$. This function had long been developed by Kintechine [4] to be $e^{-\left(\frac{\eta}{a}\right)^2}\frac{2\eta}{a^2}d\eta$.

Consequently,

$$P(\eta)d\eta = e^{\left(\frac{\eta}{a}\right)^2} \frac{2\eta}{a^2} d\eta \quad (6.2)$$

If we then rewrite (6.2) in the form

$$P(\eta) = \frac{2\eta}{a^2} e^{-\left(\frac{\eta}{a}\right)^2} \quad (6.3)$$

Instead, at the end we have

$$P(\eta)d\eta = -de^{-\left(\frac{\eta}{a}\right)^2} \quad (6.4)$$

Define $\phi(\eta)$ as the chance of finding a wave amplitude higher than a given crest elevation $\eta(x)$, thus,

$$\phi(\eta) = \int_{\eta}^{\infty} P(\eta)d\eta = e^{-(\frac{\eta}{a})^2} \quad (6.5)$$

Thus, from (6.4)

$$\phi(\eta) = e^{-(\frac{\eta}{a})^2} \quad (6.6)$$

From (6.5)

$$\frac{d\phi(\eta)}{d\eta} = -P(\eta) \quad (6.7)$$

Consequently $P(\eta)$ is a probability density function. It is usually assumed that the wave height-distributions over sea-surface have zero mean and unit standard deviation. Thus, (6.6) is realistic for the purpose of describing events such as in statistical description of sea surface waves.

7.0 Extremity in elevation associated with wave crest height with a_0 as the corresponding wave amplitude.

To progress in this development, we follow the identical reasoning previously brought in to explain the identical events of random marine wave forms [1]. Consequently, in eqn. (6.5), $\phi(\eta)$ is the chances of the wave amplitude a_0 being greater than $\eta(x)$. Thus, the chances of a_0 less than $\eta(x)$ (in height) is given by $1 - \phi(\eta)$. But a wave train consists of a large number of Vp components (say, *N*), the chance (in this case) becomes $[1 - \phi(\eta)]^N$. Because $1 - \phi(\eta) < 1.0$, the chance becomes smaller, the larger *N* becomes. On the opposite side, the chance that an a_0 should exceed $\eta(x)$ is $1 - [1 - \phi(\eta)]^N$. For completeness, consider the interval of length $d\eta$. As $d\eta \to 0$, the chances that at least one a_0 should exceed $\eta(x)$ minus at least one a_0 exceeds $\eta + \delta\eta$ is stated as

$$-d\{1 - [1 - \phi(\eta)]\}^N = -\frac{d}{d\eta}\{1 - [1 - \phi(\eta)]\}^N d\eta$$

$$= -\frac{d}{d\phi}\{1 - [1 - \phi(\eta)]\}^N P d\eta$$

$$= N[1 - \phi(\eta)]^{N-1} P d\eta$$

$$P = -\frac{d\phi}{d\eta}, \quad P_0 = N[1 - \phi(\eta)]^{N-1} P \quad (7.1)$$

Then, $P_0(\eta)$ is the probability density which describes H_{max} and the extreme high crest elevation that can characterize rogue wave events.

8.0 Conclusion

The analysis hereby presented seems to have brought to focus some of the wave parameters involved in wave crest distributions. Among these parameters is the significant wave height $H_{1/3}$. The generalised form of this parameter H_p $(0 < P \leq 1)$, is a parameter used as bases in a number of calculations. Emphasised in this study is therefore the analytical relationship of H_p with

i. H_{rms} and
ii. H_{max}

This relationship may give rise to some of the successful research findings that may explain some of the physical manifestations observed in marine wave field.

Finally, the probability function $P(\eta)$ (eqn. 6.3) is presently found useful in explaining the physical manifestations observed with rogue wave events [7].

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