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A CERTAIN CLASS OF (J, K)- SYMMETRIC FUNCTION INVOLVING SIGMOID FUNCTION DEFINED BY USING SUBORDINATION PRINCIPLE

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ABSTRACT

In this article, the authors investigated the the interplay that exist between (J,K)- symmetric function and the well known starlike function which is one of the sub-classes of univalent function. The (J,K)- symmetric function in simple term, means a function that defines a graph in such that, the sum of the function values in the closed neighbourhood of every vertex equals k and the function values are at most j. Its application is seen in graph theory. The method used in this investigation was the subordination principle that involved the sigmoid function which is equipped with some distinct properties such as it outputs the set of real number between 0 and 1, this is just one of its interesting features. The study produced new coefficient estimates and its connection with Fekete-Szego inequalities were found.

1. Introduction

Let the class of analytic and univalent function A be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \qquad (1)$$

The function in (1) is analytic in the open unit disk $E = \{z \in \mathbb{C}: |z| < l\}$ which consist of functions which are normalized in $E$and the subclass of A is usually represented by S.

The subordination concept take the form that, there are two functions say f and g which are analytic in E, then f is subordinate to g in E, provided there is an analytic function ω in E such that $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ holds. This subordinate is represented by $f \prec g$. Also if g is univalent in E, the subordination is then equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. For further details see [1].

In [1], the Hadamand product is a familiar idea in geometric function theory that involves $f, g \in A$. Here f take the form given in (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then we can have

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

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Definition 1.1 [1]: Let k be a positive integer. A domain D is said to be $k-$ fold symmetric if a rotation of D about the origin through an angle $\frac{2\lambda}{k}$ carries D onto itself. A function f is said to be $k-$ fold symmetric in E if for every z in E

$$f(ze^{\frac{2\lambda i}{k}}) = e^{\frac{2\lambda i}{k}} f(z). \tag{1.2}$$

The collection of all $k-$ fold symmetric functions is represented by S^k and for $k=2$ we get class of the odd univalent functions. The notion of $(j,k-)$ symmentical functions $(k=2,3,\ldots;j=0,1,2,\ldots,k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalizes the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. For further information see [1,2].

Definition 1.2[1]: Let $\mu = (e^{\frac{2\lambda i}{k}})$ and $j=0,1,2,\ldots,k-1$ where $k \geq 2$ is a natural number. A function $f\colon E \to \mathbb{C}$ is called $(j,k)-$ symmetrical if

$$f(\varepsilon z) = \varepsilon^{j} f(z), z \in E. \tag{1.3}$$

We note that the collection of all $(j,k)-$ symmetric functions is represented by $S^{(j,k)}$. Also, $S^{(0,2)}, S^{(1,2)},$ and $S^{(j,k)}$ are called even, odd and $k-$ symmetric functions respectively.

The following decomposition proposition below establish the mapping of $k-$ symmetric domain onto the complex plane comes handy in studying of the collection of the sequence of $k-$ symmetrical functions $f_{j,k}$.

Proposition 1.3 [2]: for every mapping $f\colon D \to \mathbb{C}$, and D is a $k-$ fold symmetric set, there exists exactly the sequence of $(j,k)-$ symmetrical functions $f_{j,k},$

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z).$$

Where $f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^{v} z),$ (1.4)

$(f \in A, k = 1,2,\ldots; j=0,1,2,\ldots,k-1)$

From (1.4) we have

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj} f(\varepsilon^{v} z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty} a_n (\varepsilon^{v} z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \varphi_n a_n z^n, a_1 = 1, \varphi_n = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, n = lk+j: \\ 0, n \neq lk+j \end{cases} \tag{1.5}$$

Definition 1.4[1]: Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be univalent starlike with respect to on 1 which maps the unit disk E onto a region in the right half-plane which is symmetric with respect to the real axis. Let $0 \leq \beta \leq \alpha \leq 1$ and $B_1 > 0$. Then the function $f(z) \in A$ is in the class $S^{j,k}(\phi)$ if

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \phi(z) \tag{1.6}$$

Remark: The suitable choices, $j, k, \alpha, \beta, \phi$ Fuad, AL-Sarari and Latha invetigated the following subclasses:

(i) $S^{(1,k)}(\phi) = S_s^{\,k}(\phi)$ the class introduce by Al-Shaqsi and Darus in [3]

(ii) $S^{(1,k)}\left(\frac{1+Az}{1+Bz}\right) = S_s{}^k[A, B]$ the class introduce by Al-Shaqsi and Darus in [3]

(iii) $S^{(1,2)}(\phi) = S_s{}^*(\phi)$ the class introduce by Shanmugam et al, in [4]

(iv) $S^{(1,2)}\left(\frac{1+z}{1-z}\right) = S_s{}^*$ the famous Sakaguchi class [5]

(v) $S^{(1,1)}(\phi) = S^*(\phi)$ the class introduce by Ma and Minda [6]

Sigmoid function is a unique special function that has the form

$$\ell(z) = \frac{1}{1+e^{-z}}, z \geq 0 . \quad (1.7)$$

The following properties are peculiar with sigmoid functions and they are:

(i) It is simply differentiable.

(ii) It outputs are real numbers that are between 0 and 1

(iii) It maps a very large input domain to a small range of output.

(iv) It ne'er losses information because its is one-to-one function.

(v) It step-ups monotonically.

The sigmoid function is the most popular activation function in the hardware implementation of Artificial Neutral Network (ANN) for more information see [7,8,9,10].

Motivation: The study in [1] as well as in [9] provided huge influence in this present work.

In this present study, the power series development for the collection of the sequence of $(j, k)-$symmetrical functions $f_{j,k}$ is of the form:

$$\frac{z\varphi f'(z)}{\varphi z+\sum_{n=2}^{\infty}\varphi_n a_n z^n} = 1 + \left(2-\frac{\varphi_2}{\varphi_1}\right)a_2 z + \left(\left(3-\frac{\varphi_3}{\varphi_1}\right)a_3 - \left(2-\frac{\varphi_2}{\varphi_1}\right)\frac{\varphi_2}{\varphi_1}a_2^2\right)z^2 +$$

$$\left(\left(4-\frac{\varphi_4}{\varphi_1}\right)a_4 + \left(\frac{2\varphi_2\varphi_3}{\varphi_1^2} - 2\frac{\varphi_3}{\varphi_1} - 3\frac{\varphi_2}{\varphi_1}\right)a_2 a_3 + \left(2-\frac{\varphi_2}{\varphi_1}\right)\frac{\varphi_2^2}{\varphi_1^2}a_2^3\right)z^3 +$$

$$\left(\left(5-\frac{\varphi_5}{\varphi_1}\right)a_5 + \left(\frac{2\varphi_1\varphi_4}{\varphi_1^2} - \frac{2\varphi_4}{\varphi_1} - \frac{4\varphi_2}{\varphi_1}\right)a_2 a_4 + \left(\frac{3\varphi_2^2}{\varphi_1^2} + \frac{4\varphi_2\varphi_3}{\varphi_1^2} - 3\frac{\varphi_3\varphi_2^2}{\varphi_1^3}\right)a_2^2 a_3 - \left(3-\frac{\varphi_3}{\varphi_1}\right)\frac{\varphi_3}{\varphi_1}a_3^2 -\right.$$

$$\left.\left(2-\frac{\varphi_2}{\varphi_1}\right)\frac{\varphi_2^3}{\varphi_1^3}a_2^4\right)z^4 + \ldots \quad (1.8)$$

The concept of bi-linear fractional transform has been widely used by so many researchers among them are Fuads, AL-Sarari and Latha [1], Ma and Minda [6], Oladipo [8], just to mention but few. On this we basis, we carry out our study on the interplay that exist between the bi-linear function and the familiar Charathedeory function and this is shown given below:

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^n + \cdots$$

Notice $v(z) = \frac{p(z)-1}{p(z)-1}$ and by simple computation we have:

$$u(z) = \frac{p_1 z}{2} + \left(\frac{p_1}{2} - \frac{p_1^2}{4}\right)z^n + \left(\frac{p_3}{2} + \frac{p_1}{4}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{4}\right)z^3 + \left(\frac{p_4}{2} + \frac{p_1^2}{8}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{p_1}{4}(p_1 p_2 - p_2) - \frac{p_2^2}{4}\right)z^4 + \cdots \quad (1.9)$$

The sigmoid function has its power series development in the form:

$$\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 \ldots \quad (1.10)$$

The power series development of the composition function involving $\ell(z)$ and $u(z)$ gives the form:

$$\ell(u(z)) = 1 - p_1 - (p_2 - \frac{7}{4}p_1^2)z^2 - (p_3 - \frac{p_1}{2}(\frac{29}{6}p_1^2 - 7p_2))z^3 - (p_4 - 3p_1p_3 - \frac{7}{4}p_2^2 + \frac{49}{6}p_2p_1^2 - \frac{217}{64}p_1^4)z^4 + \cdots \quad (1.11)$$

Definition 1.5: Let $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 \ldots$ where $\ell \in A$ is a logictic sigmoid activation function and $\ell'(0) > 0$. It is univalent and starlike with respect to which maps the unit disk e onto a region in the light half-plane which is symmetric with respect to the real axis. The function $f(z) \in A$ is in the subclass $S^{j,k}(\ell)$ if

$$\frac{z\varphi_1 p^1(z)}{\varphi_1 z + \sum_{n=2}^{\infty} \varphi_n a_n z^n} \prec \ell(\phi) \quad (1.12)$$

The following Lemmas were employed to establish our results

Lemma 1.6 [1]: *If* a function $p \in P$ is given by $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ is considered as analytic function with positive real part in E, then $|c_2 - nc_1^2| \leq 2\max\{1, |2n - 1|\}$, the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, p(z) = \frac{1+z}{1-z}. \quad (1.13)$$

Lemma 1.7 [112]: (Miller and Moeanu (2000):*If* a function $p \in P$ is given by $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots (z \in E)$ then $|p| \leq 2, k \in N$ where P is the family of all functions analytic in v for which

$$p(0) = 1 \text{ and } \mathrm{Re}\,(p(z)) > 0, (z \in E). \quad (1.14)$$

Lemma 1.8 [1,3,11]: *If* $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is analytic function with positive real point in E, then

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, \text{ if } v \leq 0, \\ 2, \text{ if } 0 \leq v \leq 1; \\ 4v - 2, \text{ if } v \geq 1 \end{cases} \quad (1.15)$$

2. MAIN RESULT

2.1 Coefficient Bounds for the class $S^{j,k}(\ell(u(z))$

Theorem 2.1: Let $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 + \cdots$ where $\ell \in A$, a logistic sigmoid activation function is and $\ell'(0) > 0$. If $f(z)$ given by (1) belongs to the class $S^{j,k}(\ell(u(z))$ where $\varphi_n = \frac{1}{k}\sum_{m=0}^{k-1} \delta^{(n-j)m} = \begin{cases} 1, n = lk + j: \\ 0, n \neq lk + j \end{cases}$, then

$$|a_2| \leq \frac{2}{|2 - \frac{\varphi_2}{\varphi_1}|},$$

$$|a_3| \leq \frac{9}{|3 - \frac{\varphi_3}{\varphi_1}|} + \frac{4}{|3 - \frac{\varphi_3}{\varphi_1}||2 - \frac{\varphi_2}{\varphi_1}|},$$

$$|a_4| \leq \frac{2}{|4 - \frac{\varphi_4}{\varphi_1}|}\left(\frac{53}{3} + \left(\frac{9}{|3 - \frac{\varphi_3}{\varphi_1}||2 - \frac{\varphi_2}{\varphi_1}|} + \frac{4}{|3 - \frac{\varphi_3}{\varphi_1}||2 - \frac{\varphi_2}{\varphi_1}|^2}\right)\left(|2\frac{\varphi_3\varphi_2}{\varphi_1^2} - 2\frac{\varphi_3}{\varphi_1} - 3\frac{\varphi_2}{\varphi_1}|\right) + \left(\frac{2}{|2 - \frac{\varphi_2}{\varphi_1}|}\right)^2 |\frac{\varphi_2}{\varphi_1}|^2\right).$$

$$|a_5| \leq \frac{1}{|5-\frac{\varphi_5}{\varphi_1}|}\Big(\frac{1687}{6}+\Big(\frac{424}{3|4-\frac{\varphi_4}{\varphi_1}||2-\frac{\varphi_2}{\varphi_1}|}\Big)\Big(\Big|\Big(\frac{\varphi_4\varphi_1}{\varphi_1^2}-\frac{\varphi_4}{\varphi_1}-2\frac{\varphi_2}{\varphi_1}\Big)\Big|\Big)+2\Big(\Big|\Big(3\frac{\varphi_2^2}{\varphi_1^2}+4\frac{\varphi_3\varphi_2}{\varphi_1^2}-3\frac{\varphi_3\varphi_2^2}{\varphi_1^3}\Big)\Big|\Big)\Big(\frac{11}{|3-\frac{\varphi_3}{\varphi_1}||2-\frac{\varphi_2}{\varphi_1}|}+\frac{8}{|3-\frac{\varphi_3}{\varphi_1}||2-\frac{\varphi_2}{\varphi_1}|^2}\Big)+4\Big(\frac{49}{|3-\frac{\varphi_3}{\varphi_1}|}+\frac{28}{|3-\frac{\varphi_3}{\varphi_1}||2-\frac{\varphi_2}{\varphi_1}|}+\frac{4}{|3-\frac{\varphi_3}{\varphi_1}||2-\frac{\varphi_2}{\varphi_1}|}\Big)\Big|\frac{\varphi_3}{\varphi_1}\Big|+\Big(\frac{16}{|2-\frac{\varphi_2}{\varphi_1}|^3}\Big)\Big|\frac{\varphi_3}{\varphi_1}\Big|+\Big(\frac{16}{|2-\frac{\varphi_2}{\varphi_1}|^3}\Big)\Big|\frac{\varphi_3}{\varphi_1}\Big|^3\Big) \qquad (2.1)$$

Where φ_n is defined by (1.5).

Proof:

Assuming Theorem 2.1 holds, then the following equalities hold:

$$\frac{z\varphi_1 f'(z)}{\varphi_1 z+\sum_{n=2}^{\infty}\varphi_n a_n z^n} = \ell(u(z)).$$

Note that if $p(z)$ is analytic with positive real in E and $p(0)=1$ and applying the LHS of eqn (1.8) and LHS of eqn (1.11) we have:

$$1+\Big(2-\frac{\varphi_2}{\varphi_1}\Big)a_2 z+\Big(\Big(3-\frac{\varphi_3}{\varphi_1}\Big)a_3-\Big(2-\frac{\varphi_2}{\varphi_1}\Big)\frac{\varphi_2}{\varphi_1}a_2^2\Big)z^2+$$

$$\Big(\Big(4-\frac{\varphi_4}{\varphi_1}\Big)a_4+\Big(\frac{2\varphi_2\varphi_3}{\varphi_1^2}-2\frac{\varphi_3}{\varphi_1}-3\frac{\varphi_2}{\varphi_1}\Big)a_2a_3+\Big(2-\frac{\varphi_2}{\varphi_1}\Big)\frac{\varphi_2^2}{\varphi_1^2}a_2^3\Big)z^3+$$

$$\Big(\Big(5-\frac{\varphi_5}{\varphi_1}\Big)a_5+\Big(\frac{2\varphi_1\varphi_4}{\varphi_1^2}-\frac{2\varphi_4}{\varphi_1}-\frac{4\varphi_2}{\varphi_1}\Big)a_2a_4+\Big(\frac{3\varphi_2^2}{\varphi_1^2}+\frac{4\varphi_2\varphi_3}{\varphi_1^2}-3\frac{\varphi_3\varphi_2^2}{\varphi_1^3}\Big)a_2^2a_3-\Big(3-\frac{\varphi_3}{\varphi_1}\Big)\frac{\varphi_3}{\varphi_1}a_3^2-\Big(2-\frac{\varphi_2}{\varphi_1}\Big)\frac{\varphi_2^3}{\varphi_1^3}a_2^4\Big)z^4+\ldots=1-p_1-\Big(p_2-\frac{7}{4}p_1^2\Big)z^2-\Big(p_3-\frac{p_1}{2}\Big(\frac{29}{6}p_1^2-7p_2\Big)\Big)z^3-\Big(p_4-3p_1p_3-\frac{7}{4}p_2^2+\frac{49}{6}p_2p_1^2-\frac{217}{64}p_1^4\Big)z^4+\cdots \qquad (2..2)$$

Further simplification gives:

$$a_2 = \frac{-p_1}{(2-\frac{\varphi_2}{\varphi_1})}$$

$$a_3 = \frac{-(p_2-\frac{7}{4}p_1^2)}{(3-\frac{\varphi_3}{\varphi_1})}+\frac{p_1^2}{(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})}$$

$$a_4 = \frac{-(p_3-\frac{p_1}{2}(\frac{29p_1^2}{6}-7p_2))}{(4-\frac{\varphi_4}{\varphi_1})}-\frac{1}{(4-\frac{\varphi_4}{\varphi_1})}\Big(\frac{2\varphi_2\varphi_3}{\varphi_1^2}-\frac{2\varphi_3}{\varphi_1}-\frac{3\varphi_2}{\varphi_1}\Big)\Big(\frac{p_1(p_2-\frac{7}{4}p_1^2)}{(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})}-\frac{p_1^2}{(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})^2}\Big)-\Big(\frac{p_1^2}{(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})}\Big)\frac{\varphi_2^2}{\varphi_1^2}$$

$$a_5 = \frac{-1}{(5-\frac{\varphi_5}{\varphi_1})}((p_4 - 3p_1p_3 - \frac{7}{4}p_2^2 + \frac{49}{6}p_2p_1^2 - \frac{217}{64}p_1^4) - (2\frac{\varphi_1\varphi_3}{\varphi_1^2} - 2\frac{\varphi_4}{\varphi_1} -$$

$$4\frac{\varphi_2}{\varphi_1})(\frac{p_1p_3+\frac{p_1^2}{2}(\frac{29p_1^2}{2}-7p_2)}{(5-\frac{\varphi_5}{\varphi_1})(4-\frac{\varphi_4}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})}) + (3\frac{\varphi_2^2}{\varphi_1^2} + 4\frac{\varphi_3\varphi_2}{\varphi_1^3} - 3\frac{\varphi_2^2\varphi_3}{\varphi_1^3})(\frac{p_2p_1^2-\frac{7}{4}p_1^3}{(5-\frac{\varphi_5}{\varphi_1})(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})} -$$

$$\frac{p_1^4}{(5-\frac{\varphi_5}{\varphi_1})(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})^2}) + (\frac{(p_2-\frac{7}{4}p_1^2)^2}{(5-\frac{\varphi_5}{\varphi_1})(3-\frac{\varphi_3}{\varphi_1})} - 2\frac{p_1^2(p_2-\frac{7}{4}p_1^2)}{(5-\frac{\varphi_5}{\varphi_1})(3-\frac{\varphi_3}{\varphi_1})(2-\frac{\varphi_2}{\varphi_1})} + \frac{p_1^4}{(5-\frac{\varphi_5}{\varphi_1})(3-\frac{\varphi_3}{\varphi_1})(5-\frac{\varphi_5}{\varphi_1})^2})\frac{\varphi_3}{\varphi_1} +$$

$$(\frac{p_1^4}{(5-\frac{\varphi_5}{\varphi_1})})\frac{\varphi_2^3}{\varphi_1^3}) \quad .$$

Applying **Lemma (1.7)** and modulus principle desired the result follows and this complete the proof.

2.2. Fekete-Szego Inequality for the class $S^{j,k}(\ell(u(z))$

Theorem 2.2: Let $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 + \cdots$ where $\ell \in A$ is a logistic sigmoid activation function and $\ell'(0) > 0$. If $f(z)$ given by (1) belongs to the class $S^{j,k}(\ell(u(z))$ and $\varphi_n = \frac{1}{k}\sum_{m=0}^{k-1} \delta^{(n-j)m} = \begin{cases} 1, n = lk + j: \\ 0, n \neq lk + j \end{cases}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(3-\frac{\varphi_3}{\varphi_1})}[5 + \frac{1}{(2-\frac{\varphi_2}{\varphi_1})} - \mu\frac{(3-\frac{\varphi_3}{\varphi_1})}{(2-\frac{\varphi_2}{\varphi_1})^2}] \ if \mu \leq k_1, \\ \frac{1}{(3-\frac{\varphi_3}{\varphi_1})}, if k_1 \leq \mu \leq k_2; \\ \frac{1}{(3-\frac{\varphi_3}{\varphi_1})}[5 + \frac{1}{(2-\frac{\varphi_2}{\varphi_1})} - \mu\frac{(3-\frac{\varphi_3}{\varphi_1})}{(2-\frac{\varphi_2}{\varphi_1})^2}] \ if \mu \geq k_2 \end{cases}$$

Where

$$k_1 = \frac{(2-\frac{\varphi_2}{\varphi_1})}{(3-\frac{\varphi_3}{\varphi_1})}[1 - \frac{7}{2}((2 - \frac{\varphi_2}{\varphi_1}))]$$

and

$$k_2 = \frac{(2-\frac{\varphi_2}{\varphi_1})}{(3-\frac{\varphi_3}{\varphi_1})}[\frac{3}{2}((2-\frac{\varphi_2}{\varphi_1}))+1]$$

Proof: Applying **Theorem 2.1** and **Lemma 1.8** will achieve the desired result.

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