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Explicit Closed-Form Solution of Black-Scholes Equation and its Application to Cash-or-nothing Binary Options

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ABSTRACT

This work deals with the explicit closed-form solution of Black-Scholes equation and its application to cash-or-nothing binary options. We first transform the Black-Scholes equation into a diffusion equation by change of variables. We then apply the Fourier Transform method to find the general solution of the diffusion equation. Finally, we establish an explicit closed-form solution for binary options. Hence, for a call (put) option, one gets the discounted risk neutral probability that the stock price is above (below) the strike price at time, T.

1.0 Introduction

One of the most important equations in finance is the Black-Scholes (B-S) equation. This is because it allows for accurate pricing of options [1]. The B-S equation is a partial differential equation that is made up of two constants and two variables. The constants are r, the risk-free interest rate, assumed to be constant in time and σ, the volatility of the underlying asset, while the variables are $W(t, s)$, the value of the tradable derivative and $s(x, t)$, the value of the underlying asset s at time t, which is assumed to follow geometric Brownian motion. In real market, some of the assumptions of B-S equation do not hold, hence, Merton [2] using stochastic calculus extended the model by removing some of the assumptions. Many ways of pricing options based on the B-S model have been investigated.

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Han and Wu [3], Ehrhardt and Mickens [4], and Jeong et al. [5] used finite difference method on American option pricing governed by the B-S equation. This method was extended by Cen and Le [6] for a generalized B-S equation. The Stratonovich calculus was applied by Perelloa et al. [7] to derive the B-S equation. The fitted finite volume spatial discretization and an implicit time stepping method for B-S governing option pricing was introduced by Wang [8]. Jodar et al. [9] applied Mellin transform to the solution of B-S equation. Ad hoc B-S approach was used by Berkowitz [10] to outperform the B-S formula out-of-sample. A new successive over-relaxation method was developed by Li and Lee [11] to calculate the B-S implied volatility. A new second-order exponential time differencing method was used by Yousuf et al. [12] for pricing American option with transaction cost. An upwind finite difference method was applied by Lesmana and Wang [13] to the solution of nonlinear B-S equation under transaction cost. The same method was applied by Tagliani and Milev [14] in discrete monitored barrier options. A reduced basis method for pricing options based on B-S and Heston models was introduced by Burkovska et al. [15]. A new operator splitting method for solving fractional B-S under American options was presented by Chen et al. [16]. Hence, there is need for a method that can analyze functions defined on unbounded domains and offering insights into their frequency components. Fourier transform is a powerful method of solving the Black-Scholes equation. The reason being that it provides a complete solution method that is normally not explored in financial mathematics. It maps a function defined on physical space to a function defined on the space of frequencies whose values quantify the amount of each periodic frequency contained in the original function. This work deals with the analytical solution of B-S equation using the Fourier Transform method and its application in the valuation of cash-or-nothing binary options.

2.0 Fourier Transform

One-dimensional Fourier transform $\mathscr{F}(g(x))(\lambda)$ of a function $g(x)$ such that $\int_{\mathbb{R}} dx\, |g(x)|^2 < \infty$ is defined by

$$\mathscr{F}(g(x))(\lambda) = \tilde{g}(\lambda)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\, e^{-i\lambda x} g(x) , \qquad (1)$$

where $i = \sqrt{-1}$. The inverse Fourier transform is defined by

$$\mathscr{F}^{-1}(\tilde{g}(\lambda))(x) = g(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\lambda\, e^{i\lambda x} \tilde{g}(\lambda) . \qquad (2)$$

The Fourier transform of the n-th derivative of a function g is given by

$$\mathscr{F}(g^{(n)}(x))(\lambda) = (i\lambda)^n \mathscr{F}(g(x))(\lambda) . \qquad (3)$$

The property in equation (3) can be proven by successive integrations by parts. That is for two functions f and g with the appropriate regularity properties,

$$\int fg' = fg - \int f'g , \qquad (3a)$$

where f' denotes the derivative of f.

Proof of Equation (3)

From equation (1) and the regularity condition $\int_{\mathbb{R}} dx\ |g(x)|^2 < \infty$, it follows that $\mathscr{F}(g^{(n)}(x))(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\, e^{-i\lambda x}\ g^{(n)}(x)$. From equation (3a),

$$\mathscr{F}(g^{(n)}(x))(\lambda) = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,(-i\lambda)\ e^{-i\lambda x}\ g^{(n-1)}(x)$$

$$= (i\lambda)(-1)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,(-i\lambda)\ e^{-i\lambda x}\ g^{(n-2)}(x)$$

$$= \cdots$$

$$= (i\lambda)^n \mathscr{F}(g(x))(\lambda),$$

which establishes equation (3).

Convolution Theorem: The convolution theorem states that the Fourier transform of the convolution product of two functions f and g is equal to the product of the Fourier transforms of f and g. Hence, if we denote by $f * g$ the convolution product of f and g:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dy\, f(x-y)g(y), \tag{4}$$

then the Fourier transform of the convolution product is

$$\mathscr{F}(f * g) = \mathscr{F}(f)\mathscr{F}(g). \tag{5}$$

Proof of equation (5)

From the right hand side of equation (1) and equation (5), we have

$$\mathscr{F}(f)\mathscr{F}(g) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} da\, e^{-i\lambda a}\ f(a)\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} db\, e^{-i\lambda b}\ g(b)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} da\, db\, e^{-i\lambda(a+b)} f(a)g(b). \tag{5a}$$

By change of variables $t = a + b$, $s = a$, $da\, db = |J| ds\, dt$ where the Jacobian of the transformation is $|J| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$, then from equation (4), equation (5a) becomes

$$\mathscr{F}(f)\mathscr{F}(g) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dt\, e^{-i\lambda t}\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} ds\, f(t-s)g(s).$$

From equation (1), $\mathscr{F}(f)\mathscr{F}(g) = \mathscr{F}(f * g)$, which establishes the convolution theorem of equation (5).

3.0 Reducing Black-Scholes equation to a diffusion equation

The Black-Scholes partial differential equation is written as [17]

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 W}{\partial s^2} + rs\frac{\partial W}{\partial s} - rW = 0,\ s \geq 0,\ t \in [0, T], \tag{6}$$

where $W(s,t)$ is the value of the option, s the price of the underlying asset, t the time, T the expiration date, σ the volatility of the underlying asset and r the risk-free interest rate. Let r and σ be constants in equation (6), then by change of variables

$$s = \lambda e^{x}, \tag{7i}$$

$$W(s,t) = \lambda w(x,\tau), \tag{7ii}$$

$$\tau = \frac{(T-t)\sigma^2}{2}. \tag{7iii}$$

We therefore obtain the partial derivatives of $W(s,t)$ and substitute them back in equation (6). From equation (7ii), $\frac{\partial W}{\partial t} = \lambda \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial t}$. But from equation (7iii), $\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}$. Hence,

$$\frac{\partial W}{\partial t} = -\lambda \frac{\sigma^2}{2}\frac{\partial w}{\partial \tau}. \tag{8i}$$

From equation (7ii)

$$\frac{\partial W}{\partial s} = \lambda \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$

$$= \lambda \frac{\partial w}{\partial x}\frac{\partial}{\partial s}\ln\left(\frac{s}{\lambda}\right)$$

$$= \frac{\lambda}{s}\frac{\partial w}{\partial x}. \tag{8ii}$$

From equation (8ii),

$$\frac{\partial^2 W}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\lambda}{s}\frac{\partial w}{\partial x}\right)$$

$$= -\frac{\lambda}{s^2}\frac{\partial w}{\partial x} + \frac{\lambda}{s}\frac{\partial}{\partial s}\frac{\partial w}{\partial x}$$

$$= -\frac{\lambda}{s^2}\frac{\partial w}{\partial x} + \frac{\lambda}{s^2}\frac{\partial^2 w}{\partial x^2}. \tag{8iii}$$

Substituting equations (8) into equation (6) gives

$$-\lambda \frac{\sigma^2}{2}\frac{\partial w}{\partial \tau} + \frac{1}{2}\sigma^2 s^2\left(-\frac{\lambda}{s^2}\frac{\partial w}{\partial x} + \frac{\lambda}{s^2}\frac{\partial^2 w}{\partial x^2}\right) + rs\frac{\lambda}{s}\frac{\partial w}{\partial x} - r\lambda w = 0 \quad . \tag{9}$$

Divide both sides of equation (9) by $\lambda\frac{\sigma^2}{2}$ to have $-\frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x} + \frac{2r}{\sigma^2}\frac{\partial w}{\partial x} - \frac{2rw}{\sigma^2} = 0$.

Hence,

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial w}{\partial x} - \frac{2r}{\sigma^2}w. \tag{10}$$

Define

$$a = \frac{2r}{\sigma^2} - 1 \tag{11i}$$

and

$$b = -\frac{2r}{\sigma^2} = -(1+a) \quad . \tag{11ii}$$

Equation (10) becomes

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + a\frac{\partial w}{\partial x} + bw \; . \tag{12}$$

We then reduce equation (12) to a diffusion equation. The solution of equation (12) is of the form

$$w(x,\tau) = l(\tau)m(x)p(x,\tau) \; . \tag{13}$$

We then obtain the partial derivatives of $w(x,\tau)$. That is

$$\frac{\partial w}{\partial \tau} = (\partial_\tau l)mp + lm\,(\partial_\tau p) \tag{14i}$$

$$\frac{\partial w}{\partial x} = l\,(\partial_x m)p + lm\,(\partial_x p) \tag{14ii}$$

$$\frac{\partial^2 w}{\partial x^2} = l\,(\partial_x^2 m)p + 2l\,(\partial_x m)(\partial_x p) + lm\,(\partial_x^2 p), \tag{14iii}$$

where $\partial_x^n l = \frac{\partial^n l}{\partial x^n}$. Substituting equations (14) in equation (12) gives

$$(\partial_\tau l)mp + lm\,(\partial_\tau p) = l\,(\partial_x^2 m)p + 2l\,(\partial_x m)(\partial_x p) + lm\,(\partial_x^2 p)$$
$$+al\,(\partial_x m)p + alm\,(\partial_x p) + bl\,(\tau)m(x)p(x,\tau) \; . \tag{15i}$$

From equation (13), we have

$$(\partial_\tau l)mp + lm\,(\partial_\tau p) = l\,(\partial_x^2 m)p + 2l\,(\partial_x m)(\partial_x p) + lm\,(\partial_x^2 p)$$
$$+al\,(\partial_x m)p + alm\,(\partial_x p) + blmp \; . \tag{15ii}$$

Equations (15) can be satisfied if l and m are of the form

$$l(\tau) = Ae^{\hat{l}(\tau)} \tag{16i}$$

$$m(x) = Be^{\hat{m}(x)} , \tag{16ii}$$

where $A \in \mathbb{R}$ and $B \in \mathbb{R}$ are constants. Therefore,

$$\partial_\tau l(\tau) = \partial_\tau \hat{l}(\tau) \cdot Ae^{\hat{l}(\tau)}$$
$$= l\,(\partial_\tau \hat{l}(\tau)) \tag{17i}$$

$$\partial_x m(x) = \partial_x \hat{m}(x) \cdot Be^{\hat{m}(x)}$$
$$= m\,(\partial_x \hat{m}) \tag{17ii}$$

$$\partial_x^2 m(x) = m\,(\partial_x^2 \hat{m}) + m\,(\partial_x \hat{m})^2 . \tag{17iii}$$

Substituting equations (16) and (17) in equation (15) gives

$$l(\partial_\tau l)mp + lm\ (\partial_\tau p) = pl\ (m\partial_x^2 \hat{m}) + lpm\ (\partial_x \hat{m})^2 + 2lm\ (\partial_x \hat{m})(\partial_x p)$$
$$+lm\ (\partial_x^2 p) + alpm\ (\partial_x \hat{m}) + alm\ (\partial_x p) + blmp\ . \qquad (18)$$

Divide both sides of equation (18) by lm and collect like terms to have

$$(\partial_\tau p) = (\partial_x^2 p) + \partial_x p\{2(\partial_x \hat{m}) + a\}$$
$$+p\{-(\partial_\tau \hat{l}) + (\partial_x^2 \hat{m}) + (\partial_x \hat{m})^2 + a(\partial_x \hat{m}) + b\}. \qquad (19)$$

In order to make equation (19) a diffusion equation, we let $2(\partial_x \hat{m}) + a = 0$ or $\partial_x \hat{m} = -\frac{a}{2}$, which gives

$$\hat{m}(x) = -\frac{ax}{2} + A \qquad (20i)$$

and

$$-(\partial_\tau \hat{l}) + (\partial_x^2 \hat{m}) + (\partial_x \hat{m})^2 + a(\partial_x \hat{m}) + b = 0$$

implies that $-(\partial_\tau \hat{l}) + \frac{a^2}{4} - \frac{a^2}{2} + b = 0$ or $\partial_\tau \hat{l} = b - \frac{a^2}{4}$. Hence,

$$\hat{l}(\tau) = (b - \frac{a^2}{4})\tau + B\ , \qquad (20ii)$$

where $A \in \mathbb{R}$ and $B \in \mathbb{R}$. From equations (11) and (16), we have

$$l(\tau) = Ae^{\hat{l}(\tau)} = Ae^{-(\frac{a^2}{4}+a+1)\tau} \bullet e^B$$

$$m(x) = Be^{\hat{m}(x)} = Be^{-(\frac{a}{2})x} \bullet e^A.$$

Equation (13) becomes

$$w(x,\tau) = Ae^{-(\frac{a^2}{4}+a+1)\tau} e^{-(\frac{a}{2})x} p(x,\tau)\ . \qquad (21)$$

Substituting equation (20) in equation (19) gives

$$\partial_\tau p = \partial_x^2 p \text{ or } \frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2}, x \in \mathbb{R}, \tau \in [0, \frac{\sigma^2 T}{2}]. \qquad (22)$$

From equations (7), (21) and (22), we have

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2}$$
$$s = \lambda e^x$$
$$\tau = (T - t)\frac{\sigma^2}{2}$$
$$W(s,t) = \lambda w\ (x,\tau) = \lambda e^{-(\frac{a^2}{4}+a+1)\tau} e^{-(\frac{a}{2})x} p(x,\tau) \qquad (23)$$
$$a = \frac{2r}{\sigma^2} - 1\ .$$

3.1 Solving the resultant diffusion equation

Equation (23) with initial condition $p(x, 0)$ is then solved using Fourier transforms. Hence, by definition

$$\mathscr{F}\left(\frac{\partial p}{\partial \tau}\right) = (i\lambda)\mathscr{F}(p(\tau))(\lambda)$$
$$= (i\lambda)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} d\tau\, e^{-i\lambda\tau}\, p(\tau)\,.$$

But $\mathscr{F}(p) = \tilde{p}(\lambda)$ which is the Fourier transform of the function p with respect to the variable τ. Hence, the Fourier transform of equation (23) is

$$\frac{\partial \tilde{p}}{\partial \tau} = (i\lambda)^2\tilde{p}$$
$$= -\lambda^2\tilde{p}. \qquad (24)$$

Solving equation (24) by separation of variables gives

$$\ln\tilde{p} = -\lambda^2\tau + c$$
$$\tilde{p}(\lambda,\tau) = \tilde{p}(\lambda,0)e^{-\lambda^2\tau}, \qquad (25)$$

where $\tilde{p}(\lambda, 0)$ is the Fourier transform of the initial condition for p which corresponds to the terminal condition at expiry $t = T$ of the option. In order to find the solution for $p(x,\tau)$, we apply the inverse Fourier transform of equation (25). Let

$$\mathscr{F}(p_1) = \tilde{p}_1 = e^{-\lambda^2\tau} \qquad (26i)$$
$$\mathscr{F}(p_2) = \tilde{p}_2 = \tilde{p}(\lambda,0). \qquad (26ii)$$

Then equation (25) becomes

$$\tilde{p}(\lambda,\tau) = \tilde{p}_1(\lambda,\tau)\tilde{p}_2(\lambda,\tau). \qquad (27)$$

By convolution theorem, the inverse Fourier transform of equation (27) becomes

$$p(x,\tau) = (p_1 * p_2)(x,\tau)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} d\zeta\, p_1(x-\zeta,\tau)\, p_2(\zeta,\tau). \qquad (28)$$

Next, we derive the inverse Fourier transforms of equations (26i) and (26ii). From the definition of inverse Fourier transform, we have

$$\mathscr{F}^{-1}(e^{-\lambda^2\tau}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} d\lambda\, e^{i\lambda x}\, e^{-\lambda^2\tau}. \qquad (29)$$

By completing the square, we require that

$$-\lambda^2\tau + i\lambda x = A(\lambda + B)^2 + c\,, \qquad (30)$$

where A, B and c are to be determined. Equating coefficients of like terms we have $-\tau = A$, $ix = 2AB$ and $AB^2 + c = 0$. That is, $A = -\tau$, $B = \frac{-ix}{2\tau}$ and $c = \frac{-x^2}{4\tau}$. Substituting the values of A, B and c in equation (30) gives $-\tau\left(\lambda - \frac{ix}{2\tau}\right)^2 - \frac{x^2}{4\tau} = -\lambda^2\tau + i\lambda x$. Hence, $e^{-\tau\left(\lambda - \frac{ix}{2\tau}\right)^2 - \frac{x^2}{4\tau}} = e^{i\lambda x - \lambda^2\tau}$. Equation (29) becomes

$$\mathscr{F}^{-1}(e^{-\lambda^2\tau}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} d\lambda\, e^{-\tau\left(\lambda - \frac{ix}{2\tau}\right)^2} e^{-\frac{x^2}{4\tau}}.$$

By factorization, we have

$$\mathscr{F}^{-1}(e^{-\lambda^2\tau}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4\tau}}\int_{\mathbb{R}} d\lambda\, e^{-\tau\left(\lambda - \frac{ix}{2\tau}\right)^2}. \qquad (31)$$

By change of variables, let $\zeta = \sqrt{\tau}\left(\lambda - \frac{ix}{2\tau}\right)$ and $\frac{d\zeta}{d\lambda} = \sqrt{\tau}$, which implies that $d\zeta = d\lambda\sqrt{\tau}$ and $d\lambda = \frac{1}{\sqrt{\tau}}d\zeta$. Equation (31) becomes

$$\mathscr{F}^{-1}(e^{-\lambda^2\tau}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4\tau}}\frac{1}{\sqrt{\tau}}\int_{\mathbb{R}} d\zeta\, e^{-\zeta^2}.$$
$$= \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{4\tau}}\int_{\mathbb{R}} d\zeta\, e^{-\zeta^2}. \qquad (32)$$

In evaluating the integral $\int_{\mathbb{R}} d\zeta\, e^{-\zeta^2}$, we note that $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$. Hence,

$$\int_{\mathbb{R}^2} d\zeta\, e^{-\zeta^2} = \int_{\mathbb{R}} d\zeta_1 \int_{\mathbb{R}} d\zeta_2\, e^{-\zeta_1^2 - \zeta_2^2}$$

$$= \int_{\mathbb{R}} d\zeta_1 e^{-\zeta_1^2} \int_{\mathbb{R}} d\zeta_2 \, e^{-\zeta_2^2}$$
$$= \left(\int_{\mathbb{R}} d\zeta \, e^{-\zeta^2} \right)^2. \quad (32i)$$

Similarly, using polar coordinates $\zeta = (\zeta_1, \zeta_2) = r(\cos\theta, \sin\theta)$ and $d\zeta_1 d\zeta_2 = rdrd\theta$ where $r \in [0, \infty)$, $\theta \in [0,2\pi]$ we have

$$\int_{\mathbb{R}^2} d\zeta \, e^{-\zeta^2} = \int_0^{\infty} dr \, r \int_0^{2\pi} d\theta e^{-r^2}$$
$$= 2\pi \int_0^{\infty} dr \, re^{-r^2}$$
$$= 2\pi \int_0^{\infty} d\left(-\frac{1}{2}e^{-r^2}\right)$$
$$= \pi. \quad (32ii)$$

From equations (32i) and (32ii),

$$\int_{\mathbb{R}} d\zeta \, e^{-\zeta^2} = \sqrt{\pi}. \quad (33)$$

Substituting equations (33) in equation (32) gives

$$\mathscr{F}^{-1}(e^{-\lambda^2\tau}) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{4\tau}}(\sqrt{\pi})$$
$$= \frac{1}{\sqrt{2\tau}} e^{-\frac{x^2}{4\tau}}.$$

That is,

$$\mathscr{F}^{-1}(\tilde{p}_1) = p_1 = \frac{1}{\sqrt{2\tau}} e^{-\frac{x^2}{4\tau}}. \quad (34i)$$

Similarly, from equation (26ii)

$$\mathscr{F}^{-1}(\tilde{p}_2) = p_2 = p(x, 0). \quad (34ii)$$

Substituting equations (34) in equation (28) gives

$$p(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\zeta \, \frac{1}{\sqrt{2\tau}} \exp\left\{-\frac{(x-\zeta)^2}{4\tau}\right\} p(\zeta, 0)$$
$$= \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} d\zeta \exp\left\{-\frac{(x-\zeta)^2}{4\tau}\right\} p(\zeta, 0). \quad (35)$$

4.0 Application to cash-or –nothing binary options

Consider a binary option whose payoff at expiry is given as:

$$\text{payoff at expiry} = \begin{cases} 1, & \text{money} \\ 0, & \text{otherwise} \end{cases}.$$

The initial condition reads

$$W(s, T) = \begin{cases} \varphi(s-\lambda) & \text{for a call,} \\ 1-\varphi(s-\lambda) & \text{for a put,} \end{cases} \quad (36)$$

where $\varphi(\alpha)$ is the Heaviside distribution which is 0 if $\alpha < 0$ and 1 if $\alpha > 0$. Equation (36) can be expressed in terms of the new variables from equation (23). That is, using the properties that $\varphi(\beta\alpha) = \varphi(\alpha)$ for $\beta \in \mathbb{R}^+$, $e^x - 1 > 0$ if and only if $x > 0$ and $1 - \varphi(x) = \varphi(-x)$, we have from equation (23) that

$$\varphi(s-\lambda) = \varphi(\lambda e^x - \lambda)$$
$$= \varphi(e^x - 1) = \varphi(x). \quad (37)$$

Hence, from equation (23), we have $W(s, t) = \lambda e^{-(\frac{a^2}{4}+a+1)\tau} e^{-(\frac{a}{2})x} p(x, \tau)$. But $\tau = 0$ implies that $\frac{(T-t)\sigma^2}{2} = 0$, which means that $T = t$. Therefore, $W(s, T) = \lambda e^{-(\frac{a}{2})x} p(x, 0)$ or $p(x, 0) = \frac{1}{\lambda} e^{(\frac{a}{2})x} W(s, T)$. From equation (37) we have

$$p(x, 0) = \frac{1}{\lambda} e^{(\frac{a}{2})x} \varphi(\eta x), \quad (38)$$

where $\eta = 1$ for a call and $\eta = -1$ for a put. Substituting equation (38) into the general solution (equation 35) gives

$$p(x,\tau) = \frac{1}{\lambda}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} d\zeta \exp\{-\frac{(x-\zeta)^2}{4\tau}\} e^{(\frac{a}{2})\zeta}\varphi(\eta\zeta) . \quad (39)$$

By change of variables, let $\varepsilon = \frac{\zeta}{\eta}$ and $d\varepsilon = \frac{d\zeta}{\eta}$. Equation (39) becomes

$$\begin{aligned} p(x,\tau) &= \frac{1}{\lambda}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \eta d\varepsilon \exp\{-\frac{(x-\varepsilon\eta)^2}{4\tau}\} e^{(\frac{a}{2})\varepsilon\eta}\varphi(\eta^2\varepsilon) \\ &= \frac{1}{\lambda}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \eta d\varepsilon \exp\{-\frac{(x-\varepsilon\eta)^2}{4\tau}\} e^{(\frac{a}{2})\varepsilon\eta}\varphi(\varepsilon) \\ &= \frac{\eta}{\lambda}\frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty} d\varepsilon \exp\{-\frac{(x-\varepsilon\eta)^2}{4\tau}\} e^{(\frac{a}{2})\varepsilon\eta} . \end{aligned} \quad (40)$$

From [18],

$$\frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty} d\varepsilon \exp\{-\frac{(x-\varepsilon\eta)^2}{4\tau}\} e^{(\frac{a}{2})\varepsilon\eta} = e^{(\frac{a}{2})(x+\frac{a\tau}{2})}\Phi(\eta\frac{x+a\tau}{\sqrt{2\tau}}), \quad (41)$$

where Φ denotes the cumulative standard normal distribution function. Substituting equation (41) in equation (40) gives

$$p(x,\tau) = \frac{\eta}{\lambda} e^{(\frac{a}{2})(x+\frac{a\tau}{2})}\Phi(\eta\frac{x+a\tau}{\sqrt{2\tau}}). \quad (42)$$

Substituting equation (42) in equation (23) gives

$$\begin{aligned} W(s,t) &= \lambda e^{-(\frac{a^2}{4}+a+1)\tau} e^{-(\frac{a}{2})x}\frac{\eta}{\lambda} e^{(\frac{a}{2})(x+\frac{a\tau}{2})}\Phi(\eta\frac{x+a\tau}{\sqrt{2\tau}}) \\ &= \eta e^{(-a-1)\tau}\Phi(\eta\frac{x+a\tau}{\sqrt{2\tau}}). \end{aligned}$$

From equation (23), we have

$$\begin{aligned} W(s,t) &= \eta e^{-r(T-t)}\ \Phi(\eta\frac{x+a\tau}{\sqrt{2\tau}}) \\ &= \eta e^{-r(T-t)}\ \Phi(\eta d_1), \end{aligned} \quad (43)$$

where $d_1 = \frac{x+a\tau}{\sqrt{2\tau}}$.

Conclusion

Fourier transform is a powerful method of solving the Black-Scholes equation. The reason being that it provides a complete solution method that is normally not explored in financial mathematics. After obtaining the solution for the general derivative via the Fourier transform, the valuation of cash-or-nothing binary options was performed. For a call option with $\eta = 1$, one gets as expected the discounted risk neutral probability that the stock price s is above λ at time T. Similarly, for a put option with $\eta = -1$, one gets the discounted risk neutral probability that the stock price s is below λ at time T.

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