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INFLUENCE OF DAMPING COEFFICIENTS ON THE RESPONSE TO MOVING DISTRIBUTED MASSES OF RAYLEIGH BEAMS RESTING ON BI-PARAMETRIC SUBGRADE

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Abstract

This paper is concerned with the dynamic analysis of damping coefficients on the response to moving distributed masses of uniform Rayleigh beams resting on bi-parametric subgrade.
In the first instance, The fifth order partial differential equation governing the dynamical system is subjected to the finite Fourier sine integral transform to reduce it to a couple second order ordinary differential equation. This resulting coupled ordinary differential equation is then simplified using the Struble's Asymptotic techniques to form amenable to Integral transformation techniques. The Closed form Solution thereby obtained is analysed and results in plotted curves revealed that an increase in the value of the damping due to transverse displacement and damping due to strain velocity for a fixed value of shear modulus, foundation stiffness, axial force and rotatory inertia factor decreases the response amplitudes of the beam. However, damping due to resistance to transverse displacement has a more noticeable effects in reducing the response amplitudes of the damped beam than the damping due to strain velocity. It is also found that the critical speed of the moving distributed load which brings about a resonance decreases as the values of damping coefficients increases.

1.0 Introduction

Over the years, the dynamic response of structural members under the influence of moving loads is a problem that has attracted research activities and scientific investigations by numerous researchers because of its relevance in diverse areas [1-3]. The analysis of structures having uniform cross-sections and subjected to concentrated loads is very common in literature for example [4-7]. In all of these afore mentioned literature, in the formulation of the beam-type members of dynamical problems, no consideration was given to mechanisms which absorb energy from structure during its dynamic response.

An undamped system vibrates freely with constant amplitude for an indefinite period. Real system does not behave like this because their movement involves the dissipation of energy, and the energy has to be drawn from energy of vibration. This dissipation prevents the direct reacceptances of real system from ever being infinite.

In [8] infinite beams under the action of moving loads, considering nonlinear behavior and the viscous damping of supporting poor soil subgrade system was examined. It was clearly highlighted that the response of systems is greatly affected by magnitude and velocity of applied load viscous damping and ultimate resistance of poor soil. Similarly, in [9] the response of viscously damped Euler-Bernoulli beam to uniform partially distributed moving load was investigated. The researcher used numerical technique to present solutions to the complex dynamical system whereas the analytical methods are desirable as solutions so obtained have inherent vital information about the vibrating system. Other researchers worthy of mention in this area of study are in [10 - 12] to mention but a few. It is however noted that in all these aforementioned investigations that the methods and solutions are restricted to numerical simulation and Beams considered are limited to the classes of beams with lenght-span ratios lower than about 1/10 . Also, in all of these considerations, the structures considered are either not on elastic foundation or resting on the winkler elastic foundation which has suffered much

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contradiction in literature . Thus, in this paper the influence of damping coefficients on the response to moving distributed masses of simply supported Rayleigh beams resting on bi-parametric subgrade is considered. In particular the Uniform Rayleigh beam transversed by moving distributed masses is taken to rest on Pasternak Elastic foundation and all the components of inertial term relevant to the dynamical system are included in the governing partial differential equation. The objective of the study is to obtain a closed form solutions to this dynamical problem as such solutions often shed light on vital information about the vibrating system. Subsequently, the effects of the various structural parameters on the dynamic system are obtained and presented in plotted curves.

2.0 Mathematical Model

Consider the transverse motion of a homogeneous isotropic axially pre-stressed and damped Rayleigh beam resting on bi-parametric Pasternak elastic foundation and subjected to travelling masses. The mass $\overline{m}$ is assumed to touch the beam at time $t = 0$ and advances on the beam from end $x = 0$ and $x = L$ of the beam. According to the classical theory of beam flexure, the governing equation of motion of the damped dynamical system given by.

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 V(x,t)}{\partial x^2} + C_S\frac{\partial^3 V(x,t)}{\partial x^2 \partial t}\right] + \overline{m}\frac{\partial^2 V(x,t)}{\partial t^2} - \overline{m}R^0\frac{\partial^4 V(x,t)}{\partial x^4} + C\frac{\partial V(x,t)}{\partial t} - N\frac{\partial^2 V(x,t)}{\partial x^2}$$

$$+ kV(x,t) - \frac{G\partial^2 V(x,t)}{\partial x^2} = P(x,t) \qquad (2.0)$$

where x is the spatial coordinate, t is the time, *V(x,t)* is the transverse displacement, *E* is Young's modulus, I is the constant Moment of inertia of the beam, $\overline{m}$ is the constant mass per unit length of the beam, $R^o$is the measure of rotatory inertia correction factor, *K* is the elastic foundation constant, G is the shear modulus and *P(x,t)* is the uniform distributed load acting on the beam. For this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam.

For this analysis, the beam under consideration is taken to be simply supported. Thus, the boundary conditions are given by

$$V(0,t)=V(L,t)=0 \qquad \frac{\partial^2 V(0,t)}{\partial x^2}=\frac{\partial^2 V(L,t)}{\partial x^2} \qquad (2.1)$$

And the initial conditions, without any loss of generality, is taken as

$$V(x,0)=0=\frac{\partial V(x,0)}{\partial t} \qquad (2.2)$$

3.0 Transformation of Equation

The equation (2.0) is a fifth order partial differential equation. Firstly, by virtue of the boundary conditions the fifth order partial differential equation will be reduced to second order ordinary differential equation by applying the finite sine integral transform with respect to x to $V(x,t)$. The pertinet integral transform is defined as

$$\overline{V}(m,t) = \int_o^l V(x,t)\, Sin\frac{m\pi x}{L}dx \qquad (3.0)$$

with the inverse $V(x,t)=\sum_{m=1}^{\infty}\frac{\overline{m}}{V_m}\overline{V}(m,t)\,Sin\frac{m\pi x}{L}dx$ (3.1)

Introducing (3.0) and (3.1) in equation (2.0) yields

$$\overline{V}_{tt}(m,t)+\left(\frac{C}{\overline{m}}-\frac{\omega_m^2}{EI}\frac{C_s I}{L}\right)\overline{V}_t(m.t)+\left(\omega_m+\frac{k}{\overline{m}}\right)\overline{V}(m,t)-\frac{R^0}{\overline{m}L}\sum_{k=1}^{\infty}L\overline{V}_{tt}(k,t)B_A(k,m)$$

$$-\frac{N}{\overline{m}L}\sum_{K=1}^{\infty}\overline{V}(k,t)B_B(k,m)-\frac{G}{\overline{m}L}\sum_{K=1}^{\infty}\overline{V}(k,t)B_B(k,m)+\frac{M}{\overline{m}L}\sum_{k=1}^{\infty}L\left[\frac{1}{4}\overline{V}_{tt}(k,t)B_C(k,m)\right.$$

$$+\frac{M}{\overline{m}\pi L}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ut}{2n+1}\overline{V_{tt}}(k,t)B_D(n,k,m)-\frac{M}{\overline{m}\pi L}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ut}{2n+1}\overline{V_{tt}}(k,t)B_E(n,k,m)$$

$$+\frac{MU}{2\overline{m}}\overline{V}_t(k,t)B_F(k,m)$$
$$+\frac{2MU}{\overline{m}\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ut}{2n+1}\overline{V}_t(k,t)B_G(n,k,m)-\frac{2Mu}{\overline{m}\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ut}{2n+1}\overline{V}_t(k,t)B_H(n,k,m)$$
$$+\frac{MU^2}{4\overline{m}}\overline{V}(k,t)B_I(k,m)$$
$$+\frac{MU^2}{\overline{m}\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ut}{2n+1}\overline{V}(k,t)B_J(n,k,m)-\frac{Mu^2}{\overline{m}\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ut}{2n+1}\overline{V}(k,t)B_K(n,k,m)\Bigg]$$
$$=Mg\frac{L}{\overline{m}\lambda_m}\sin\frac{\lambda_m ut}{L} \tag{3.2}$$

where, $$\overline{e_k}(x)=\int_0^L Sin^2\frac{k\pi x}{L}dx \tag{3.3}$$

and $$B_A(k,m)=-\int_O^L\frac{1}{\overline{e_k}(x)}Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.4}$$

$$B_B(k,m)=-\int_O^L\frac{1}{\overline{e_k}(x)}Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.5}$$

$$B_c(k,m)=\int_O^L\frac{1}{\overline{e_k}(x)}Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.6}$$

$$B_D(n,k,m)=\int_O^L\frac{1}{\overline{e}_k}\sin(2n+1)\pi x\,Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.7}$$

$$B_E(n,k,m)=\int_O^L\frac{1}{\overline{e}_k}\cos(2n+1)\pi x Sin\frac{m\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.8}$$

$$B_F(n,k,m)=\int_O^L\frac{1}{\overline{e}_k}Sin\frac{\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.9}$$

$$B_G(n,k,m)=\int_O^L\frac{1}{\overline{e}_k}\sin(2n+1)\pi x\,Sin\frac{k\pi x}{L}Cos\frac{m\pi x}{L}\,dx \tag{3.10}$$

$$B_H(n,k,m)=\int_O^L\frac{1}{\overline{e}_k}Cos(2n+1)\pi x\,Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.11}$$

$$B_I(k,m)=\int_O^L\frac{1}{\overline{e}_k}Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.12}$$

$$B_I(k,m)=\int_O^L\frac{1}{\overline{e}_k}\;Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}\,dx \tag{3.13}$$

$$B_J(n,k,m) = -\int_O^L \frac{1}{\bar{e}_k} Sin(2n+1)\pi x \; Sin\frac{k\pi x}{L} \; Sin\frac{m\pi x}{L} \; dx \tag{3.14}$$

$$B_K(n,k,m) = -\int_O^L \frac{1}{\bar{e}_k} Cos(2n+1)\pi x \; Sin\frac{m\pi x}{L} \; Sin\frac{m\pi x}{L} \; dx \tag{3.15}$$

The equation (3.2) is the transformed equation of the problem. Next, we evaluate the integrals in (3.3) to (3.15) and substitute into (3.2) after simplification and rearrangements, to obtain

$$\left[I + R^0\left(\frac{m\pi}{L}\right)^2\right]\overline{V}_{tt}(m,t) + \left[\frac{c}{\bar{m}} - \frac{c_s}{\bar{m}}\left(\frac{m\pi}{L}\right)^4\right]\overline{V}_t(m,t) + \left[\left(\frac{EI}{\bar{m}}\left(\left(\frac{m\pi}{L}\right)^4 + \frac{k}{\bar{m}}\right)\right) + \left(\frac{G-N}{\bar{m}L}\right)\left(\frac{m\pi}{L}\right)^2\right]\overline{V}(m,t) \quad (4.5)$$

$$+\varepsilon_0 L\left\{\left[\frac{1}{4} + \frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^2 Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - 4m^2} - \frac{Cos(2n+1)\pi L - 1}{[(2n+1)L]^2}\right)\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.\right.$$

$$\left.-\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{[(2n+1)L]^2} - \frac{(-1)^{2m}}{[(2n+1)L]^2 - 4m^2}\right)\frac{Sin(2n+1)\pi ut}{(2n+1)}\right]\overline{V}_{tt}(m,t) + \frac{2um}{2\pi}$$

$$\left[\left(\sum_{n=0}^{\infty}\frac{2m(-1)^{2m} Sin(2n+1)\pi L}{[(2n+1)L]^2 - 4m^2}\;\frac{Cos(2n+1)\pi ut}{(2n+1)} - \left(\sum_{n=0}^{\infty}\frac{2m(-1)^{2m} Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - 4m^2}\;\frac{Sin(2n+1)\pi ut}{(2n+1)}\right)\right.\right]\overline{V}_t(m,t)$$

$$+\left(\frac{u^2 m\pi}{L}\right)^2\left[\frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^2 Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - 4m^2} - \frac{Cos(2n+1)\pi L - 1}{[(2n+1)L]^2}\right)\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.$$

$$\left.\left.-\left(\frac{L^2}{4\pi^2}\right)\sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{[(2n+1)L]^2} - \frac{(-1)^{2m}}{[(2n+1)L]^2 - 4m^2}\right)\frac{Sin(2n+1)\pi ut}{(2n+1)} Sin(2n+1)\pi L - \frac{1}{4}\right]\overline{V}(m,t)\right\}$$

$$+\varepsilon_0 L\left\{\sum_{\substack{k=1\\k\neq m}}^{\infty}\left[\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\frac{(-1)^{k-m} Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (k+m)^2} - \frac{\left((-1)^{k-m} Cos(2n+1)\pi L - 1\right)}{[(2n+1)L]^2 - (k-m)^2}\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.\right.$$

$$\left.-\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1) Sin(2n+1)\pi L\left(\frac{(-1)^{k-m}}{[(2n+1)L]^2 - (k-m)^2} - \frac{(-1)^{k+m}}{[(2n+1)L]^2 - (k+m)^2}\right)\frac{\sin(2n+1)\pi ut}{2n+1}\right]\overline{V}_{tt}(k,t)$$

$$+\left[\frac{2uL}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{(k+m)(-1)^{k-m}\sin(2n+1)\pi L}{[(2n+1)L]^2 - (k+m)^2} - \frac{\left((k-m)(-1)^{k-m}\sin(2n+1)\pi L\right)}{[(2n+1)L]^2 - (k-m)^2}\right)\frac{\cos(2n+1)\pi ut}{2n+1} - \frac{u}{2}\left(\frac{km}{k^2+m^2}\right)\right.$$

$$\left.-\frac{2u}{\pi}\left(\frac{L}{2\pi}\right)\left(\frac{(k+m)(-1)^{k+m} Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (k+m)^2} - \frac{\left((k-m)(-1)^{k-m} Cos(2n+1)\pi L - 1\right)}{[(2n+1)L]^2 - (k-m)^2}\right)\frac{Sin(2n+1)\pi ut}{2n+1}\right]\overline{V}_t(k,t)$$

$$\frac{u^2k^2}{2\pi L}\left[\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^{k-m} Cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (k+m)^2} - \frac{\left((-1)^{k-m} Cos(2n+1)\pi L - 1\right)}{[(2n+1)L]^2 - (k-m)^2}\right)\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.$$

$$\left.-\sum_{n=0}^{\infty}(2n+1) Sin(2n+1)\pi L\left(\frac{(-1)^{k-m}}{[(2n+1)L]^2 - (k-m)^2} - \frac{(-1)^{k+m}}{[(2n+1)L]^2 - (k+m)^2}\right)\frac{\sin(2n+1)\pi ut}{2n+1}\right]\overline{V}(k,t)$$

$$= \frac{MgL}{\bar{m}\,\pi m}\left[-(-1)^m + \cos\frac{m\pi ut}{L}\right] \tag{3.16}$$

Where $$\varepsilon_o = \frac{M}{\bar{m} L} \qquad (3.17)$$

Evidently, a closed form solution to this equation is impossible. Consequently, in what follows two special cases of the equation shall be considered namely,**(i) moving force problem and (ii) moving mass problem.**

3.1 Case I: Moving Force Problem

If the inertia term is replaced with zero, the classical case termed moving force problem is obtained. Under this assumption $\varepsilon_0 = 0$ and equation (3.16) after some simplifications and arrangements yields.

$$\left[I + R^0\left(\frac{m\pi}{L}\right)^2\right]\bar{V}_{tt}(m,t) + \left[\frac{c}{\bar{m}} - \frac{c_s}{\bar{m}}\left(\frac{m\pi}{L}\right)^4\right]\bar{V}_t(m,t) + \left[\left(\frac{EI}{\bar{m}}\left(\left(\frac{m\pi}{L}\right)^4 + \frac{k}{\bar{m}}\right)\right) + \left(\frac{G-N}{\bar{m}L}\right)\left(\frac{m\pi}{L}\right)^2\right]\bar{V}(m,t)$$
$$= \frac{MgL}{\bar{m}\pi m}\left[-(-1)^m + \cos\frac{m\pi ut}{L}\right] \qquad (3.18)$$

The equation (3.18) is an approximate model, which assumes the inertia effects of the moving mass as negligible with further simplification and rearrangements of Equation (3.18), the equation becomes

$$\overline{V_{tt}}(m,t) + \Upsilon_1 \overline{V_t}(m,t) + \delta^2_{mm}\bar{V}(m,t) = a + y_m \cos\frac{m\pi ut}{L} \qquad (3.19)$$

where

$$\Upsilon_1 = \frac{\left[\frac{c}{\bar{m}} - \frac{c_s}{\bar{m}}\left(\frac{m\pi}{L}\right)^4\right]}{I + R^0\left(\frac{m\pi}{L}\right)^2} \qquad (3.20)$$

$$\delta^2_{mm} = \frac{\left[\left(\frac{EI}{\bar{m}}\left(\frac{m\pi}{L}\right)^4 + \frac{k}{\bar{m}}\right) + \frac{G-N}{\bar{m}L}\left(\frac{m\pi}{L}\right)^2\right]}{I + R^0\left(\frac{m\pi}{L}\right)^2} \qquad (3.21)$$

$$a = \frac{-MgL\ (-1)^m}{\bar{m}\pi m\left(I + R^0\left(\frac{m\pi}{L}\right)^2\right)} \qquad (3.22)$$

$$y_m = \frac{MgL}{\bar{m}\pi m\left(I + R^0\left(\frac{m\pi}{L}\right)^2\right)} \qquad (3.23)$$

It is noted that the equation (3.19) is amenable to method of Laplace transformation. Thus, subject it to Laplace transform in conjunction with the initial conditions (2.2) and inverting, to obtain

$$V(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\left\{\frac{1}{(\beta_{bf} - \beta_{af})}\left[\left(\frac{a}{\beta_{bf}}\left(e^{\beta_{bf}t} - 1\right) - \frac{a}{\beta_{af}}\left(e^{\beta_{af}t} - 1\right)\right) + y_m\frac{\beta_{bf}^2}{\beta_{bf}^2 + \phi^2}\left(-\frac{\operatorname{Cos}\phi t}{\beta_{bf}} + \frac{e^{\varphi_{bf}t}}{\beta_{bf}} + \frac{\phi}{\beta_{bf}}\operatorname{Sin}\phi t\right)\right.\right.$$
$$\left.\left. - y_m\frac{\beta_{af}^2}{\beta_{af}^2 + \phi^2}\left(-\frac{\operatorname{Cos}\phi t}{\beta_{af}} + \frac{e^{\varphi_{af}t}}{\beta_{af}} + \frac{\phi}{\beta_{af}}\operatorname{Sin}\phi t\right)\right]\right\}\left(\operatorname{Sin}\frac{m\pi}{L}x\right) \qquad (3.24)$$

Equation (3.24), is the transverse displacement response of the beam due to moving force when it is resting on a bi-parametric foundation.

3.2 CASE II: Moving Mass Problem

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus $\varepsilon_0 \neq 0$ and one is required to solve the entire equation. (3.16) This is termed the moving mass problem. To this end. Equation (3.16) is rearrangement to take the form.

$$\overline{V}_{tt}(m,t)+\delta_{mm}^2\overline{V}(m,t)+\frac{\varepsilon_0 L}{\left[1+R^0\left(\frac{m\pi}{L}\right)^2\right]}\left\{\left[\frac{1}{4}+\frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^2 Cos(2n+1)\pi L-1}{[(2n+1)L]^2-4m^2}-\frac{Cos(2n+1)\pi L-1}{[(2n+1)L]^2}\right)\right.\right.$$

$$\frac{Cos(2n+1)\pi ut}{(2n+1)}-\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{[(2n+1)L]^2}-\frac{(-1)^{2m}}{[(2n+1)L]^2-4m^2}\right)\frac{Sin(2n+1)\pi ut}{(2n+1)}\Bigg]\overline{V_{tt}}(m,t)$$

$$+\frac{2um}{2\pi}\left[\left(\sum_{n=0}^{\infty}\frac{2m(-1)^{2m}Sin(2n+1)\pi L}{[(2n+1)L]^2-4m^2}\ \frac{Cos(2n+1)\pi ut}{(2n+1)}-\sum_{n=0}^{\infty}\frac{2m(-1)^{2m}Cos(2n+1)\pi L-1}{[(2n+1)L]^2-4m^2}\right.\right.$$

$$\left.\left.\frac{Sin(2n+1)\pi ut}{(2n+1)}\right)+\left(\frac{c}{\overline{m}}-\frac{c_s}{\overline{m}}\left(\frac{m\pi}{L}\right)^4\right)\right]\overline{V}_t(m,t)$$

$$+\left(\frac{u^2 m\pi}{L}\right)^2\left[\frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^2 Cos(2n+1)\pi L-1}{[(2n+1)L]^2-4m^2}-\frac{Cos(2n+1)\pi L-1}{[(2n+1)L]^2}\right)\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.$$

$$\left.\left.-\left(\frac{L^2}{4\pi^2}\right)\sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{[(2n+1)L]^2}-\frac{(-1)^{2m}}{[(2n+1)L]^2-4m^2}\right)\frac{Sin(2n+1)\pi ut}{(2n+1)}Sin(2n+1)\pi L-\frac{1}{4}\right]\overline{V}(m,t)\right\}$$

$$+\varepsilon_0 L\left\{\sum_{\substack{k=1\\k\neq m}}^{\infty}\left[\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{[(2n+1)L]^2-(k+m)^2}-\frac{\left((-1)^{k-m}Cos(2n+1)\pi L-1\right)}{[(2n+1)L]^2-(k-m)^2}\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.\right.$$

$$\left.-\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)Sin(2n+1)\pi L\left(\frac{(-1)^{k-m}}{[(2n+1)L]^2-(k-m)^2}-\frac{(-1)^{k+m}}{[(2n+1)L]^2-(k+m)^2}\right)\frac{\sin(2n+1)\pi ut}{2n+1}\right]\overline{V}_{tt}(k,t)$$

$$+\left[\frac{2uL}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{(k+m)(-1)^{k-m}\sin(2n+1)\pi L}{[(2n+1)L]^2-(k+m)^2}-\frac{\left((k-m)(-1)^{k-m}\sin(2n+1)\pi L\right)}{[(2n+1)L]^2-(k-m)^2}\right)\frac{\cos(2n+1)\pi ut}{2n+1}-\frac{u}{2}\left(\frac{km}{k^2+m^2}\right)\right.$$

$$\left.-\frac{2u}{\pi}\left(\frac{L}{2\pi}\right)\left(\frac{(k+m)(-1)^{k+m}Cos(2n+1)\pi L-1}{[(2n+1)L]^2-(k+m)^2}-\frac{\left((k-m)(-1)^{k-m}Cos(2n+1)\pi L-1\right)}{[(2n+1)L]^2-(k-m)^2}\right)\frac{Sin(2n+1)\pi ut}{2n+1}\right]\overline{V}_t(k,t)$$

$$\frac{u^2k^2}{2\pi L}\left[\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{[(2n+1)L]^2-(k+m)^2}-\frac{\left((-1)^{k-m}Cos(2n+1)\pi L-1\right)}{[(2n+1)L]^2-(k-m)^2}\right)\frac{Cos(2n+1)\pi ut}{(2n+1)}\right.$$

$$\left.-\sum_{n=0}^{\infty}(2n+1)Sin(2n+1)\pi L\left(\frac{(-1)^{k-m}}{[(2n+1)L]^2-(k-m)^2}-\frac{(-1)^{k+m}}{[(2n+1)L]^2-(k+m)^2}\right)\frac{\sin(2n+1)\pi ut}{2n+1}\right]\overline{V}(k,t)$$

$$= \frac{MgL}{\bar{m}\pi m\left[I + R^0\left(\frac{m\pi}{L}\right)^2\right]}\left[-(-1)^m + \cos\frac{m\pi ut}{L}\right] \quad (3.24)$$

$$\delta_{mm}^2 = \frac{\left[\left(\frac{EI}{\bar{m}}\left(\frac{m\pi}{L}\right)^4 + \frac{k}{\bar{m}}\right) + \frac{G-N}{\bar{m}L}\left(\frac{m\pi}{L}\right)^2\right]}{I + R^0\left(\frac{m\pi}{L}\right)^2}$$

Evidently, unlike in the case of moving force problem an exact analytical solution to equation (3.24) is not possible. Though the equation yields readily to numerical technique on analytical approximate method is desirable as solutions so obtain often shed light on vital information about the vibrating system. To this end, we are going to use a modification of the asymptotic method due to Struble. By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (3.24). Thus, we set the right-hand side of (3.24) to zero and consider a parameter $\varepsilon_0 < 1$for any arbitrary ratio

$$\varepsilon_0 = \frac{\varepsilon}{1+\varepsilon} \quad (3.25)$$

so that

$$\varepsilon = \varepsilon_0 + O\left(\varepsilon_0{}^2\right) \quad (3.26$$

Setting $\varepsilon_0 = 0$ a situation corresponding to the case in which the moving mass is regarded as negligible is obtained, then the solution of (3.24) can be written as

$$\bar{V}(m,t) = C_{mf} \cos\left[\omega_{mf} t - \alpha_{mf}\right] \quad (3.29)$$

where C_{mf} , ω_{mft} and α_{mf} are constants.

Since $\varepsilon_0 < 1$Struble's technique required that asymptotic solution of the homogeneous part of equation(3.24) be of the form [14]

$$\bar{V}(m,t) = \beta(m,t) \cos\left[\omega_{mf} t - \Omega(m,t)\right] + \varepsilon V(m,t) + O\left(\varepsilon_0{}^2\right) \quad (3.30)$$

where $\beta(m,t)$ and $\Omega(m,t)$ are slowly varying functions of time.

Substituting equation (3.30) and its derivatives into the homogeneous part of equation (3.24) while taking into account (3.28) and retaining terms to O (ε^2), one obtains.

$$-2\dot{\beta}(m,t)\,\omega_{mf} \sin\left[\omega_{mf} t - \Omega(m,t)\right] + 2\beta(m,t)\,\dot{\Omega}(m,t)\omega_{mf} \cos\left[\omega_{mf} t - \Omega(m,t)\right]$$

$$-\beta(m,t)\,\omega^2_{mf} \cos\left[\omega_{mf} t - \Omega(m,t)\right] + \varepsilon\ddot{V}_1 + \varepsilon F_1 \dot{\beta}(m,t) \cos\left[\omega_{mf} t - \Omega(m,t)\right]$$

$$-\varepsilon F_1 \beta(m,t)\,\omega_{mf} \sin\left[\omega_{mf} t - \Omega(m,t)\right] - \varepsilon F_1 \beta(m,t)\,\dot{\Omega}(m,t) \sin\left[\omega_{mf} t - \Omega(m,t)\right]$$

$$+ \omega^2_{mf} \beta(m,t) \cos\left[\omega_{mf} t - \Omega(m,t)\right] + \varepsilon\,\omega^2_{mf} V_1(m,t)$$

$$+ \varepsilon L\, B_B(m,m)\omega^2_{mf} \beta(m,t) \cos\left[\omega_{mf} t - \Omega(m,t)\right]$$

$$-\varepsilon \sum_{\substack{K=1\\K=m}} \left\{ L\, \beta(k,t)\left[-2\dot{\beta}(k,t)\,\omega_{kf} \sin\left[\omega_{kf} t - \Omega(k,t)\right] - \beta(k,t)\,\omega^2_{kmf} \cos\left[\omega_{kf} t - \Omega(k,t)\right]\right.\right.$$

$$+2\ \beta(k,t)\cos\left[\omega_{kf}t-\Omega(k,t)\right]+F_0\ \beta(k,t)\ \cos\ \left[\omega_{kf}t-\Omega(k,t)\right]+V_1(k,t)\left]\ \right\}=0 \qquad (3.31)$$

Retaining terms to O (ε) only, thevariational equations are obtained by equating the coefficients of $\sin\left[\omega_{mf}t-\ \Omega(m,t)\right]$ and $\cos\ \left[\omega_{mf}t-\Omega(m,t)\right]$ terms on both side of equation (3.31) to zero. Thus, noting the following trigonometric identities

$$\frac{\cos(2n+1)\pi ut}{2n+1}\cos(\omega_{bj}t-\phi(m,t))=\frac{1}{2}\left[\cos\left[(\omega_{bj}t-\phi(m,t))+\frac{(2n+1)\pi ut}{2n+1}\right]+\cos\left[(\omega_{bj}t-\phi(m,t))-\frac{(2n+1)\pi ut}{2n+1}\right]\right]$$

$$\frac{\sin(2n+1)\pi ut}{2n+1}\cos(\omega_{bj}t-\phi(m,t))=\frac{1}{2}\left[\sin\left[(\omega_{bj}t-\phi(m,t))+\frac{(2n+1)\pi ut}{2n+1}\right]+\sin\left[(\omega_{bj}t-\phi(m,t))-\frac{(2n+1)\pi ut}{2n+1}\right]\right]$$

$$\frac{\cos(2n+1)\pi ut}{2n+1}\sin(\omega_{bj}t-\phi(m,t))=\frac{1}{2}\left[\sin\left[(\omega_{bj}t-\phi(m,t))+\frac{(2n+1)\pi ut}{2n+1}\right]-\sin\left[(\omega_{bj}t-\phi(m,t))-\frac{(2n+1)\pi ut}{2n+1}\right]\right]$$

$$\frac{\sin(2n+1)\pi ut}{2n+1}\sin(\omega_{bj}t-\phi(m,t))=\frac{1}{2}\left[\cos\left[(\omega_{bj}t-\phi(m,t))-\frac{(2n+1)\pi ut}{2n+1}\right]\right.$$
$$\left.-\cos\left[(\omega_{bj}t-\phi(m,t))+\frac{(2n+1)\pi ut}{2n+1}\right]\right]$$

and neglecting terms which do not contribute to the variatinal equation, equation (3.31) yields

$$-2\dot{\beta}(m,t)\omega_{mf}-\varepsilon F_1\beta(m,t)\omega_{mf}+\varepsilon F_1\beta(m,t)\dot{\Omega}(m,t)=0 \qquad (3.32)$$

And $$+2\beta(m,t)\dot{\Omega}(m,t)\omega_{mf}+\varepsilon F_1\dot{\beta}(m,t)+\varepsilon\omega_{mf}^2\ LB_B(m,m)\beta(m,t)=0 \qquad (3.33)$$

Solving equations (3.32) and (3.33) simultaneously gives

$$\beta(m,t)=Ae^{-\tau_1 t} \quad \text{and} \quad \Omega(m,t)=\left(\frac{2\varepsilon\omega_{mf}^3 LB_B(m,m)-\varepsilon^2F_1^2\omega_{mf}}{4+\varepsilon^2F_1}\right)t+\alpha_{mf} \qquad (3.34)$$

Where A and α_{mf} are constants.

Therefore, when the effect of the moving mass is considered, the first approximation to the homogeneous system is.

$$\overline{V}(m,t)=\ \Omega(m,t)\cos\left[\alpha_{mm}t-\alpha_{mf}\right] \qquad (3.35)$$

where $$\alpha_{mm}=\delta_{mm}\left[1-\frac{\delta_{mm}^2\vartheta_A L-\vartheta_A L\Upsilon_1-\varepsilon_0u^2m^2\pi^2}{2\vartheta_A L\ \ \delta_{mm}}\hbar_A(m,n,t)\right] \qquad (3.36)$$

and $$\vartheta_A=1+R^0\left(\frac{m\pi}{L}\right)\text{and}\ \Upsilon_1=\frac{\left[\frac{c}{m}-\frac{c_s}{m}\left(\frac{m\pi}{L}\right)^4\right]}{I+R^0\left(\frac{m\pi}{L}\right)^2} \qquad (3.37)$$

Expression (3.36) represents the modified natural frequency of the free system due to the presence of the moving mass. Thus, to solve the non-homogeneous equation (3.24), the differential operator which act on $\overline{V}(m,t)$ and $\overline{V}(k,t)$ are replaced by equivalent free system operator defined by the modified frequency given by

$$\overline{V}_{tt}(m,t)\ +\ \alpha^2{}_{mm}\ \overline{V}(m,t)=b+z_m\cos\frac{m\pi ut}{L} \qquad (3.38)$$

Where $$\vartheta_A = 1 + R^0 \left(\frac{m\pi}{L} \right)^2, \qquad b_m = \frac{MgL}{\overline{m}\pi \; m\vartheta_A} \frac{(-1)^{m+1}}{1 + \frac{\mu^0 \; L\hbar_A(m,n,t)}{\vartheta_A}},$$

$$z_m = \frac{MgL}{\overline{m}\pi m\vartheta_A} \frac{1}{1 + \frac{\mu^0 \; L\hbar_A(m,n,t)}{\vartheta_A}}$$

Equation (3.37) when solved using the Laplace transformation techniques and convolution theory in conjunction with the initial conditions gives $\overline{V}(m,t) = \left[\frac{b_m}{\alpha_{mm}^2}(1 - Cos\alpha_{mm}t) + \frac{z_m}{\left(\frac{m\pi u}{L}\right)^2 - \alpha_{mm}^2} \left(Cos\alpha_{mm}t - Cos\left(\frac{m\pi u}{L}\right)\right)\right]$

(3.39)

which on inversion yields

$$V(x,t) = \frac{2}{L} \sum_{m=1}^{\infty} \frac{1}{\left(\frac{m\pi u}{L}\right)^2 - \alpha_{mm}^2} \left[\frac{b_m}{\alpha_{mm}^2} \left(\left(\frac{m\pi u}{L}\right)^2 - \alpha_{mm}^2 \right)(1 - \text{Cos}\,\alpha_{mm}t) + z_m \left(\text{Cos}\,\alpha_{mm}t - \text{Cos}\left(\frac{m\pi u}{L}\right)\right) t \right] \sin\frac{m\pi x}{L} \qquad (3.39)$$

Equation (3.39) represents the transverse displacement response to masses moving at constant velocity of a simply supported uniform damped beam on bi-parametric elastic foundation.

4.0 Analysis of Closed Form Solutions

The response amplitude of a dynamical system such as this may grow without bound.Conditions under which this happens are termed resonance condition. Equation (3.24) clearly shows that the simply supported elastic bean resting on bi-parametric elastic foundation and traversed by moving force experiences resonance effect whenever.

$$\beta_{af} = \beta_{bf} \qquad (4.1)$$

while equation (3.28) indicates that the same beam under the action of moving mass will experience resonance effect whenever

$$\alpha_{mm} = f_m \qquad (4.2)$$

where $$\alpha_{mm} = \delta_{mm} \left[1 - \frac{\delta_{mm}^2 \vartheta_A L - \vartheta_A L \Upsilon_1 - \varepsilon_0 u^2 m^2 \pi^2}{2\vartheta_A L \;\; \delta_{mm}} \hbar_A(m,n,t) \right] \qquad (4.3)$$

and $$\delta_{mm} \left[\frac{2L\delta_{mm}\left[1 + R^0 \left(\frac{m\pi}{L}\right)^2\right] - \frac{\varepsilon_0}{2}\left(\frac{\delta_{mm}^2 - \Upsilon_1}{\delta_{mm}\varepsilon_0} + u^2 m^2 \pi^2 \hbar_A(m,n,t) \right)}{2L\delta_{mm}\left[1 + R^0\left(\frac{m\pi}{L}\right)^2\right]} \right] = f_m \qquad (4.3)$$

From equations above, it is evident that for the same national frequency the critical velocity for the system of a uniform simply supported damped bean resting on an elastic foundation and traversed by partially distributed forces moving at constant velocity is greater than that traversed by moving distributed masses problem. Thus for the same natural frequency of a uniform bean, resonance is reached earlier in the moving distributed mass problem than in the moving force problem.

5.0 Numerical Result and Discussion

In order to illustrate the theory in this work numerically, it is assumed that the damped uniform beam is of length 12.192m, modulus of elasticity $E = 3.1 \times 10^{10} N/m^2$ and the moment of inertial I is 2.87698x$10^{-3}$$m^4$. Furthermore, the mass per unit length of the beam $\overline{m} = 2758.291 kg/m$, mass ratio $\varepsilon = 0.25$ and the load velocity u= 8.128m/s. Values of C and C_s ranging between 0 and 10 were used, values of axial force N between 0N and 2000000N were used, value of shear modulus G was varied between 0 N/m^2 and 4000000N/m^2 , value of rotatory inertia correction factor R^0 was varied between 0 and 50 while the value of K was varied between 0N/m^3 and 40000000 N/m^3.The transverse deflections of the damped beam were calculated and plotted against time for various values of damping due to resistance to strain velocity C, damping due to resistance to transverse displacement C_s, foundation modulus K, shear modulus G and axial force N.The results are presented in plotted curves.

Figure 3.1 displays the transverse displacement response of a simply supported damped beam under the action of distributed forces moving at constant velocity for various values of damping due to resistance to transverse displacement C_s for fixed values of damping due to resistance to strain velocity C, foundation modulus, shear modulus G, rotatory inertia correction factor R^0 and axial force N, The figure shows that as C_s increases the amplitude of the damped beam decreases. The same results and analyses are obtained for moving distributed mass as shown in figure 3.7.

Similarly, from figure 3.2, for various values of damping due to resistance to strain velocity C, and fixed values of damping due to resistance to transverse displacement C_s, foundation modulus K, shear modulus G, rotatory inertia factor R^0 and axial force N. It is observed that higher values of damping coefficient C reduces the deflection profile of the damped beam. Figure 3.9 depicts the comparison of the effects of damping due to resistance to strain velocity C and damping due to transverse displacement Cs of a simply supported damped beam under the action of distributed force for fixed values of Axial force N, shear modulus G and foundation modulli K. Clearly, the response amplitudes when both Cs and C are not considered is higher than the response amplitudes when C and Cs are considered separately.

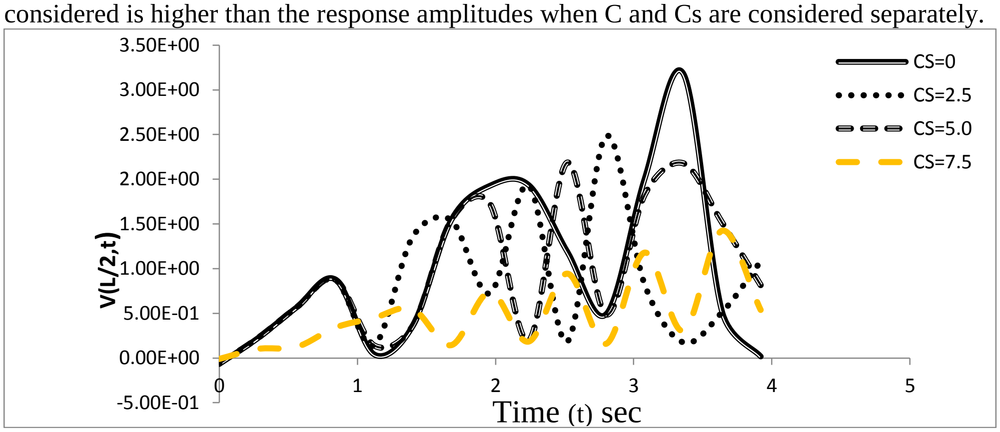

Fig. 3.1: Response amplitude of a damped uniform simply supported beam on bi-parametric elastic foundation and traversed by moving distributed force for K=40000, N=20000,G=10000, R0=10, C=5 and various values of C_s.

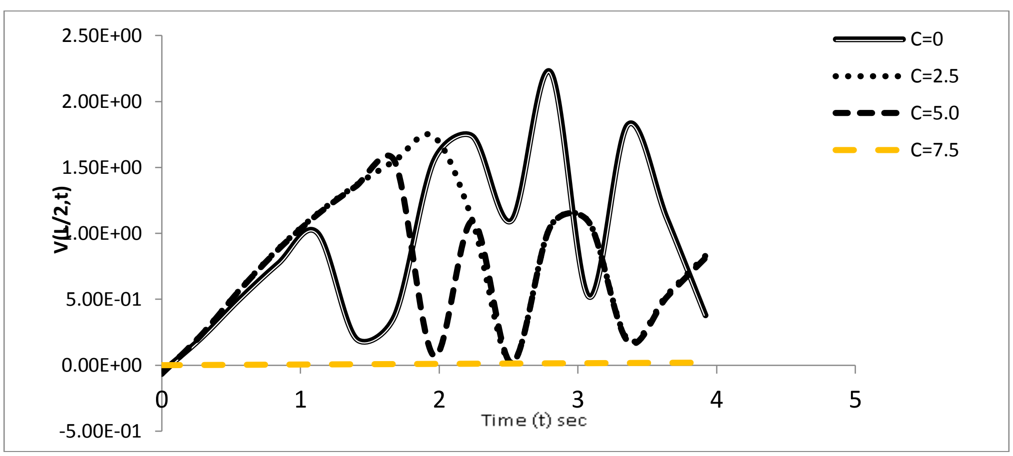

Fig. 3.2: Response amplitude of uniform simply damped supported damped beam on bi-parametric elastic foundation and traversed by moving distributed force for K=40000, N=20000,G=10000, R0=10, C_s =5 and various values of C.

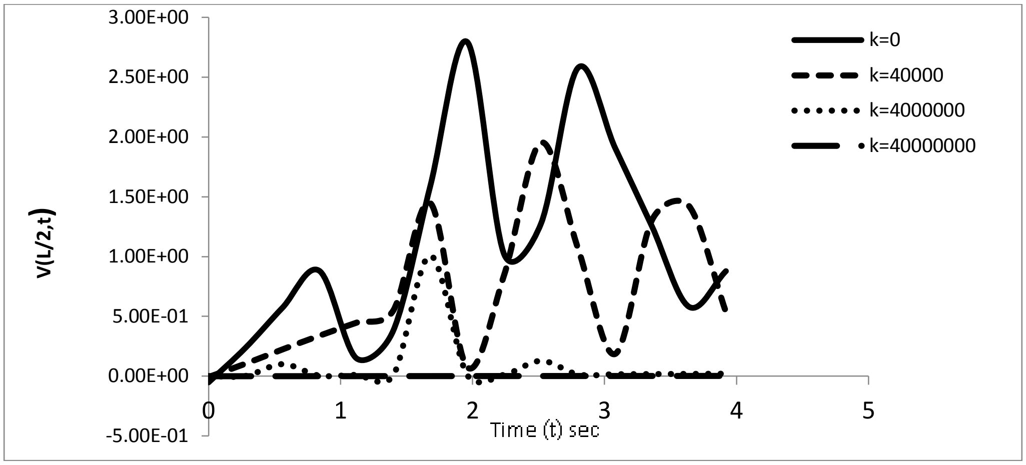

Fig 3.3: Deflection profile of a damped uniform simply supported beam on bi- parametric elastic foundation and traversed by moving distributed force for G=10000, R0=10, N=20000,K=40000, C_s =5, C=5 and various values of Foundation Moduli K.

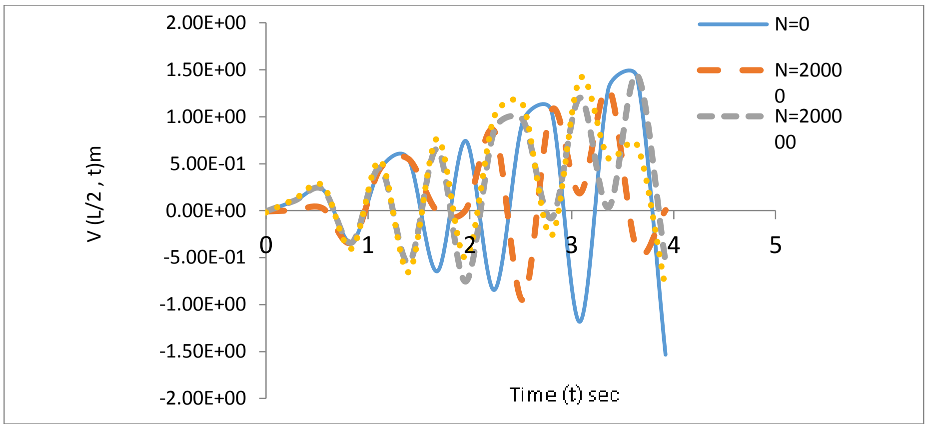

Fig. 3.4: Deflection profile of uniform simply supported damped beam on bi- parametric elastic foundation and traversed by moving distributed forces for K=40000, G=10000, R0=10, C_s =5, C=5 and various values of axial force N.

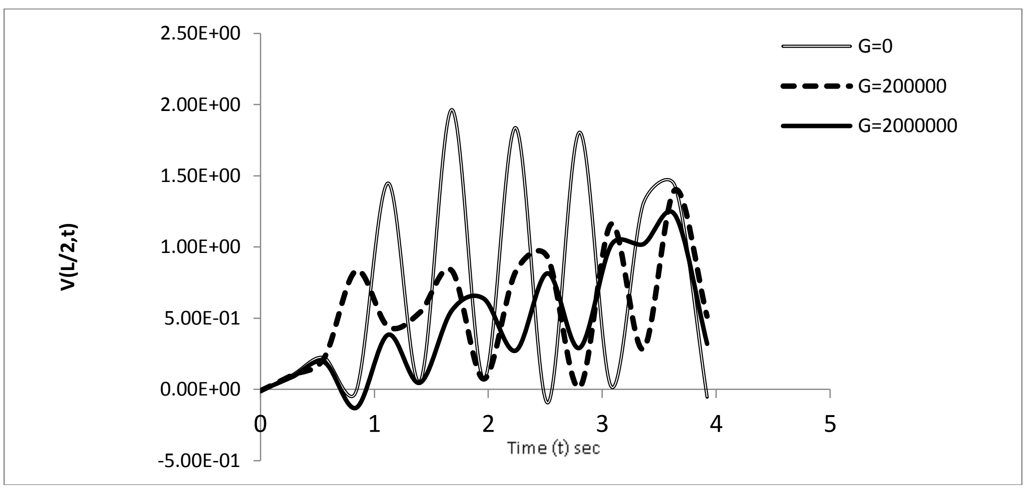

Fig. 3.5: Transverse displacement response of uniform simply supported damped beam on bi-parametric elastic foundation and traversed by moving distributed forces for K=40000, N=20000,R0=10, C_s =5, C=5 and various values of G.

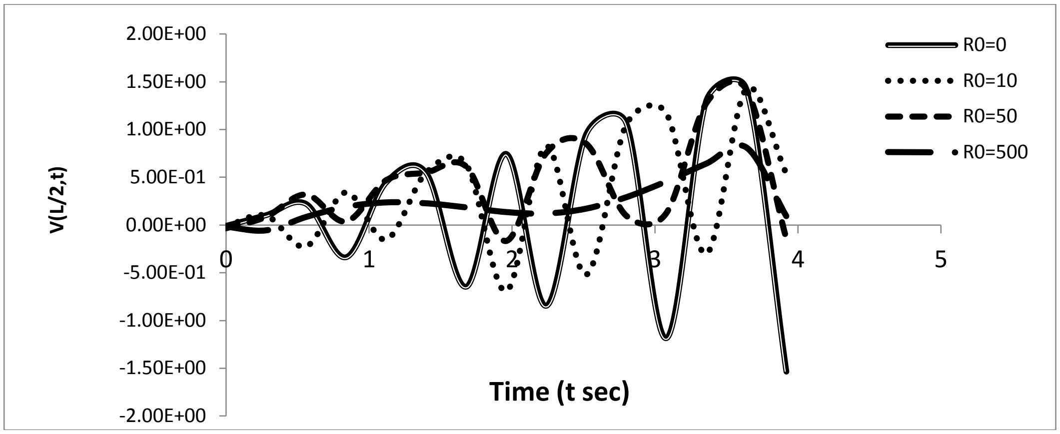

Fig. 3.6: Deflection profile of a uniform simply supported damped beam on bi- parametric elastic foundation and traversed by moving distributed forces for K_0=40000, G=10000, N=20000, C_s =5, C=5 and various values of R0.

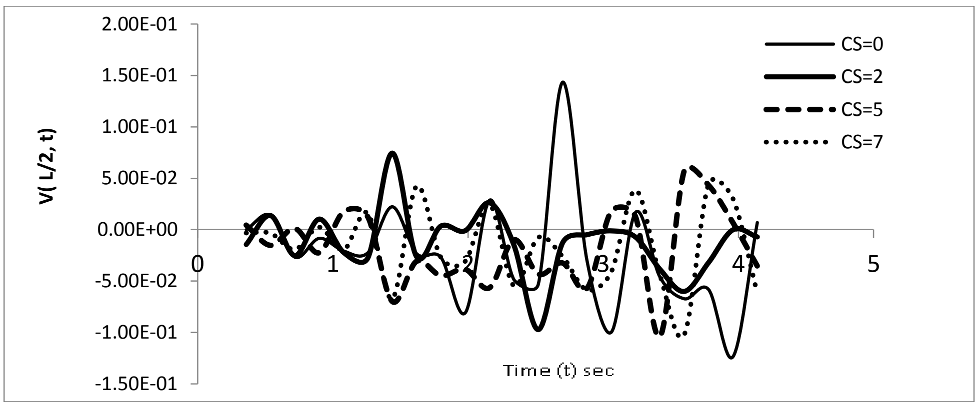

Fig 3.7. : Response amplitude of uniform simply supported damped beam on bi-parametric elastic foundation and traversed by moving distributed mass for K=40000, N=20000,G=10000, R0=10,C =5 and various values of C_s.

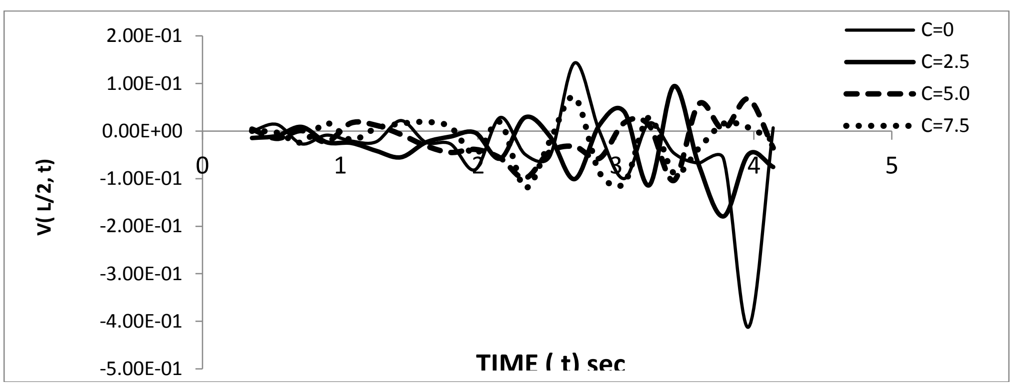

Fig 3.8 : Response amplitude of uniform simply supported damped beam on bi-parametric elastic foundation and traversed by moving distributed Masses for K=40000, N=20000,G=10000, R0=10, C_s =5 and various values of C.

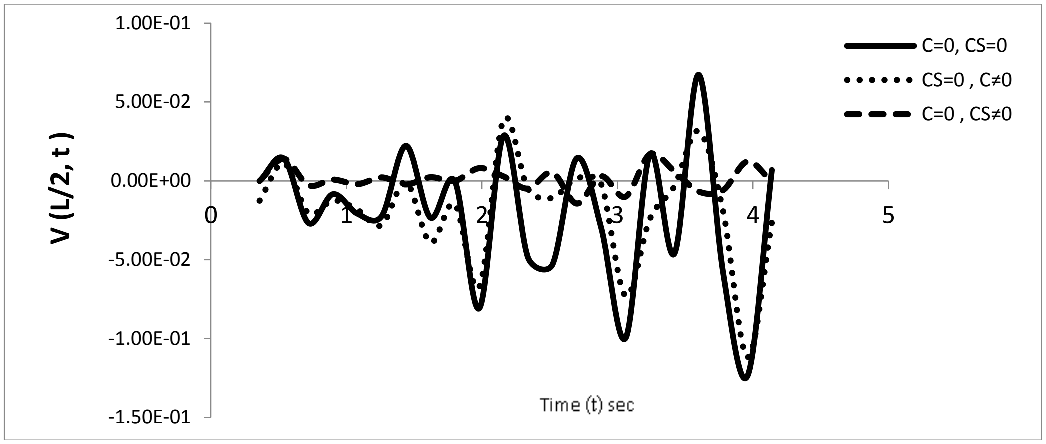

Fig. 3.9. Comparison of the displacement response of a damped simply supported beam under the action of moving distributed masses for cases (i) when both C and CS are set to be zero (ii) when C=0 but Cs ≠ 0 (iii) when Cs=0 but C ≠ 0 and for fixed value of K_0=40000, G=10000, and N=20000.

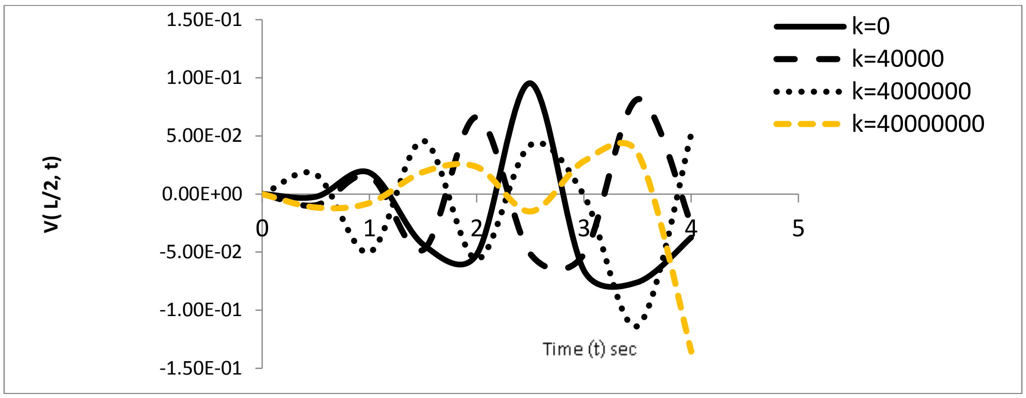

Fig 3.10: Deflection profile of a damped uniform simply supported beam on bi- parametric elastic foundation and traversed by moving distributed masses for G=10000, R0=10, N=20000,K=40000, C_s =5, C=5 and various values of Foundation Moduli K.

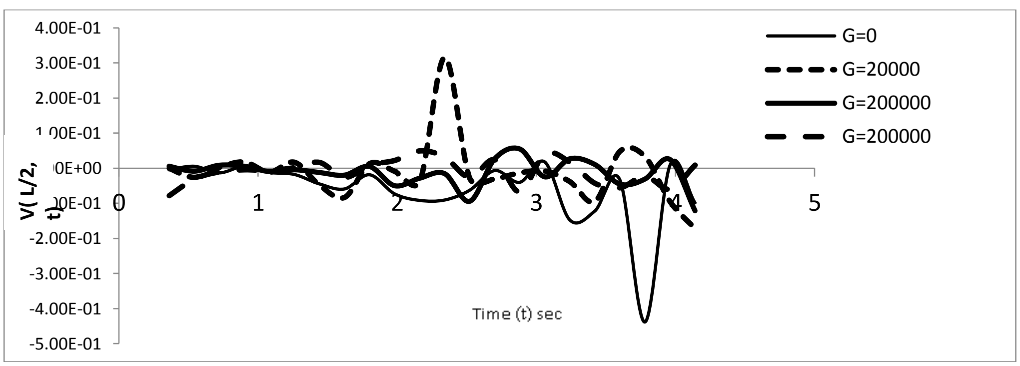

Fig 3.11: Response amplitude of a uniform simply supported damped beam on bi- parametric elastic foundation and traversed by moving distributed masses for G=10000, R0=10, N=20000, K= 40000, C_s =5, C=5 and various values of G.

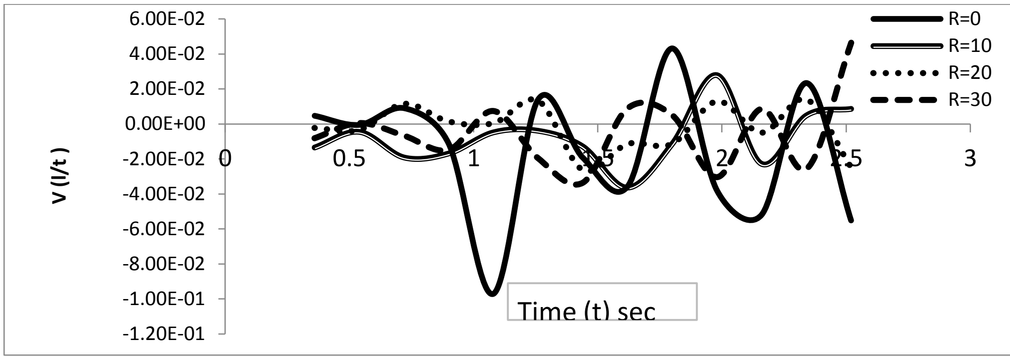

Fig 3.12: Transverse displacement response amplitude of a uniform simply supported damped beam on bi- parametric elastic foundation and traversed by moving distributed masses for G=10000, R0=10, N=20000, K=40000, C_s =5, C=5 and various values of R^0.

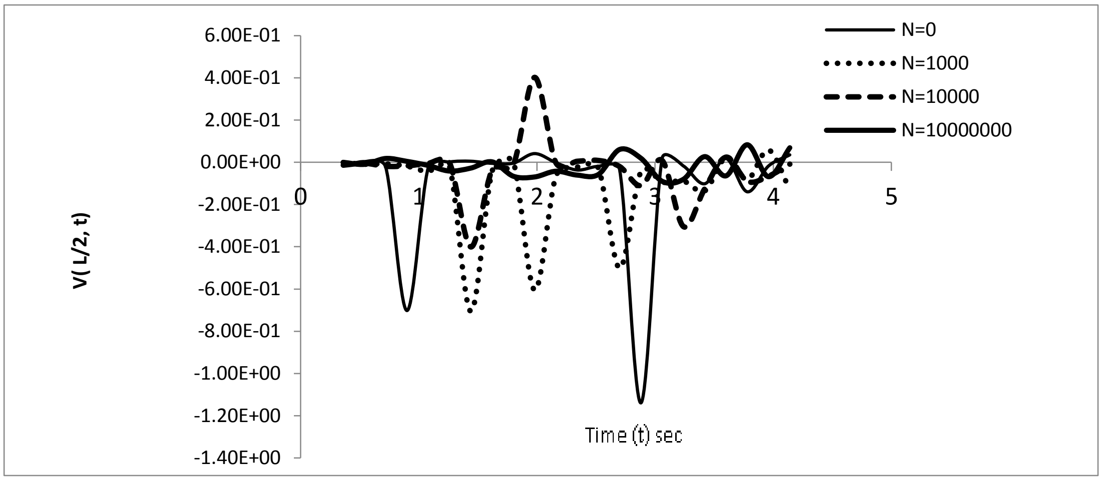

Fig 3.13: Transverse amplitude of a uniform simply supported damped beam under the actions of distributed masses travelling at constant velocity for various values of N and for fixed values of G=10000, R0=10, K=40000 C_s =5, and C=5.

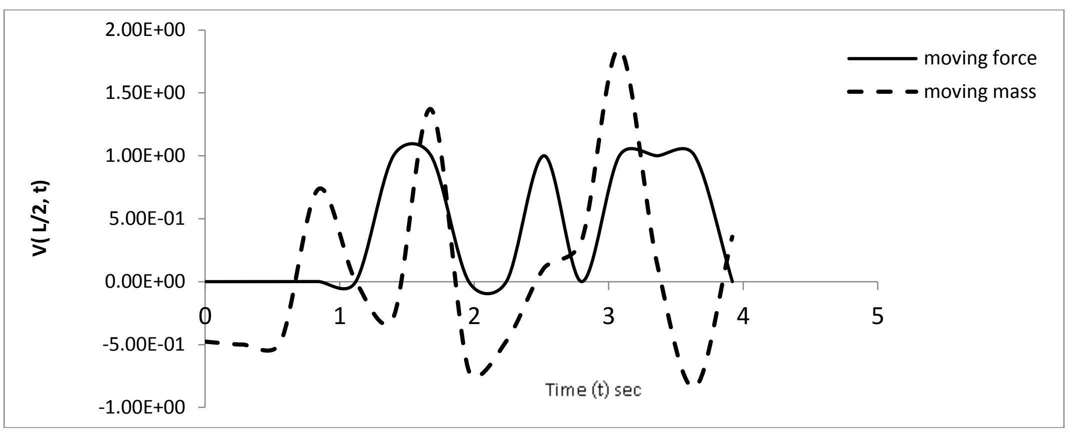

Fig. 3.14: Comparison of deflection of moving force and moving mass cases of a damped uniform simply supported beam on bi- parametric elastic foundation for K=40000, N=20000,G=10000, R0=10, C=5 C_s=5.

Figures 3.6 and 3.12 display the effect of rotatory inertia correction factor R^0 on the transverse deflection of simply supported uniform damped beam respectively in both cases of moving distributed force and moving distributed mass. The graph shows that the response amplitude decreases as the value of the rotatory inertia factor increases for fixed values of the foundation modullis K, shear modulus G, axial force N, damping due to resistance to strain velocity C, damping due to resistance to transverse displacement C_s and for various values of rotatory inertia factor R^0.

The effect of axial force on the transverse deflection in both cases of moving force and moving mass displayed in figures 3.4 and 3.13 respectively show that an increase in the value of axial force N decrease the deflection of the simply supported damped beam for fixed values of foundation modulli K, shear modulus G, rotatory inertia factor R^0,dampingdue to resistance to strain velocity C, and damping due to resistance to transverse displacement C_s. The effect of shear modulus G on the transverse deflection in both cases of moving distributed force and moving distributed mass displayed in figure 3.5 and 3.11 respectively show that an increase in the value of shear modulus G, decreases the deflection of the simply supported damped beam for fixed values of foundation modulliK, axial force N, rotatory inertia factor R^0, damping due to resistance to strain velocity C, and damping due to resistance to transverse displacement C_s.
The effect of foundation stiffness K on the transverse displacement response of a simply supported uniform damped beam under the action of partially distributed forces moving at constant velocity for various values of foundation stiffness K and for fixed values of axial force N, shear modulus G and rotatory inertia correction factor R^0 is display in figure 3.3. It is clearly show that the response amplitude of the damped beam decreases as the values of foundation stiffness K increases. Similar results and analyses are obtained when the simply supported damped beam is subjected to a partially distributed mass travelling at constant velocity as shown in figure 3.10.
Furthermore, figure 3.14 shows the comparison of the transverse displacement response of moving distributed force and moving distributed mass cases for simply supported uniform damped beam traversed by a moving load travelling at constant velocity for fixed value of damping due to resistance to strain velocity C, and damping due to resistance to transverse displacement C_s, foundation stiffness K, axial force N and shear modulus G and rotatory inertia correction factor R^0. It is clearly shown that the amplitude of moving force solution is higher than that of the moving mass solution.

6.0 Conclusion

The problem of the dynamic analysis of elastic structures resting on bi-parametric foundation and under distributed masses moving at constant velocity is considered in this paper. The governing equations are non-homogeneous fifth order partial differential equations with variable and singular coefficients. A Closed form solutions to the damped dynamical problems are obtained and numerical computations and analyses using various values of structural parameters in the dynamical system are also presented in plotted curve. The analysis exhibits the following interesting results.

(i) As the damping due to strain velocity C increases, the response amplitudes of the damped uniform beam decrease for the cases of moving distributed force and moving distributed mass problems for fixed values of other parameters.

(ii) As the damping due to transverse displacement C_S increases, the response amplitudes of the damped uniform beam decreases for the cases of moving distributed force and moving distributed mass problems and for fixed values of other parameters. However, damping due to resistance to transverse displacement C_S has a more pronounced effect in reducing the response amplitudes of the damped beam than the damping due to strain velocity C.

(iii) Generally as foundation modullik, axial force N, rotating inertial factor R, and shear modulus are increased, the response amplitudes of the vibrating system decrease. Also in the illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence resonance is reached earlier in the moving mass problem. Thus, accurate evolution of the moving mass problem is desirable as approximation by the moving force solution is highly misleading.

(iv) In summary, analytical solutions have been provided for this class of dynamical problems of damped beam-type structure under the action of moving distributed loads resting on bi-parametric for simply supported boundary conditions.

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