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CRACK PROPAGATION AND ARREST IN AN ORTHOTROPIC MATERIAL USING AN ELLIPTIC STOP HOLE

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ABSTRACT

In this paper, we investigated the influence of elliptical stop hole on mode-III deformation behavior of a semi-infinite crack in a homogeneous, elastic orthotropic material subjected to antiplane loading. To make the problem perceptive to analysis within the realm of two-dimensional classical theory of elasticity, we made use of Galilean transformation to convert the governing wave equation to Laplace equation which is time independent. By employing conformal mapping and integral transformation techniques, a solution of the displacement is obtained leading to closed form expression for mode III stress intensity factor. The results obtained shows that the elliptical stop hole is a veritable means of arresting crack propagation.

1. Introduction

Broberg [1] defines crack in a structure as a material separation comprising of disjoint upper and lower faces with the separation distance substantially smaller than the separation extant. This separation if it occurs in a structure, and no action is taken to arrest its growth, the crack may propagate until the structure breaks. This breakage may entail loss of life depending of the usage. To avert this occurrence, many researchers have proffered several methods of arresting crack propagation [2-9]. One of the efficient methods is to install a circular crack breaker(stop-hole) at the end of crack to prevent or delay its growth [10-15]. The effectiveness of the stop hole method depends on a proper understanding of the influence of the hole size and shape.

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A survey of most work in literature in this regard shows that the numerical techniques were extensive employed while the theoretical methods were few. The theoretical analysis of a crack problem plays a significant role in the understanding of the mechanism of failure of structural components as well as serve as a benchmark for the purpose of judging the accuracy of various numerical techniques. To contribute to the knowledge of theoretical methods of solutions of crack arrest,

we investigated the influence of the elliptic crack breaker on mode III deformation behavior of a semi-infinite crack in a homogeneous, elastic orthotropic material subjected to longitudinal shear loads starting with the formulation the governing boundary value problem for anti-plane deformation and constructing two analytic functions that transforms the initial configuration of the problem to one analyzable by method of integral transform, which transform the governing equations into algebraic equations using complex variable and Mellin-integral transform techniques .Closed form solutions for the displacement and stress fields are obtained leading to mode III stress intensity factor (SIF) at the boundary of the crack breaker.

2 Theoretical Analysis

a. Formulation of the problem

Consider an infinite elastic orthotropic material containing a semi- infinite crack which occupies the region $-\infty < x' < 0$ in a moving coordinate system (x', y', z') . A pair of longitudinal shear loads of magnitude Q is applied along the crack surface on an interval $[-a, -d]$ of length L . An elliptical crack breaker (stop hole) with minor and major axes $m_1 = \sinh\alpha$ and $m_2 = \cosh\alpha$ respectively is introduced at the center of the orthotropic material which is at the origin of a fixed coordinate system (x, y, z) . Fig. 1 illustrates the configuration of the problem under consideration. Suppose that, at time $t = 0$, the crack tip starts to move with constant velocity v along the x'-direction and ends up at the crack breaker , attaining a displacement vt. Suppose , also, that the disturbance due to the load is anti-plane so that it creates only an out of plane displacement and stresses in the z -direction . The problem is to investigate the influence of the elliptical stop hole on the stress intensity factor at the crack tip

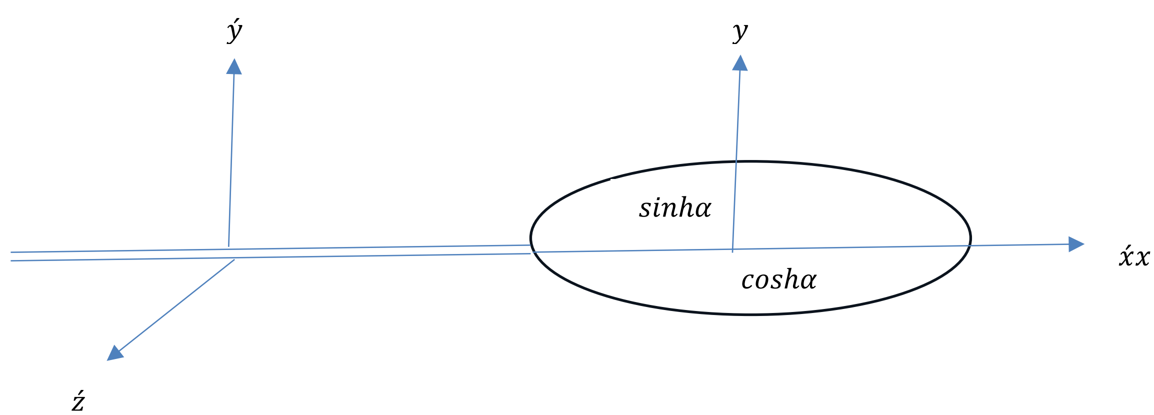

Fig 1. Geometry of the problem

Under the prevalence of anti-plane strain condition, i.e. the only nonzero component of the displacement vector is the one the z' direction, here denoted by w. The state depends only on the coordinates x' and y', so that derivatives with respect to z' vanish. Consequently, the only non-zero stress components $\sigma_{x'z}$ and $\sigma_{y'z}$ are given by

$$\sigma_{x'z} = c_{44}\frac{\partial w}{\partial x'}, \quad \sigma_{y'z} = c_{55}\frac{\partial w}{\partial y'} \tag{2.1}$$

where c_{44} and c_{55} are the shear moduli in the x' and y' directions.

Accordingly, the equation of motion takes the form

$$\frac{\partial \sigma_{x'z}}{\partial x'} + \frac{\partial \sigma_{y'z}}{\partial y'} = \rho\frac{\partial^2 w}{\partial t^2} \tag{2.2}$$

Where ρ is the mass density of the elastic material

Substituting the stress-displacement relations (2.1) into equation (2.2) and simplifying the result leads to the two-dimensional wave equation

$$\frac{\partial^2 w}{\partial x'^2} + \frac{1}{\eta^2}\frac{\partial^2 w}{\partial y'^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{2.3}$$

where $\eta = \left(\frac{C_{44}}{C_{55}}\right)^{\frac{1}{2}}$ and $c = \left(\frac{C_{44}}{\rho}\right)^{\frac{1}{2}}$ is the wave speed.

The corresponding boundary conditions on the crack surface under anti-plane strain loading Q are as follows:

$$C_{44}\frac{\partial W}{\partial x'}(x', 0) = \begin{cases} \pm Q & , \quad a \le x' \le d \\ 0 & , otherwise \end{cases} \tag{2.4}$$

$$\frac{\partial W}{\partial y'}(b, 0) = 0 \quad , \mathrm{b} > 0 \tag{2.5}$$

$$w(x, 0) = 0 \qquad , x > 0 \tag{2.6}$$

3. Methods

For a crack moving with constant velocity v in the x'-direction, it is convenient to introduce the Galilean transformation

$$x = x' - vt \quad , \quad y = \eta y' \quad , \quad t' = t \tag{3.1}$$

With this transformation, the wave equation becomes independent of time and eq. (2.3) reduces to Laplace's two-dimensional equation

$$\nabla^2 w(x, y) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{3.2}$$

In terms of polar coordinates (r, θ) which are related by

$x = r\cos\theta$, $y = r\sin\theta$, the non-zero stresses are

$$\sigma_{rz}(r,\theta) = C_{55}\frac{\partial W}{\partial r}(r,\theta) \ , \sigma_{\theta z}(r,\theta) = C_{44}\frac{1}{r}\frac{\partial W}{\partial \theta}(r,\theta) \tag{3.3}$$

and eqn (3.2) becomes

$$\frac{\partial^2 w(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial w(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}=0, \qquad r\ge b\ ,-\pi<\theta<\pi \tag{3.4}$$

subject to the boundary conditions

$$\frac{\partial w}{\partial \theta}(r,\pm\pi)=\begin{cases}\dfrac{\pm rQ}{c_{44}} & a\le r\le d\\[2ex] 0 & Otherwise\ (r<a, r>d\end{cases} \tag{3.5}$$

$$\frac{\partial w(b,\theta)}{\partial r}=0, \quad b=1(unit\ radius) \tag{3.6}$$

b. Transformation of problem

Because of the nature of the geometry of the problem, we shall transform the configuration twice using two holomorphic functions. Since

$$w=\frac{1}{2}\left(ze^{-\alpha}+z^{-1}e^{\alpha}\right) \quad \text{where } \alpha \text{ is real} \tag{3.7a}$$

maps a circle $|z|=1, \quad z=e^{i\theta}$ on to an ellipse with the lengths of the major and minor axes as $2\cosh\alpha$ and $2\sinh\alpha$ respectively , the inverse function $f(w)=\frac{1}{2}\left(we^{-\alpha}+w^{-1}e^{\alpha}\right)$ maps back the ellipse into a unit circle. We then use the function

$$\xi(z)=\frac{1}{2}\left(z+\frac{1}{z}\right)-1 \qquad ,z=re^{i\theta} \tag{3.8}$$

to map the circular hole onto a line with the edge terminating at the origin.

Resulting in

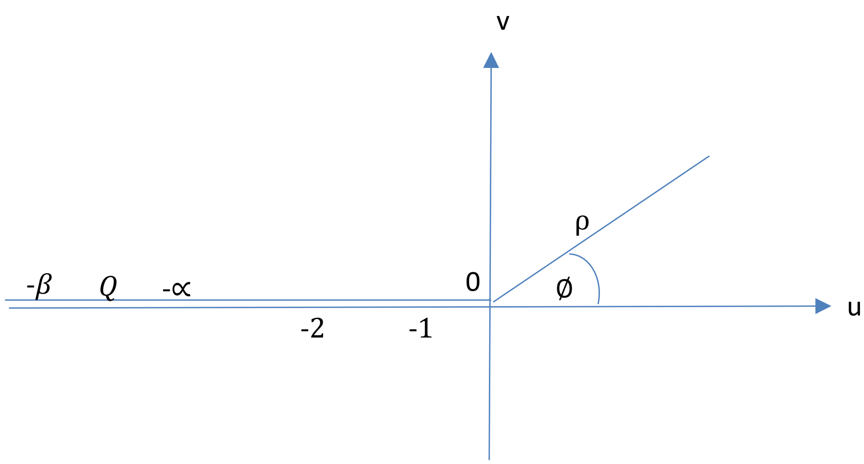

Fig 2: The transformed configuration of the original problem

Then, using the holomorphic function defined in equation (3.8),Eqns. (3.4),(3.5) and (3.6) transforms to

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)W(\rho,\phi)=0 \qquad \rho>0, \quad 0\le\phi\le\pi \tag{3.9}$$

subject to the boundary conditions

$$\frac{\partial W(\rho,\pi)}{\partial\phi}=\begin{cases}\frac{Q\rho}{c_{44}}\left[\frac{(\rho-1)}{\sqrt{\rho(\rho-2)}}+1\right] & ,\alpha<\rho<\beta \\ 0 & otherwise\end{cases} \tag{3.10}$$

$$\frac{\partial W}{\partial\phi}(\rho,\phi)=0 \qquad 0\le\phi\le\pi \tag{3.11}$$

$$W(\rho,0)=0 \qquad \rho>0 \tag{3.12}$$

c. Analytic solution of the transformed problem

Employing Mellin integral transform of $W(\rho,\phi)$ defined by $\tilde{W}(s,\phi)=\int_0^\infty W(\rho,\phi)\rho^{s-1}\partial\rho$,to eqns. (3.9) -(3.12), the differential equation derived is

$$\frac{d^2}{d\phi^2}\tilde{W}(s,\phi)+s^2\tilde{W}(s,\phi)=0, \qquad -\frac{1}{2}<\operatorname{Re}s<0 \tag{3.13}$$

The solution of eqn. (3.13), subject to the boundary conditions (3.10)- (3.12) is

$$\tilde{W}(s,\phi)=\frac{K}{c_{44}}\Im(\beta,\alpha;s)\frac{\sin\phi s}{s\cos\pi s} \tag{3.14}$$

where

$$\Im(\beta,\alpha;s)=\int_\sigma^\tau\left(\rho^s\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}-\rho^{s-1}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}+\rho^s\right)\partial\rho \tag{3.15}$$

Given the inverse Mellin transform of $\tilde{W}(s,\phi)$ as $W(\rho,\phi)=\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\tilde{W}(s,\phi)\rho^{-s}ds$

$$W(\rho,\phi)=\frac{K}{c_{44}}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\Im(\beta,\alpha;s)\frac{\sin\phi s}{s\cos\pi s}\rho^{-s}dsW(\rho,\phi)=\frac{Q}{c_{44}}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\Im(\beta,\alpha;s)\frac{\sin\phi s}{s\cos\pi s}\rho^{-s}ds\ ,$$

(3.16)

Then term by term evaluation of the integrals in (3.15) using the convergent series, Nnadi [16]

$$(1-t)^{-\frac{1}{2}}=\sum_{k=0}^\infty j_k t^k, \qquad |t|<1, \tag{3.17}$$

where the coefficients are defined by

$$j_k=\frac{(2k)!}{2^{2k}(k!)^2} \tag{3.18}$$

For the first term, we have

$$\int_\alpha^\beta\rho^s\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}\partial\rho=\sum_{k=0}^\infty j_k 2^k\int_\alpha^\beta\rho^{s-k}\partial\rho$$

$$=\sum_{k=0}^\infty j_k 2^k\left[\frac{\rho^{s-k+1}}{s-k+1}\right]_\alpha^\beta=\sum_{k=0}^\infty j_k 2^k\left[\frac{\beta^{s-k+1}-\alpha^{s-k+1}}{s-k+1}\right] \tag{3.19}$$

For the second term,

$$\int_{\alpha}^{\beta}\rho^{s}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}\partial\rho=\sum_{k=0}^{\infty}j_{k}2^{k}\int_{\alpha}^{\beta}\rho^{s-k}\partial\rho$$

$$=\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\rho^{s-k}}{s-k}\right]_{\alpha}^{\beta}=\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\beta^{s-k}-\alpha^{s-k}}{s-k}\right] \tag{3.20}$$

For the third term,

$$\int_{\alpha}^{\beta}\rho^{s}\partial\rho=\left[\frac{\rho^{s+1}}{s+1}\right]_{\alpha}^{\beta}=\frac{\beta^{s+1}}{s+1}-\frac{\alpha^{s+1}}{s+1} \tag{3.21}$$

Hence,

$$\Im(\beta,\alpha;s)=\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\beta^{s-k+1}-\alpha^{s-k+1}}{s-k+1}\right]-\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\beta^{s-k}-\alpha^{s-k}}{s-k}\right]+\left[\frac{\beta^{s+1}}{s+1}-\frac{\alpha^{s+1}}{s+1}-\right] \tag{3.22}$$

Inserting eqn. (3.22) into eqn. (3.16), we obtain

$$W(\rho,\phi)=\frac{Q}{c_{44}}\left(I_{\alpha\beta}^{(1)}-I_{\alpha\beta}^{(2)}+I_{\alpha\beta}^{(3)}\right) \tag{3.83}$$

Term by term evaluation of the three terms in eqn. (3.21) yields

$$\Im(\beta,\alpha;s)=\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\beta^{s-k+1}-\alpha^{s-k+1}}{s-k+1}\right]-\sum_{k=0}^{\infty}j_{k}2^{k}\left[\frac{\beta^{s-k}-\alpha^{s-k}}{s-k}\right]+\left[\frac{\beta^{s+1}}{s+1}-\frac{\alpha^{s+1}}{s+1}-\right] \tag{3.26}$$

Inserting eqn. (3.26) into eqn. (3.23), we obtain

$$W(\rho,\phi)=\frac{Q}{c_{44}}\left\{I_{\beta}^{(1)}-I_{\beta}^{(2)}+I_{\beta}^{(3)}\right\}-\frac{Q}{c_{44}}\left\{I_{\alpha}^{(1)}-I_{\alpha}^{(2)}+I_{\alpha}^{(3)}\right\} \tag{3.27}$$

where

$$I_{\beta}^{(1)}=\sum_{k=0}^{\infty}j_{k}2^{k}\beta^{1-k}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s-k+1)s\cos\pi s}\left(\frac{\rho}{\beta}\right)^{-s}ds \tag{3.28}$$

$$I_{\alpha}^{(1)}=\sum_{k=0}^{\infty}j_{k}2^{k}\alpha^{1-k}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s-k+1)s\cos\pi s}\left(\frac{\rho}{\alpha}\right)^{-s}ds \tag{3.29}$$

$$I_{\beta}^{(2)}=\sum_{k=0}^{\infty}j_{k}2^{k}\beta^{-k}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s-k)s\cos\pi s}\left(\frac{\rho}{\beta}\right)^{-s}ds \tag{3.30}$$

$$I_{\alpha}^{(2)}=\sum_{k=0}^{\infty}j_{k}2^{k}\alpha^{-k}\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s-k)s\cos\pi s}\left(\frac{\Upsilon}{\alpha}\right)^{-s}ds \tag{3.31}$$

$$I_{\beta}^{(3)}=\beta\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s+1)s\cos\pi s}\left(\frac{\Upsilon}{\beta}\right)^{-s}ds \tag{3.32}$$

$$I_{\alpha}^{(3)}=\alpha\frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty}\frac{\sin\phi s}{(s+1)s\cos\pi s}\left(\frac{\Upsilon}{\alpha}\right)^{-s}ds \tag{3.33}$$

Evaluating the integrals $I_{\beta}^{(j)}$ *and* $I_{\alpha}^{(j)}$, $j=1,2,3$ using residue theory and Jordan lemma, we have I_{β}^{1} , I_{β}^{2} , I_{β}^{3} , I_{α}^{1} , I_{α}^{2} , I_{α}^{3}

The displacement for $\rho<\beta$

$$\frac{Q}{c_{44}}\left(I_{\beta}^{(1)}-I_{\beta}^{(2)}+I_{\beta}^{(3)}\right)=\frac{Q}{c_{44}}\left\{-2\sin\phi\rho+\frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\right.$$

$$=\frac{Q}{c_{44}}\left\{-2\sin\phi\rho+\frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\right.$$

$$+\left(j_3 2^3-j_4 2^4\right)\frac{1}{\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\Upsilon}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}+\left(j_4 2^4-j_5 2^5\right)\frac{1}{\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\Upsilon}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}$$

$$\left.+\left(j_5 2^5-j_6 2^6\right)\frac{1}{\pi\beta^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\Upsilon}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)}+\left(j_6 2^6-j_7 2^7\right)\frac{1}{\pi\beta^6}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\Upsilon}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{11}{2}\right)\left(n-\frac{1}{2}\right)}+\ldots\right\} \quad (3.34)$$

For $\rho > \alpha$

$$\frac{Q}{c_{44}}\left(I_{\alpha}^{(1)}-I_{\alpha}^{(2)}+I_{\alpha}^{(3)}\right)=\frac{Q}{c_{44}}\left\{-2\sin\phi\rho+\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)\left(n-\frac{1}{2}\right)}\right.$$

$$+\left(j_1 2^1-j_2 2^2\right)\frac{1}{\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}+\left(j_2 2^2-j_3 2^3\right)\frac{1}{\pi\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}$$

$$\left.+\left(j_3 2^3-j_4 2^4\right)\frac{1}{\pi\alpha^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}+\left(j_4 2^4-j_5 2^5\right)\frac{1}{\pi\alpha^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\ldots\right\} \quad (3.35)$$

Experimental work

Since the loading split the upper half $\rho\phi$-plane into three regions denoted by R_1 , R_{11} and R_{111} defined as follows:

$R_1=\{(\rho,\phi)\,|\,0<\rho<\alpha\ ,\ 0<\phi<\pi\}$, $R_{11}=\{(\rho,\phi)\,|\,\alpha<\rho<\beta\ ,\ 0<\phi<\pi\}$ and

$R_{111} = \{(\rho,\phi) \mid \beta < \rho < \infty \ , 0 < \phi < \pi\}$

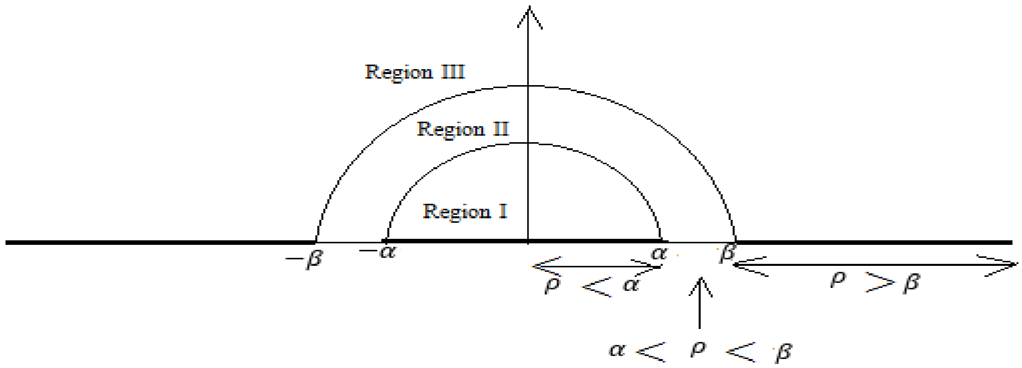

Fig.3. Solution regions induced by loading

Since stresses concentrates at the crack tip, we shall proof that our derived displacement

$$W(\rho,\phi) = \frac{bQ}{c_{44}}\left\{I_{\beta}^{(1)} - I_{\beta}^{(2)} + I_{\beta}^{(3)}\right\} - \frac{bQ}{c_{44}}\left\{I_{\alpha}^{(1)} - I_{\alpha}^{(2)} + I_{\alpha}^{(3)}\right\}, \qquad \rho < \beta, \rho > \alpha \tag{3.35}$$

satisfies both the governing Laplace equation

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} = 0 \tag{3.36}$$

and the boundary conditions

$$W(\rho,0) = 0, \qquad \alpha < \rho < \beta \tag{3.37}$$

$$\frac{\partial W}{\partial \phi}(\rho,\pi) = \frac{bQ\rho}{c_{44}}\left[\frac{(\rho-1)}{\sqrt{\rho(\rho-2)}} + 1\right], \quad \alpha < \rho < \beta, \quad \rho > 2 \tag{3.38}$$

Now using eqn.(3.35) for $\rho > \alpha$ (Region II (A))

$$W(\rho,\phi) = \frac{bQ}{c_{44}}\left\{\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} + \frac{1}{2}\sin\phi\rho^{-1} + \frac{1}{2\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}\right.$$

$$\left. + \frac{\sin 2\phi}{2}\rho^{-2} + \frac{1}{\pi\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)} + \frac{5}{8}\frac{\sin 3\phi}{3}\rho^{-3} + \frac{15}{8\pi\alpha^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\alpha}\right)^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)} + \ldots\right\} \tag{3.39}$$

Therefore

$$W(\rho,0) = 0 \tag{3.40}$$

Rewriting eqn. (3.39), we have

$$W(\rho,\phi)=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{1}{2}\sin\phi\rho^{-1}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{\sin 2\phi}{2}\rho^{-2}\right.$$
$$\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{15}{8}\frac{\sin 3\phi}{3}\rho^{-3}+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\ldots\right\} \quad (3.41)$$

Differentiating eqn.(3.41) ,we have

$$\frac{\partial W(\rho,\phi)}{\partial\phi}=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}+\frac{1}{2}\cos\phi\rho^{-1}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{3}{2}\right)}+\cos 2\phi\rho^{-2}+\ldots\right\} \quad (3.42)$$

Therefore

$$\frac{\partial W(\rho,\pi)}{\partial\phi}=\frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1}+\rho^{-2}-\frac{15}{8}\rho^{-3}+\ldots\right\} \quad (3.43)$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2}=\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-1}-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)}-2\sin 2\phi\rho^{-2}\right.$$
$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{45}{8}\sin 3\phi\rho^{-3}-\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n+\frac{1}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\} \quad (3.44)$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2}=\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-3}\right.$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}-\sin 2\phi\rho^{-4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{45}{8}\sin 3\phi\rho^{-5}$$
$$\left.-\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\} \quad (3.45)$$

$$\frac{\partial W(\rho,\phi)}{\partial\rho}=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-2}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(-n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\ldots\right\}$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial\rho^2}=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}+\sin\phi\rho^{-3}\right.$$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}+3\sin 2\phi\rho^{-4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}+\frac{15}{2}\sin 3\phi\rho^{-5}$$

$$\left.+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\} \qquad (3.46)$$

$$\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial\rho}=\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-3}-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}-\sin 2\phi\rho^{-4}\right.$$

$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{15}{8}\sin 3\phi\rho^{-5}+\frac{5}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\} \qquad (3.47)$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2}=\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-3}\right.$$

$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}-\sin 2\phi\rho^{-4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{45}{8}\sin 3\phi\rho^{-5}$$

$$\left.-\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\} \qquad (3.48)$$

Therefore

$$\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}=\frac{bQ}{c_{44}}\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}+\sin\phi\rho^{-3}\right.$$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}+3\sin 2\phi\rho^{-4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}$$

$$\left.+\frac{15}{2}\sin 3\phi\rho^{-5}+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n+\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\}$$

$$+\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-3}-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}-\sin 2\phi\rho^{-4}\right.$$

$$\left.-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{15}{8}\sin 3\phi\rho^{-5}+-\frac{5}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}+\ldots\right\}$$

$$+\frac{bQ}{c_{44}}\left\{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\alpha^{n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{1}{2}\sin\phi\rho^{-3}-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{3}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{3}{2}\right)}-\sin 2\phi\rho^{-4}\right.$$

$$\left.-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{5}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{5}{2}\right)}-\frac{45}{8}\sin 3\phi\rho^{-5}-\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\alpha}\right)^{-n+\frac{7}{2}}\rho^{-n-\frac{3}{2}}}{\left(n-\frac{7}{2}\right)}=0\right\} \quad (3.49)$$

Now for $\rho < \beta$ (Region II (B))

$$W(\rho,\phi)=\frac{bQ}{c_{44}}\left\{-2\sin\phi\rho+\frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}-\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right.$$

$$-\frac{1}{\pi\tau^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{15}{8\pi\tau^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)\left(n-\frac{1}{2}\right)}$$

$$-\frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)\left(n-\frac{1}{2}\right)}+\frac{63}{8\pi\beta^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)\left(n-\frac{1}{2}\right)}+\ldots\Bigg\} \tag{3.50}$$

$$\frac{\partial W(\rho,\phi)}{\partial\phi}=\frac{bQ}{c_{44}}\Bigg\{-2\cos\phi\rho+\frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)}-\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}$$

$$-\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)}+\frac{15}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$-\frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)}+\frac{63}{8\pi\beta^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\} \tag{3.51}$$

Therefore

$$\frac{\partial W(\rho,\pi)}{\partial\phi}=\frac{bQ}{c_{44}}\{2\rho\} \tag{3.52}$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2}=\frac{bQ}{c_{44}}\Bigg\{2\sin\phi\rho-\frac{2\beta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(\frac{3}{2}-n\right)}+\frac{1}{2\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{1}{2}\right)}$$

$$+\frac{1}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{3}{2}\right)}-\frac{15}{8\pi\beta^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$+\frac{7}{2\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{7}{2}\right)}-\frac{63}{8\pi\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{\rho}{\beta}\right)^{n-\frac{1}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\} \tag{3.53}$$

$$\frac{1}{\rho^2}\frac{\partial^2 W(\rho,\phi)}{\partial\phi^2} = \frac{bQ}{c_{44}}\left\{2\sin\phi\rho^{-1} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}\right.$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$\left.+\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)} - \frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)} + \ldots\right\} \quad (3.54)$$

$$\frac{\partial W(\rho,\phi)}{\partial\rho} = \frac{bQ}{c_{44}}\left\{-2\sin\phi\rho + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{3}{2}}}{\left(\frac{3}{2}-n\right)} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{1}{2}\right)} -\right.$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{3}{2}\right)} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{5}{2}\right)} -$$

$$\left.\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{7}{2}\right)} + \frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{3}{2}}}{\left(n+\frac{9}{2}\right)} + \ldots\right\} \quad (3.55)$$

$$\frac{1}{\rho}\frac{\partial W(\rho,\phi)}{\partial\rho} = \frac{bQ}{c_{44}}\left\{-2\sin\phi\rho^{-1} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}\right.$$

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)} + \frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$-\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)}+\frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\} \quad (3.56)$$

$$\frac{\partial^2 W(\rho,\phi)}{\partial\rho^2}=\frac{bQ}{c_{44}}\Bigg\{-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\,\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}$$

$$-\frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}$$

$$+\frac{\frac{15}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}-\frac{\frac{7}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)}$$

$$+\frac{\frac{63}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\} \quad (3.57)$$

Therefore

$$\frac{\partial^2 W}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\phi^2}=\frac{bQ}{c_{44}}\Bigg\{-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\,\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}$$

$$-\frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}-\frac{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}$$

$$+\frac{\frac{15}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}-\frac{\frac{7}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)}$$

$$+\frac{\frac{63}{8\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(n-\frac{3}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\}$$

$$+\frac{bQ}{c_{44}}\Bigg\{-2\sin\phi\rho^{-1}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$-\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)}+\frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}+\ldots\Bigg\}$$

$$+\frac{bQ}{c_{44}}\Bigg\{2\sin\phi\rho^{-1}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\beta^{\frac{3}{2}-n}\rho^{n-\frac{5}{2}}}{\left(\frac{3}{2}-n\right)}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{1}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{1}{2}\right)}$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{3}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{3}{2}\right)}+\frac{15}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{5}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{5}{2}\right)}$$

$$+\frac{7}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{7}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{7}{2}\right)}-\frac{63}{8\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n-\frac{1}{2}\right)\sin\left(n-\frac{1}{2}\right)\phi\left(\frac{1}{\beta}\right)^{n+\frac{9}{2}}\rho^{n-\frac{5}{2}}}{\left(n+\frac{9}{2}\right)}\Bigg\}=0 \quad (3.58)$$

6. Conversion of the boundary condition for region II to series form

Recall the boundary condition for region II

$$\frac{\partial W}{\partial\phi}(\rho,\pi)=\frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\sqrt{\rho((\rho-2))}}+\rho\right],\quad \alpha<\rho<\beta,\quad \rho>2 \qquad (3.59)$$

We convert the above equation to series form using the formula

$$(1-t)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}J_k t^k \qquad (3.60)$$

Where

$$J_k=\frac{(2k)!}{2^{2k}(k!)^2} \qquad (3.61)$$

Now

$$\frac{\partial W}{\partial \phi}(\rho,\pi)=\frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\sqrt{\rho(\rho-2)}}+\rho\right]=\frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\rho^{\frac{1}{2}}(\rho-2)^{\frac{1}{2}}}+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[\frac{\rho(\rho-1)}{\rho^{\frac{1}{2}}(\rho-2)^{\frac{1}{2}}}+\rho\right]=\frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}(\rho-1)(\rho-2)^{-\frac{1}{2}}+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}\rho(\rho-2)^{-\frac{1}{2}}-\rho^{\frac{1}{2}}(\rho-2)^{-\frac{1}{2}}+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[\rho^{\frac{1}{2}}\rho\rho^{-\frac{1}{2}}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}-\rho^{\frac{1}{2}}\rho^{-\frac{1}{2}}\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}-\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}+\rho\right] \tag{3.62}$$

But

$$(1-t)^{-\frac{1}{2}}=\sum_{k=0}^{\infty} j_k t^k \Rightarrow \qquad \left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}=\sum_{k=0}^{\infty} c_j\left(\frac{2}{\rho}\right)^k=\sum_{k=0}^{\infty} j_k 2^k \rho^{-k}$$

Hence

$$\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}=\sum_{k=0}^{\infty} j_k 2^k \rho^{1-k} \tag{3.63}$$

Therefore

$$\frac{\partial W}{\partial \phi}(\rho,\pi)=\frac{bQ}{c_{44}}\left[\rho\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}-\left(1-\frac{2}{\rho}\right)^{-\frac{1}{2}}+\rho\right]=\frac{bQ}{c_{44}}\left[\sum_{k=0}^{\infty} j_k 2^k \rho^{1-k}-\sum_{k=0}^{\infty} j_k 2^k \rho^{-k}+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[\left(j_0 2^0\rho-j_0 2^0\rho^0\right)+\left(j_1 2^1\rho^0-j_1 2^1\rho^{-1}\right)+\left(j_2 2^2\rho^{-1}-j_2 2^2\rho^{-2}\right)+\left(j_3 2^3\rho^{-2}-j_3 2^3\rho^{-3}\right)+\left(j_4 2^4\rho^{-3}-j_4 2^4\rho^{-4}\right)+\ldots+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[(\rho-1)+\left(1-\rho^{-1}\right)+\left(\frac{3}{8}\times 4\rho^{-1}-\frac{3}{8}\times 4\rho^{-2}\right)+\left(\frac{5}{16}\times 8\rho^{-2}-\frac{5}{16}\times 8\rho^{-3}\right)+\left(\frac{35}{128}\times 16\rho^{-3}-\frac{35}{128}\times 16\rho^{-4}\right)+\ldots+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[(\rho-1)+\left(1-\rho^{-1}\right)+\left(\frac{3}{2}\rho^{-1}-\frac{3}{2}\rho^{-2}\right)+\left(\frac{5}{2}\rho^{-2}-\frac{5}{2}\rho^{-3}\right)+\left(\frac{35}{8}\rho^{-3}-\frac{35}{8}\rho^{-4}\right)+\ldots+\rho\right]$$

$$=\frac{bQ}{c_{44}}\left[2\rho+\frac{1}{2}\rho^{-1}+\rho^{-2}+\frac{15}{8}\rho^{-3}+\ldots\right] \tag{3.64}$$

From region II (A)

$$\frac{\partial W(\rho,\pi)}{\partial \phi}=\frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1}+\rho^{-2}-\frac{15}{8}\rho^{-3}+\ldots\right\}\ ,\rho>\alpha \tag{3.65}$$

From region II(B)

$$\frac{\partial W(\rho,\pi)}{\partial \phi}=\frac{bQ}{c_{44}}\{2\rho\} \qquad ,\rho<\beta \tag{3.66}$$

Hence by superposition principle

$$\frac{\partial W(\rho,\pi)}{\partial \phi} = \frac{bQ}{c_{44}}\{2\rho\} - \frac{bQ}{c_{44}}\left\{-\frac{1}{2}\rho^{-1} + \rho^{-2} - \frac{15}{8}\rho^{-3} + \ldots\right\} \tag{3.67}$$

$$= \frac{bQ}{c_{44}}\left\{2\rho + \frac{1}{2}\rho^{-1} - \rho^{-2} + \frac{15}{8}\rho^{-3} + \ldots\right\}$$

Which satisfies the boundary condition in series form.

Results and discussion

Having established the validity of our solution, we now investigate the tip of the crack for the stress fields using the solution obtained above

The tip is at the origin and is approached as $\rho \to 0$. The asymptotic value is obtained as

$$W(\rho,\phi) = \frac{4Q}{\pi c_{44}}\left(\frac{\chi(\beta)}{\sqrt{\beta}} - \frac{\chi(\alpha)}{\sqrt{\alpha}}\right)\sin\frac{\phi}{2}\rho^{\frac{1}{2}} \tag{3.68}$$

Next we introduce local polar coordinate (R,ψ) at the intersection of the crack breaker boundary and the x-axis to obtain the required displacement field as

$$w(R,\psi) = \frac{4Q}{\pi c_{44}}\left(\frac{\chi(\beta)}{\sqrt{\beta}} - \frac{\chi(\alpha)}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{2}}R\sin\psi = \frac{k_{III}}{\sqrt{\pi}}\frac{1}{c_{44}}R\sin\psi \tag{3.69}$$

where

$$K_{111} = \frac{2\sqrt{2}Q}{\sqrt{\pi}}\left[\frac{\beta-\alpha}{\sqrt{\beta}+\sqrt{\alpha}} + \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{\frac{2}{\beta}}\sqrt{\frac{2}{\alpha}}\frac{(\beta-\alpha)}{2}}{\sqrt{\frac{\alpha}{2}-1}+\sqrt{\frac{\beta}{2}-1}}\right) + \frac{\beta-\alpha}{\sqrt{\beta-2}+\sqrt{\alpha-2}}\right] \tag{3.70}$$

is the mode III stress intensity factor. This is a very important parameter in analyzing crack growth and has the ability to predict whether catastrophic failure will occur due to unstable crack propagation.

Now

$$\frac{\partial w(R,\psi)}{\partial \psi} = \frac{K_{III}}{\sqrt{\pi}}\frac{1}{c_{44}}R\cos\psi\ , \tag{3.71}$$

Hence the near crack-tip stress field is given by

$$\sigma_{\psi z}(R,\psi) = c_{44}\frac{1}{R}\frac{\partial w(R,\psi)}{\partial \psi} = \frac{k_{III}}{\sqrt{\pi}}\cos\psi \tag{3.72}$$

CONCLUSION

Arresting crack initiation and propagation using the stop hole technique has been adopted by many researchers. But a survey of their method of solutions shows that the finite element methods have been mostly used to derived the stress fields and the stress intensity factors. As we know, analytical solutions in closed-form are desired for accurate analysis and design due to their many advantages over numerical and approximate solutions. Moreover, an analytical solution can serve as a benchmark for the purpose of judging the accuracy and efficiency of various numerical and approximate techniques. In this regard, this study has made the following contributions to knowledge

i. we have been able to use analytical means to derive closed form solutions which agrees with the numerical results in the literature.

ii. The construction of the mapping functions

$$f(w)=\frac{1}{2}\left(we^{-\alpha}+w^{-1}e^{\alpha}\right)$$ maps the ellipse into a circular hole

$$\xi(z)=\frac{1}{2}\left(z+\frac{1}{z}\right)-1$$ maps the hole unto a straight line

Thereby making the problem analyzable by method of integral transform.

iii. We can see from Fig.2 that infinite crack terminates at the origin, it was not able to cross the origin establishing the fact that the elliptical stop hole (crack breaker) actually arrested the advancing crack.

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