Journal of the Nigerian Association of Mathematical Physics
Volume 65, (October 2022 – August 2023 Issue), **pp125 – 138**

AN ECONOMIC ORDER QUANTITY MODEL FOR DELAYED DETERIORATING ITEMS WITH LINEAR DEMAND RATE AND TWO STORAGE FACILITIES

Usman Lawal, Murtala Sani and Babangida Bature

Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Umaru Musa Yar'adua University, Katsina, Nigeria

Abstract

Many inventory models are developed under the presumption that goods begin to deteriorate as soon as they arrive at the warehouse. However, some products, such as dried fruits, cereal grains, etc., have a shelf-life and begin to deteriorate after a period of time. This phenomenon is known as non-instantaneous or delayed deterioration. In addition, situations like price reductions, inexpensive storage, and high demand occur, and one may choose to acquire a large number of goods, creating a storage problem. One should hire a different warehouse to keep the excess inventory because one's own warehouse has a limited storage capacity. In this research, an economic order quantity model for delayed deteriorating items with linear demand and two storage facilities has been investigated. The demand before deterioration sets in is assumed to be time dependent linear demand rate and that after deterioration sets in is assumed to be constant. Shortages are allowed and are completely backlogged. The values of optimal time at which the inventory level reaches zero in OW and optimal cycle length and optimal ordering policy are determined so as to minimize the total variable cost. The optimal solutions' existence and uniqueness are provided with the necessary and sufficient conditions. For each case, various numerical examples are provided to illustrate how the models are applied. The managerial implications of the sensitivity analysis of specific model parameters on optimal solutions are then examined. In the discussions, recommendations are made for lowering the inventory system's total variable cost.

***Keywords*:** Economic Order Quantity, Delayed Deterioration, Linear Demand Rate, Two Storage Facilities.

1. Introduction

Inventory issues affect practically all industries and commercial establishments. Because everyone in today's global market wants to keep themselves at the top, which necessitates a well-formulated and perfect inventory policy, inventory management has become an essential component of manufacturing, distribution, and retail infrastructure. As a result, it has become a significant and expanding area of research interest. In order to select an appropriate inventory policy with a view to minimizing total inventory costs or maximizing total profit for the system, we typically develop a mathematical model based on the real-life situation with the incorporation of various inventory parameters like demand, deterioration, and storage capacity, etc. Inventory management's goal is to make sure that the proper number, timing, quality, and price of commodities are always available to the customers. The quantity of the replenishment order and the interval between orders must both be decided upon in order to accomplish this goal. Inventory control considers deterioration carefully. How to manage and keep inventory for deteriorating items is one of the key components of the inventory problem. Inventories are the idle stock of tangible commodities with economic worth that are kept by an organization in a variety of forms, including as raw materials, goods that are still being manufactured, and finished goods that are awaiting packaging, transit, usage, or sale in the future. The management of these commodities, which make up a considerable amount of the company's investment, is crucial for maximizing profit or minimizing cost. Many small businesses are unable to identify the specific losses brought on by bad inventory control.

Corresponding Author**:** Usman L., Email: lawalkurami@gmail.com, Tel: +2348130876752

The demand rate for any product is always fluctuating on the real market. The assumption that time-dependent demand is linear, quadratic, or exponential was used by several researchers to develop their models. Furthermore, a uniform change, a steady increase or decrease, and rapid changes in demand rate are required for linear demand, quadratic demand, and exponential demand rates, respectively. [1] attempted to modify the basic assumptions of the main EOQ model developed by [2] for a scenario with a time-varying demand rate. Later, a heuristic approach for selecting lot size quantities in the general case of a deterministic time-varying demand rate and discrete opportunities for replenishment was discussed by [3]. Many inventory models are developed on the assumption that the demand rate of items varies either linearly or exponentially over time.

The issue of no-shortage inventory replenishment with a linearly time-varying demand rate across a finite time horizon was proposed by [4] as a fully analytical solution. For a time-dependent linear trended demand rate, [5] created a straightforward inventory replenishment decision method with the goal of minimizing the overall replenishment and carrying costs. In order to reduce the overall variable cost, [6] created an EOQ model for deteriorating goods with a linear trend in demand rate. The replenishment inventory problem for a depreciating good over a finite horizon with a linear trend in demand rate was studied by [7]. For the replenishment of instantaneously deteriorating items with a time-varying linear trended demand rate and shortages in all cycles.[8] investigated a heuristic approach.

Deterioration plays a significant role in many inventory systems. Most physical goods undergo decay or deterioration over time, examples being medicines, volatile liquids, blood banks, and so on. So decay or deterioration of physical goods in stock is a very realistic factor and there is a big need to consider this in inventory modelling. A model with exponentially decaying inventory was initially proposed by [9]. [10] formulated model with variable deteriorating rate of two-parameter Weibull distribution. [11] generalized this model by taking three-parameter Weibull distribution. [12] developed an optimal production inventory model for deteriorating items, where manufacturers sold the goods to multiple markets with varying demands. [13] analysed an inventory model for deteriorating items assuming constant deterioration rate with expiry date and time varying cost. Similarly, studies on inventory models within instantaneous deterioration can be found in [14], [15], [16], [17] and so on.

Many researchers, such as [9], [11], [13] and so on, assumed that the deterioration of the items in inventory starts from the instant of their arrival. However, items such as food items, pharmaceuticals, chemicals, blood, alcohol, gasoline, radioactive and so on deteriorate very rapidly over time and the loss from deterioration in these items cannot be ignored. [18] defined the term ‘‘non-instantaneous’’ for such deteriorating items. He gave an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. [19] studied an inventory policy for non-instantaneous deteriorating items under permissible delay in payments. Shortages were allowed and partially backlogged. The backlogging rate was a variable and dependent on the waiting time for the next replenishment. [20] developed an inventory model for non-instantaneously deteriorating items where suppliers provided a permissible delay in payment schedule linked to order quantity. [21] developed an inventory model with linear holding cost and stock-dependent demand for non-instantaneous deteriorating items. The holding cost was considered as linearly increasing function with respect to time and shortages were not allowed. [22] developed a two-warehouse system for non-instantaneous deterioration products with promotional effort and inflation over a finite time horizon. The demand depends on the sale team’s initiatives and shortages were partially backlogged at a rate dependent on the duration of waiting time up to the arrival of next lot. [23] developed an EOQ model for non-instantaneous deteriorating items with two levels of shortage under trade credit policy. The supplier offers the retailer a trade credit period to settle the amount and shortages were not allowed. Similarly, some inventory models for non-instantaneous deteriorating items can be found in [24], [25], [26] and so on.

Most of the researchers developed their inventory model for a single warehouse which has unlimited capacity. This assumption is not applicable in real-life situation. When an attractive price discount for bulk purchase is available, the management decides to purchase a huge quantity of items at a time. These goods cannot be stored in the existing storage (the owned warehouse with limited capacity). Furthermore, to take advantage, it may be profitable for the retailer to hire another storage facility called the rented warehouse. Units are continuously transferred from rented warehouse to owned and sold from owned warehouse. Usually, the holding cost in rented warehouse is higher than that in owned warehouse, due to the non-availability of better preserving facility which results in higher deterioration rate. Hence to reduce the holding cost, it is more economical to consume the goods of rented warehouse at the earliest. At present, appropriate distribution channels, as well as optimal logistics and warehousing facilities, are part and parcel of any effective marketing system. Two-warehousing systems, in general, facilitate the reduction in costs, ensure smooth and better supply management, and allow for safe and secured upkeep of inventories for final deliveries to retail destinations. Here we are citing some notable researchers who worked on a two-warehouse system. Inventory model with double storage facility OW and RW was first developed by [27] in which he assumed that the holding cost in rented warehouse (RW) is greater than that in own warehouse (OW), therefore, items in RW are first transferred to OW to meet the demand until the stock level in RW drops to zero and then items in OW are released. After his pioneering contribution, several other researchers have attempted to extend his work to various other realistic situations. [28] extended [27] model to cover the transportation cost from RW to OW that is considered to be a fixed constant independent of the quantity being transported. But he did not consider shortages in his model. [29] further developed the model with or without shortages by assuming that the demand varies over time with linearly increasing trend and that the transportation cost from RW to OW depends on the quantity being transported. In their model, the stock was transferred from RW to OW in an intermittent pattern. However, their work is for non-deteriorating items. In addition, a great deal of research efforts has been devoted to inventory models of deteriorating items in two warehousing area. [30] have derived an optimal replenishment policy for a two-warehouse inventory model under a conditionally permissible delay in payments. [31] have developed a two-warehouse inventory model with the increasing demand and time-varying deterioration. [32] studied a two-warehouse inventory model for deteriorating items having different deterioration rates and permissible delay with exponentially increasing trend in demand. [33] developed EOQ model for constant deteriorating items with cubic demand rate and salvage Value. [34] has developed a production-inventory model for a two-stage supply chain, consisting of one manufacturer and one retailer, and derived a solution for the profit. [35] developed two-warehouse inventory model for non-instantaneous deteriorating items with optimal credit period and partial backlogging under inflation. They proposed the model from seller's perspective by incorporating the fact that granting the trade credit from the seller to its buyer not only increases sales and revenue but also opportunity cost and default risk, shortages were allowed and partially backlogged and the backlogging rate was dependent on the waiting time for the next replenishment. Similarly, [36] developed a two-warehouse inventory model for non-instantaneous deteriorating items with partial backlogging and permissible delay in payments under inflation. [37] dealt with exponential demand rate and deterioration rate is constant in both warehouses. [38] proposed a two-warehouse inventory model in which demand rate varies exponentially with time and deterioration of items follows two-parameter Weibull distribution under the effect of inflation. [39] presented a two-warehouse inventory model wherein demand depends on stocks using genetic algorithm under the effect of inflation.

In most inventory model, shortages are not allowed. But sometimes customers' demands cannot be satisfied by the supplier and this situation is known as stock out or shortages. [6] established an EOQ model for instantaneous deteriorating items with time-dependent linear demand rates and shortages under inflation and time discounting. [40] developed an inventory model for non-instantaneous deteriorating items with stock-dependent demand rate, time-varying holding cost and shortages that are completely backlogged. [41] developed an EOQ model for delayed deteriorating items with inventory-level-dependent demand rate and constant partially backlogged shortages. [42] developed an inventory model for deteriorating items with stock-dependent demand rate and time-varying deterioration. Shortages are allowed and partially backlogged; the backlogging rate depends on the waiting time for the next replenishment. [43] discussed a time-dependent partially backlogged inventory model for deteriorating items with a time-varying demand rate and holding cost. Similarly, some inventory models with shortages can be found in [18], [44] and so on.

The problem of managing deteriorating inventory has received a considerable attention in recent years. Generally, deterioration is defined as damage, spoilage, decay, obsolescence, evaporation, pilferage, etc., that result in decreasing the usefulness of the original one. There is hardly any need for considering the effect of deterioration in the determination of the economic lot size for the items having low deterioration rate such as steel, hardware, glassware, toys, etc. However, items such as food items, pharmaceuticals, chemicals, blood, alcohol, gasoline, radioactive etc. deteriorate very rapidly over time and the loss from deterioration in these items cannot be ignored. In this direction, [45] considered an EOQ model for delayed deteriorating items with linear time dependent demand rate. [45] considered single warehouse in their model. However, this assumption may not always be applicable. Large quantity of items may be ordered or purchased due to attractive price discount, avoiding shortages/ stoke out condition, uncertainty in demand/ supply, fear increase in price among others. These items cannot store in a single warehouse with limited capacity. The retailer may rent another storage facility called rented warehouse. Units are continuously transferred from rented warehouse to owned and sold from owned warehouse. Usually, the holding cost in rented warehouse is higher than that in owned warehouse because of better preserving facility and a lower deterioration rate. Hence to reduce the holding cost, it is more economical to consume the goods of rented warehouse at the earliest.

In this research, an effort has been made to extend the work of [45] by considering two storage facilities. The demand before deterioration sets in is assumed to be time dependent linear demand rate and that after deterioration sets in is assumed to be constant. The analytical solution of the model is obtained and the solution is illustrated with the help of numerical examples. Finally, sensitivity analysis is carried out to show the effect of changes in some model parameters on decision variables. This is followed by discussions and conclusion.

2 Notation and Assumptions

2.1 Notation

The model is developed using the following notation.

A The ordering cost per order.
C_d The deterioration cost per unit per unit time (\$/unit/ year).
C_s Shortage cost per unit per unit of time.
h The Holding cost per unit per unit time in OW (\$/unit/ year).
f The Holding cost per unit per unit time in RW (\$/unit/ year).
α Deterioration rate in RW, where$0 < \alpha < 1$.
β Deterioration rate in OW, where$0 < \beta < 1,\ \beta < \alpha$
t_d The length of time in which the product exhibits no deterioration.
t_r Time at which the inventory level reaches zero in RW.
t_w Time at which the inventory level reaches zero in OW.
T The length of the replenishment cycle time (time unit).
W Capacity of the owned warehouse
Z The maximum inventory level per cycle
B_m The backorder level during the shortage period.
Q The order quantity where $Q = (Z + B_m)$.
$I_O(t)$ Inventory level in the OW at any time$t$ where $0 \leq t \leq T$.
$I_r(t)$ Inventory level in the RW at any time$t$ where $0 \leq t \leq T$.
$S(t)$ Shortage level at any time$t$ where $t_w \leq t \leq T$.

2.2 Assumption

The inventory system is developed based on the following assumptions.

1. The replenishment rate is infinite.
2. The lead time is zero.
3. A single delayed deteriorating item is considered.
4. The OW has a fixed capacity of W units; the RW has unlimited capacity.
5. The unit inventory holding cost per unit time in RW is higher than that in OW and the deterioration rate in RW is less than that in OW.
6. Demand before deterioration begins is linear function of time t and is given by $a + bt\ a \geq 0, b \neq 0$.
7. Demand after deterioration sets in is assumed to be constant and is given by d.
8. Shortages are allowed and completely backlogged.

3. Formulation of the model $t_d < t_r$

The inventory system is developed as follows. During the time interval $[0,\ t_d]$, the inventory level $I_r(t)$ in RW is depleting gradually due to market demand only and it is assumed to be linear function of time t whereas in OW inventory level remains unchanged. At time interval $[t_d,\ t_r]$ the inventory level $I_r(t)$ in RW is depleting due to combined effects of demand from the customers and deterioration and the demand at this time is reduced to d, a constant demand and the inventory in OW gets depleted due to deterioration alone. At time $t = t_w$, the inventory level $I_o(t)$ in OW depletes to zero the due to combined effects of demand from the customers and deterioration. Shortages occur at the time $t = t_w$ and are completely backlogged. However, the demand is backlogged in the interval$[t_w,\ T]$. The whole process of the inventory is repeated. Figure 3.1 describes the behaviours of model over the time interval$[0,\ T]$.

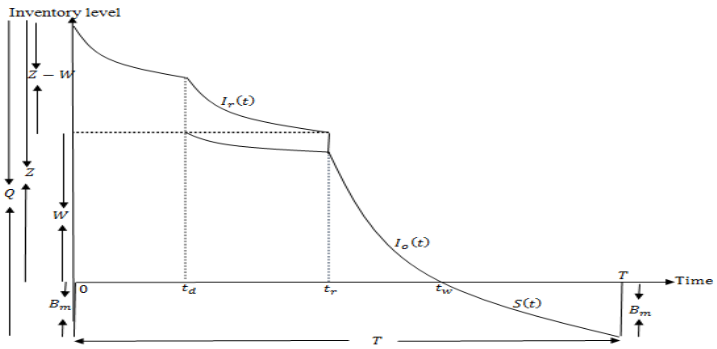

Figure 3.1: Two-warehouse inventory system when $t_d < t_r$

The differential equations that describe the inventory level in both RW and OW at any time t over the period$[0,\ T]$ are given by:

$$\frac{dI_r(t)}{dt} = -(a+bt), \qquad 0 \le t \le t_d \qquad (3.1)$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -d, \qquad t_d \le t \le t_r \qquad (3.2)$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0, \qquad t_d \le t \le t_r \qquad (3.3)$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -d, \qquad t_r \le t \le t_w \qquad (3.4)$$

$$\frac{dS(t)}{dt} = -d, \qquad t_w \le t \le T \qquad (3.5)$$

With boundary conditions

$I_r(0) = Z - W, I_r(t_r) = 0, I_o(t_d) = W, I_o(t_W) = 0, and\ S(t_w) = 0$

The solution of (3.1), (3.2), (3.3), (3.4) and (3.5) using the above conditions are given bellow.

$$I_r(t) = Z - W - \left(at + b\frac{t^2}{2}\right) \qquad 0 \le t \le t_d \qquad (3.6)$$

$$I_r(t) = \frac{d}{\beta}\left(e^{\beta(t_r - t)} - 1\right), \qquad t_d \le t \le t_r \qquad (3.7)$$

$$I_o(t) = We^{\alpha(t_d - t)} \qquad t_d \le t \le t_r \qquad (3.8)$$

$$I_o(t) = \frac{d}{\alpha}\left(e^{\alpha(t_w - t)} - 1\right), \qquad t_r \le t \le t_w \qquad (3.9)$$

$$S(t) = d(t_w - t) \qquad t_w \le t \le T \qquad (3.10)$$

Considering continuity of $I_r(t)$ at $t = t_d$, it follows from (3.6) and (3.7) that

$$Z = W + \left(at_d + b\frac{t_d^2}{2}\right) + \frac{\lambda}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) \qquad (3.11)$$

Considering continuity of $I_o(t)$ at $t = t_r$, it follows from (3.8) and (3.9) that

$$t_w = t_r + \frac{1}{\alpha} ln\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d}\right) \qquad (3.12)$$

The maximum backordered inventory B_m is obtained at $t = T$, and then from (3.10), we have

$$B_m = d(T - t_w) \qquad (3.13)$$

Thus the order size during total time interval $[0,\ T]$ is

$$Q = Z + B_m = W + \left(at_d + b\frac{t_d^2}{2}\right) + \frac{d}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) + d(T - t_w) \qquad (3.14)$$

(iii) The deterioration cost is given by

$$DC = C\left[\beta\int_{t_d}^{t_r} I_r(t)dt + \alpha\int_{t_d}^{t_r} I_o(t)dt + \alpha\int_{t_r}^{t_w} I_o(t)dt\right]$$

$$= C\left[\frac{d}{\beta}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\} + W\left\{1 - e^{-\alpha(t_r-t_d)}\right\} + \frac{d}{\alpha}\left\{\left(e^{\alpha(t_w-t_r)} - 1 - \alpha(t_w - t_r)\right)\right\}\right] \quad (3.15)$$

(iv) The fixed ordering cost per order is given by A

(v) The inventory holding cost per cycle in RW is given by

$$C_{h_r} = f\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right]$$

$$= f\left[Zt_d - Wt_d - \left(a\frac{t_d^2}{2} + b\frac{t_d^3}{6}\right) + \frac{d}{\beta^2}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\}\right] \quad (3.16)$$

(v) The inventory holding cost per cycle in OW is given by

$$C_h = h\left[\int_0^{t_d} W dt + \int_{t_d}^{t_r} I_o(t)dt + \int_{t_r}^{t_w} I_o(t)dt\right]$$

$$= h\left[Wt_d + \frac{W}{\alpha}\left\{1 - e^{-\theta_1(t_r-t_d)}\right\} + \frac{d}{\alpha^2}\left\{e^{\alpha(t_w-t_r)} - 1 - \alpha(t_w - t_r)\right\}\right] \quad (3.17)$$

(vi) The backordered cost per cycle is given by

$$SC = C_s\int_{t_w}^{T} -I_s(t)dt$$

$$= \frac{C_s d}{2}(T - t_w)^2 \quad (3.18)$$

(vii) The average total cost per unit time for a replenishment cycle (denoted by $Z(T)$ is given by

$Z(t_w, T) = \frac{1}{T}${Ordering cost + inventory holding cost for RW+ inventory holding cost for OW + deterioration cost + backordered cost }

$$= \frac{1}{T}\Bigg\{A + f\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3}\right) + \frac{dt_d}{\beta}\left(e^{\beta(t_r-t_d)} - 1\right) + \frac{d}{\beta^2}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\}\right]$$
$$+ h\left[Wt_d + \frac{W}{\alpha}\left\{1 - e^{\alpha(t_d-t_r)}\right\} + \frac{d}{\alpha^2}\left\{e^{\alpha(t_w-t_r)} - 1 - \alpha(t_w - t_r)\right\}\right]$$
$$+ C_d\left[\frac{d}{\beta}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\} + W\left\{1 - e^{\alpha(t_d-t_r)}\right\} + \frac{d}{\alpha}\left\{\left(e^{\alpha(t_w-t_r)} - 1 - \alpha(t_w - t_r)\right)\right\}\right] + \frac{C_s dT^2}{2}$$
$$- C_s d t_w T + \frac{C_s d t_w^2}{2}\Bigg\} \quad (3.19)$$

Substituting t_w in (3.19) from (3.12), we have

$$= \frac{1}{T}\Bigg\{A + f\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3}\right) + \frac{dt_d}{\beta}\left(e^{\beta(t_r-t_d)} - 1\right) + \frac{d}{\beta^2}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\}\right]$$
$$+ h\left[Wt_d + \frac{W}{\alpha}\left\{1 - e^{\alpha(t_d-t_r)}\right\} + \frac{d}{\alpha^2}\left\{\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d} - 1 - ln\left(\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d}\right)\right\}\right]$$
$$+ C_d\Bigg[\frac{d}{\beta}\left\{e^{\beta(t_r-t_d)} - 1 - \beta(t_r - t_d)\right\} + W\left\{1 - e^{\alpha(t_d-t_r)}\right\}$$
$$+ \frac{d}{\alpha}\left\{\left(\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d} - 1 - ln\left(\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d}\right)\right)\right\}\Bigg] + \frac{C_s dT^2}{2}$$
$$- C_s d\left(t_r + \frac{1}{\alpha}ln\left(\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d}\right)\right)T + \frac{C_s d}{2}\left(t_r + \frac{1}{\alpha}ln\left(\frac{d + W\alpha e^{\alpha(t_d-t_r)}}{d}\right)\right)^2\Bigg\} \quad (3.20)$$

Let $X_1 = A + f\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3}\right) + \frac{dt_d}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) + \frac{d}{\beta^2}\left\{e^{\beta(t_r - t_d)} - 1 - \beta(t_r - t_d)\right\}\right]$,

$$X_2 = h\left[Wt_d + \frac{W}{\alpha}\left\{1 - e^{\alpha(t_d - t_r)}\right\} + \frac{d}{\alpha^2}\left\{\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d} - 1 - ln\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d}\right)\right\}\right],$$

$$X_3 = C_d\left[\frac{d}{\beta}\left\{e^{\beta(t_r - t_d)} - 1 - \beta(t_r - t_d)\right\} + W\left\{1 - e^{\alpha(t_d - t_r)}\right\} + \frac{d}{\alpha}\left\{\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d} - 1 - ln\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d}\right)\right)\right\}\right] + \frac{C_s d}{2}\left(t_r + \frac{1}{\alpha}ln\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d}\right)\right)^2$$

Therefore equation (3.20) becomes

$$Z(T\ ) = \frac{1}{T}\left\{X_1 + X_2 + X_3 + + \frac{C_s dT^2}{2} - C_s d\left(t_r + \frac{1}{\alpha}ln\left(\frac{d + W\alpha e^{\alpha(t_d - t_r)}}{d}\right)\right)T\right\} \qquad (3.21)$$

3.1 Optimal Decision

In order to find the optimal ordering policies that minimize the total variable cost per unit time, we establish the necessary and sufficient conditions. The necessary condition for the total variable cost per unit time $Z(T)$ to be minimum is obtained by differentiating $Z(T)$ with respect T and equates to zero. The optimum value of T for which the sufficient condition $\frac{d^2Z(T)}{dT^2} > 0$ is satisfied gives a minimum for the total variable cost per unit time $Z(T)$. The necessary and sufficient conditions that minimize $Z(T)$ are respectively, $\frac{dZ(T)}{dT} = 0$ and $\frac{d^2Z(T)}{dT^2} > 0$

The first derivatives of the total variable cost, in (3.21), with respect to T is as follows.

$$\frac{dZ(T)}{dT} = -\frac{1}{T^2}\left\{X_1 + X_2 + X_3 - \frac{C_s dT^2}{2}\right\}$$

Therefore, $\frac{dZ_1(T)}{dT} = 0$ gives the following nonlinear equation in terms T

$$\left\{X_1 + X_2 + X_3 - \frac{C_s dT^2}{2}\right\} = 0 \qquad (3.22)$$

which implies

$$T = \sqrt{\frac{2}{C_s d}\{X_1 + X_2 + X_3\}} \qquad (3.23)$$

Theorem 3.1

The total variable cost $Z(T)$ is convex and reaches its global minimum at the point T^*, where T^* is the point which satisfies (3.22).

Proof:

$$\left.\frac{d^2Z(\ T)}{dT^2}\right|_{T^*} = \frac{1}{T^*}C_s d > 0 \qquad (3.24)$$

Thus, the EOQ corresponding to the optimal cycle length T^* will be computed as follows:

EOQ^* =The maximum inventory +the backorder level during the shortage period.

$$= W + (\alpha t_d + bt_d^2) + \frac{d}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) + d(T^* - t_w^*) \qquad (3.25)$$

3.2 Numerical Examples

This section will provide some numerical examples to illustrate the applicability of models developed by considering the following examples.

Example 3.1

Consider an inventory system with the following input parameters: $A = \$300$/order, $C_d = \$10$/unit/year, $C_s = \$15$/unit/year, $h = 5$days, $f = 8$days, $W = 300$units, $a = 100, b = 120$, $d = 20$ units, $\alpha = 0.2$units/year, $\beta = 0.15$units/year, $t_d = 4$ d ays, $t_r = 6$ days. Substituting the above values in equations (3.12), (3.21), (3.23) and (3.25) to obtain the values of the optimal length of time with positive inventory t_w^* =21.4577 days, the optimal cycle length T^* =27.56103days, the optimal average total cost $Z(T) = \$1830.998$ per year, and the economic order quantity EOQ_1^* =1848.278 units per year respectively.

3.3 Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-20\%, -10\%$, $+10\%$ to 20% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these system parameter values on the optimal cycle length, optimal total cost and the optimal order quantity per cycle are discussed

Table 3.1: Sensitivity analysis for example 3.1 ($t_d < t_r$)

Parameters	% Change in parameter	% Change in t_w^*	% Change in T^*	% Change in EOQ^*	% Change in $Z(T^*)$
A	−20	0.0000	−0.0263	−0.0079	−0.1189
	−10	0.0000	−0.0132	−0.0039	−0.0595
	+10	0.0000	0.0132	0.0039	0.0594
	+20	0.0000	0.0263	0.0079	0.1189
C_s	−20	0.0000	4.8076	1.4338	−2.6320
	−10	0.0000	2.1647	0.6456	−1.2025
	+10	0.0000	−1.8066	−0.5388	1.0262
	+20	0.0000	−3.3378	−0.9955	1.9126
f & h	−20	0.0000	−3.7307	−1.1126	−16.8468
	−10	0.0000	−1.8476	−0.5510	−8.3433
	+10	0.0000	1.8141	0.5410	8.1920
	+20	0.0000	3.5964	1.0726	16.2405
α & β	−20	17.6524	11.6234	0.0026	−9.5728
	−10	7.8541	5.1138	0.1187	−4.5204
	+10	−6.4391	−4.0952	−0.0187	4.1456
	+20	−11.8161	−7.4548	−0.1079	7.8785
a	−20	0.0000	−0.5633	−4.4963	−2.5436
	−10	0.0000	−0.2812	−2.2481	−1.2700
	+10	0.0000	0.2805	2.2478	1.2664
	+20	0.0000	0.5601	4.4954	2.5294
b	−20	0.0000	−1.8139	−10.9290	−8.1909
	−10	0.0000	−0.9028	−5.4633	−4.0767
	+10	0.0000	0.8947	5.4609	4.0402
	+20	0.0000	1.7815	10.9194	8.0450

4. Formulation of the model when ($t_r < t_d$).

The inventory system is developed as follows. During the time interval $[0,\ t_r]$, the inventory level $I_r(t)$ in RW is depleting gradually due to market demand only and it is assumed to be linear function of time$t$ whereas in OW inventory level remains unchanged. At time interval $[t_r,\ t_d]$ the inventory level $I_o(t)$ in OW is depleting due to demand from the customers and is also assumed to be quadratic function of time$t$. At time $t = t_w$, the inventory level $I_o(t)$ in OW depletes to zero due to combined effects of demand from the customers and deterioration. Shortages occur at the time $t = t_w$ and are completely backlogged. However, the demand is backlogged in the interval$[t_w, T]$. The whole process of the inventory is repeated. Figure 4.1 describes the behaviours of model over the time interval$[0, T]$.

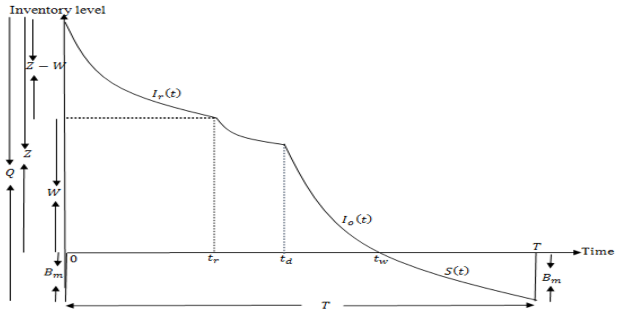

Figure 4.1: Two-Warehouse Inventory System When $t_d > t_r$

Note: The assumptions here are the same as in section 2.2 except that

The differential equations that describe the inventory level in both RW and OW at any time t over the period$[0,\ T]$ are given by:

$$\frac{dI_r(t)}{dt} = -(a+bt), \qquad 0 \le t \le t_r \qquad (4.1)$$

$$\frac{dI_o(t)}{dt} + \beta I_o(t) = -(a+bt), \qquad t_r \le t \le t_d \qquad (4.2)$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0, \qquad t_d \le t \le t_w \qquad (4.3)$$

$$\frac{dS(t)}{dt} = -d, \qquad t_w \le t \le T \qquad (4.4)$$

with boundary conditions $I_r(t_r) = 0$, $I_o(t_r) = W$, $I_o(t_W) = 0$ and $S(t_w) = 0$.

The solution of equations (4.1), (4.1), (4.1) and (4.1) using the above conditions are as follows.

$$I_r(t) = a(t_r - t) + \frac{b}{2}(t_r^2 - t^2) \qquad 0 \le t \le t_r \qquad (4.5)$$

$$I_o(t) = W + a(t_r - t) + \frac{b}{2}(t_r^2 - t^2) \qquad t_r \le t \le t_d \qquad (4.6)$$

$$I_o(t) = \frac{d}{\alpha}\left(e^{\alpha(t_w - t)} - 1\right), \qquad t_d \le t \le t_w \qquad (4.7)$$

$$I_s(t) = d(t_w - t) \qquad (4.8)$$

Considering continuity of $I_o(t)$ at $t = t_d$, it follows from (4.6) and (4.7) that

$$t_w = t_d + \frac{1}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right] \qquad (4.9)$$

Now, at $t = 0$ when $I_r(t) = Z - W$ and solving (4.5) we get the maximum inventory level per cycle is

$$Z = W + at_r + \frac{b}{2}t_r^2 \qquad (4.10)$$

The maximum backordered inventory B_m is obtained at $t = T$, and then from (4.8) we have

$$B_m = d(T - t_w) \qquad (4.11)$$

Thus, the order size during total time interval $[0,\ T]$ is

$$Q = Z + B_m = W + at_r + \frac{b}{2}t_r^2 + d(T - t_w) \qquad (4.12)$$

(iii) The deterioration cost is given by

$$DC = C_d\left[\alpha\int_{t_d}^{t_w} I_o(t)dt\right]$$

$$= C_d\left[\frac{d}{\alpha}\left\{e^{\alpha(t_w - t_d)} - 1 - \alpha(t_w - t_d)\right\}\right] \qquad (4.13)$$

(iv) The fixed ordering cost per order is given by A

(v) The inventory holding cost per cycle in RW is given by

$$C_{h_r} = f\left[\int_0^{t_r} I_r(t)dt\right]$$

$$= f\left\{\frac{a}{2}t_r^2 + \frac{b}{3}t_r^3\right\} \qquad (4.14)$$

(v) The inventory holding cost per cycle in OW is given by

$$C_{h_{ro}} = h\left[\int_0^{t_r} W dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_w} I_o(t)dt\right]$$

$$= h\left[Wt_d + \frac{a}{2}\left(2t_r t_d - t_d^2 - t_r^2\right) + \frac{b}{6}\left(3t_r^2 t_d - t_d^3 - 2t_r^3\right) + \frac{d}{\alpha^2}\left\{e^{\alpha(t_w - t_d)} - 1 - \alpha\left(t_w - t_d\right)\right\}\right] \quad (4.15)$$

(vi) The backordered cost per cycle is given by

$$SC = C_s \int_{t_w}^{T} -I_s(t)dt$$

$$= \frac{C_s d}{2}\left(T - t_w\right)^2, \quad (4.16)$$

(vii) The average total cost per unit time for a replenishment cycle (denoted by $Z(T)$ is given by

$Z(t_{w,}T) = \frac{1}{T}${Ordering cost + inventory holding cost for RW+ inventory holding cost for OW + deterioration cost + backordered cost }

$$= \frac{1}{T}\left\{A + f\left\{\frac{a}{2}t_r^2 + \frac{b}{3}t_r^3\right\} + h\left[Wt_d + \frac{a}{2}(2t_r t_d - t_d^2 - t_r^2) + \frac{b}{6}(3t_r^2 t_d - t_d^3 - 2t_r^3) + \frac{d}{\alpha^2}\{e^{\alpha(t_w - t_r)} - 1 - \alpha(t_w - t_r)\}\right] + C_d\left[\frac{d}{\alpha}\{e^{\alpha(t_w - t_d)} - 1 - \alpha(t_w - t_d)\}\right] + \frac{C_b d}{2}(T - t_w)^2\right\} (57) \quad (4.17)$$

Substituting t_w in (4.17) from (4.9), we have

$$= \frac{1}{T}\left\{A + f\left\{\frac{a}{2}t_r^2 + \frac{b}{3}t_r^3\right\} + h\left[Wt_d + \frac{a}{2}(2t_r t_d - t_d^2 - t_r^2) + \frac{b}{6}(3t_r^2 t_d - t_d^3 - 2t_r^3) + \frac{1}{\alpha}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\} - \frac{d}{\alpha^2} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right] + C_d\left[\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\} - \frac{d}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right] + \frac{C_s d T^2}{2} - C_s d\left\{t_d + \frac{1}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right\}T + \frac{C_s d}{2}\left\{t_d + \frac{1}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right\}^2\right\} \quad (4.18)$$

Let $Y_1 = A + f\left\{\frac{a}{2}t_r^2 + \frac{b}{3}t_r^3\right\}$

$$Y_2 = h\left[Wt_d + \frac{a}{2}(2t_r t_d - t_d^2 - t_r^2) + \frac{b}{6}(3t_r^2 t_d - t_d^3 - 2t_r^3) + \frac{1}{\alpha}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\} - \frac{d}{\alpha^2} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right]$$

$$Y_3 = C_d\left[\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\} - \frac{d}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right] + \frac{C_s d}{2}\left\{t_d + \frac{1}{\alpha} ln\left[1 + \frac{\alpha}{d}\left\{W + a(t_r - t_d) + \frac{b}{2}(t_r^2 - t_d^2)\right\}\right]\right\}^2\right]$$

Therefore equation (4.18) becomes

$$Z(T) = \frac{1}{T}\left\{Y_1 + Y_2 + Y_3 + \frac{C_s d}{2}T^2 - C_s dT\left(t_d + \frac{1}{\alpha}\ln\left[1 + \frac{\alpha}{d}\left\{W + a\left(t_r - t_d\right) + \frac{b}{2}\left(t_r^2 - t_d^2\right)\right\}\right]\right)\right\} \quad (4.19)$$

4.1 Optimal Decision

In order to find the optimal ordering policies that minimize the total variable cost per unit time, we establish the necessary and sufficient conditions. The necessary condition for the total variable cost per unit time $Z(T)$ to be minimum is obtained by differentiating $Z(T)$ with respect T and equates to zero. The optimum value of T for which the sufficient condition $\frac{d^2 Z(T)}{dT^2} > 0$ is satisfied gives a minimum for the total variable cost per unit time $Z(T)$.

The necessary and sufficient conditions that minimize $Z(T)$are respectively,$\frac{dZ(T)}{dT} = 0$ and $\frac{d^2 Z(T)}{dT^2} > 0$

The first derivatives of the total variable cost, in (4.19), with respect to T is as follow

$$\frac{dZ(T)}{dT} = -\frac{1}{T^2}\left\{Y_1 + Y_2 + Y_3 - \frac{C_b d T^2}{2}\right\}$$

Therefore,$\frac{dZ_1(T)}{dT} = 0$ gives the following nonlinear equation in terms T

$$\left\{Y_1 + Y_2 + Y_3 - \frac{C_s d T^2}{2}\right\} = 0 \qquad (4.20)$$

which implies

$$T = \sqrt{\frac{2}{C_s d}\{Y_1 + Y_2 + Y_3\}} \qquad (4.21)$$

Theorem 4.1. The total variable cost $Z(T)$is convex and reaches its global minimum at the point T^*, where T^* is the point which satisfies (4.21).

Proof:

$$\left.\frac{d^2 Z(T)}{dT^2}\right|_{T^*} = \frac{1}{T^*} C_s d > 0 \qquad (4.22)$$

Thus, the EOQ corresponding to the optimal cycle length T^* will be computed as follows:
$EOQ^* =$ The maximum inventory $+$ the backorder level during the shortage period.

$$= W + at_r + \frac{b}{2} t_r^2 + d(T^* - t_w^*) \qquad (4.23)$$

4.2 Numerical Example

This section will provide some numerical examples to illustrate the applicability of models developed by considering the following example.

Example 4.1 (When $t_r < t_d$)

The parameters used here are the same as in example 3.1 except $t_d = 10$ days and $t_r = 9.8$ days. Substituting the above values in equations (4.9), (4.21), (4.19) and (4.23) to obtain the values of the optimal length of time with positive inventory t_w^* =17.98836 days, the optimal cycle length T^* =24.58105 days, the optimal average total cost $Z(T) = \$1977.807$ per year, and the economic order quantity EOQ_1^* =1828.191 units per year respectively.

4.3 Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-20\%, -10\%$, $+10\%$ to 20% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these system parameter values on the optimal cycle length, optimal total cost and the optimal order quantity per cycle are discussed.

Table 4.1: Sensitivity analysis for example 4.1 ($t_r < t_d$)

Parameters	% Change in parameter	% Change in t_w^*	% Change in T^*	% Change in EOQ^*	% Change in $Z(T^*)$
A	-20	0.000	-0.008	-0.001	-0.016
	-10	0.0000	-0.004	-0.001	-0.008
	$+10$	0.0000	0.004	0.001	0.008
	$+20$	0.0000	0.008	0.001	0.016
C_s	-20	0.0000	11.185	1.488	-1.776
	-10	0.0000	5.118	0.681	-0.619
	$+10$	0.0000	-4.391	-0.584	0.163
	$+20$	0.0000	-8.211	-1.092	0.067
f & h	-20	0.0000	-9.932	-1.321	-20.227
	-10	0.0000	-4.836	-0.643	-9.850
	$+10$	0.0000	4.613	0.614	9.395
	$+20$	0.0000	9.031	1.201	18.392
α & β	-20	14.701	0.059	-0.987	-15.118
	-10	6.535	0.027	-0.439	-6.718
	$+10$	-5.348	-0.024	0.359	5.495
	$+20$	-9.806	-0.046	0.658	10.072

a	-20	0.0000	-0.945	-2.727	-1.924
	-10	0.0000	-0.471	-1.363	-0.959
	$+10$	0.0000	0.469	1.363	0.954
	$+20$	0.0000	0.935	2.726	1.904
b	-20	0.0000	-7.375	-16.277	-15.019
	-10	0.0000	-3.602	-8.127	-7.336
	$+10$	0.0000	3.444	8.106	7.014
	$+20$	0.0000	6.738	16.192	13.723

5. Results and Discussion of Sensitivity Analysis

Based on the computational results shown in Tables 3.1 and 4.1 the following managerial insights are obtained.

1. It can be seen that when the ordering cost (A) increases, the optimal cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $(Z(T^*))$ increase and vice versa. This implies that the retailer should order less quantity of items to shorten the cycle length when the ordering cost increases.
2. It can be seen that when the shortage cost per unit per unit of time (C_s) increases, the optimal cycle length(T^*) and economic order quantity (EOQ^*) decrease while total variable cost $(Z(T^*))$ increase. It implies that when the shortages cost increase, the number of backordered items reduce drastically which in turn lead to the decrease of order quantity and cycle length.
3. It can be seen that when the holding cost per unit per unit time in both rented and owned warehouses $(f \text{ and } h)$ increases, the optimal cycle length(T^*), economic order quantity (EOQ^*) and total variable cost$(Z(T^*))$ increase and vice versa. It implies that when holding cost increases, the retailer shall order large quantity in order to cover the cost of storage.
4. When the rate of deterioration $(\alpha \text{ and } \beta)$ increases, the optimal time at which the inventory level reaches zero in OW(t_w^*), cycle length (T^*) and economic order quantity (EOQ^*) decrease, while the total variable cost $\left(Z(T^*)\right)$ increases and vice versa. This implies that the retailer shall order less quantity to avoid items being deteriorating when the deterioration rate increases.
5. It can be seen that when the demand rate $(a)\, or (b)$ increases, the optimal cycle length (T^*), economic order quantity (EOQ^*) and total variable cost$(Z(T^*))$ increase and vice versa. It implies that when the demand rate increases, the retailer will order more quantity in order to obtain the maximum profit per unit time.

6. Conclusion

In this chapter, an economic order quantity model for delayed deteriorating items with linear demand and two storage facilities (when $t_d < t_r$ and $t_r < t_d$) has been investigated. The demand before deterioration sets in is assumed to be time dependent linear demand rate and that after deterioration sets in is assumed to be constant. Shortages are allowed and are completely backlogged. The values of optimal time at which the inventory level reaches zero in OW and optimal cycle length and optimal ordering policy are determined so as to minimize the total variable cost. The optimal solutions' existence and uniqueness are provided with the necessary and sufficient conditions. For each case, various numerical examples are provided to illustrate how the models are applied. The managerial implications of the sensitivity analysis of specific model parameters on optimal solutions are then examined. In the discussions, recommendations are made for lowering the inventory system's total variable cost.

REFERENCES

[1]. Silver, E. A. and Meal, H. C. (1969). A simple modification of the EOQ for the case of a varying demand rate. *Production and Inventory Management*, **10**(4), 52–65.

[2]. Harris, F. (1913). How many parts to make at once, factory. *The Magazine of Management*, **10**(2), 135–136.

[3]. Silver, E. A. and Meal, H. C. (1973). A heuristic for selecting lot size quantities for the case of a deterministic time-varying demand rate and discrete opportunities for replenishment. *Production and Inventory Management*, **14**(1), 64–74.

[4]. Donalson, W. A. (1977). Inventory replenishment policy for a linear trend in demand-an analytical solution. *Operational Research Quarterly*, **28**(3), 663–670.

[5]. Silver, E. A. (1979). A simple inventory replenishment decision rule for a linear trend in demand. *Journal of the Operational Research Society*, **30**(1), 71–75.

[6]. Goswami, A. and Chaudhuri, K. S. (1991). An EOQ model for deteriorating items with a linear trend in demand. *Journal of the Operational Research Society*, **42**(12), 1105–1110.

[7]. Chakrabarti, T. and Chaudhuri, K. S. (1997). An EOQ model for deteriorating items with a linear trend in demand and shortages in all cycles. *International Journal of Production Economics*, **49**(3), 205–213.

[8]. Chakrabarti, T., Giri, B. C. and Chaudhuri, K. S. (1998). A heuristic for replenishment of deteriorating items with time-varying demand and shortages in all cycles. *International Journal of Systems Science*, **29**(6), 551–555.

[9]. Ghare, P. M. and Schrader, G. F. (1963). A model for an exponentially decaying inventory. *Journal of Industrial Engineering*, **14**(5), 238–243.

[10]. Covert, R. P. and Philip, G. C. (1973). An EOQ model for items with Weibull distribution deterioration. *AIIE Transaction*, **5**, 323-326.

[11]. Philip, G. C. (1974). A generalized EOQ model for items with Weibull distribution. *AIIE Transactions*, **6**(2), 159–162.

[12]. He, Y., Wang, S.Y. and Lai, K.K. (2010). An optimal production-inventory model for deteriorating items with multiple-market demand. *European Journal of Operational Research*, **203**(3), 593–600.

[13]. Sharma, S., Singh, S. and Singh, S. R. (2018). An inventory model for deteriorating items with expiry date and time varying holding cost. *International Journal of Procurement Management*, **11**(5), 650-666.

[14]. Taleizadeh, A. A. (2014). An EOQ model with partial back-ordering and advance payments for an evaporating item. *International Journal of Production Economics*, **155**(c), 185–193.

[15]. Baraya, Y. M. and Sani, B. (2016). A deteriorating inventory model for items with constant demand rate under warehouse capacity constraint. *ABACUS: Journal of the Mathematical Association of Nigeria*, **43**(2), 180–192.

[16]. Tavakoli, S. and Taleizadeh, A. A. (2017). An EOQ model for decaying item with full advanced payment and conditional discount. *Annals of Operations Research*, **259**(3), 1–22.

[17]. Chen, Z. (2018). Optimal inventory replenishment and pricing for a single-manufacturer and multi-retailer system of deteriorating items. *International Journal of Operational Research*, **31**(1), 112–139.

[18]. Wu, K. S., Ouyang, L. Y. and Yang, C. T. (2006). An optimal replenishment policy for non-instantaneous deterioration items with stock dependent demand and partial backlogging. *International Journal of production Economics*, **101**(2), 369–384.

[19]. Geetha, K. V. and Uthayakumar, R. (2010). Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments. *Journal of Computational and Applied Mathematics*, **233**(10), 2492–2505.

[20]. Chang, C. T., Cheng, M. and Ouyang, L. Y. (2015). Optimal pricing and ordering policies for non-instantaneously deteriorating items under order-size-dependent delay in payments. *Applied Mathematical Modelling*, **39**(2), 747–763.

[21]. Kumar, S., Malik, A. K., Abhishek, S., Yadav, S. K. and Yashveer, S. (2016). An inventory model with linear holding cost and stock- dependent demand for non-instantaneous deteriorating items. *Advancement in Science and Technology AIP Conf. Proc.*, 1715, 020058.

[22]. Palanivel, M., Priyan, S. and Mala, P. (2018). Two-warehouse system for non-instantaneous deterioration products with promotional effort and inflation over a finite time horizon. *Journal of Industrial Engineering International*, **14**, 603-612.

[23]. Udayakumar, R. and Geetha, K. V. (2018). An EOQ model for non-instantaneous deteriorating items with two levels of storage under trade credit policy. *International Journal of Industrial Engineering*, **14**(2), 343–365.

[24]. Shah, N. H., Soni, H. N. and Patel, K. A. (2013). Optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. *Omega*, **41**(2), 421–430.

[25]. Jaggi, C. K., Sharma, A. and Tiwari, S. (2015). Credit financing in economic ordering policies for non-instantaneous deteriorating items with price dependent demand under permissible delay in payments: A new approach. *International Journal of Industrial Engineering Computations*, **6**(4), 481–502.

[26]. Tsao, Y. C., Zhang, Q., Fang, H. P. and Lee, P. L. (2017). Two-tiered pricing and ordering for non-instantaneous deteriorating items under trade credit. *Operational Research International Journal*, **19**(3), 833–852.

[27]. Hartley, R.V., (1976). Operations research—a managerial emphasis. *Good Year Publishing Company, Santa Monica*, **12**,315–317.

[28]. Sarma, K. V. S. (1987). A deterministic order level inventory model for deteriorating items with two storage facilities. *European Journal of Operational Research*, **29**(1), 70–73.

[29]. Goswami, A. and Chaudhuri, K. S. (1992). An economic order quantity model for items with two-levels of storage for a linear trend in demand. *Journal of the Operational Research Society*, **43**(2), 157–167.

[30]. Liang, Y. and Zhou, F. (2011). A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment. *Applied Mathematical Modelling*, **35**(5), 2221–2231.

[31]. Sett, B. K., Sarkar, B., Goswami, A. (2012). A two-warehouse inventory model with increasing demand and time varying deterioration. *Scientia Iranica*, **19**(6), 1969–1977.

[32]. Ranja-n, R. S. and Uthayakumar, R. (2015). A two–warehouse inventory model for deteriorating items with permissible delay under exponentially increasing demand. *International Journal of Supply and Operation Management*, **2**(1), 662 -682.

[33]. Karthikeyan, K. and Santhi, G. (2015). An inventory model for constant deteriorating items with cubic demand and salvage value. *International Journal of Applied Engineering Research*, **10**, 3723-3728.

[34]. Sana, S. S. (2016). Optimal Production lot size and reorder point of a two stage supply chain while random demand is sensitive with sales teams' initiatives. *International Journal of Systems Science*, **47**(2), 450–465.

[35]. Palanivel, M. and Uthayakumar R. (2016). Two-warehouse inventory model for non-instantaneous deteriorating items with optimal credit period and partial backlogging under inflation. *Journal of Control and Decision*, **3**(2), 132-150.

[36]. Palanivel, M. and Uthayakumar, R. (2017). Two-warehouse inventory model for non-instantaneous deteriorating items with partial backlogging and permissible delay in payments under inflation. *International Journal of Operational Research*, **28**(1), 35–69.

[37]. Kaliraman, N. K., Raj, R., Chandra, S. and Chaudhary, H. (2017). Two warehouse inventory model for deteriorating item with exponential demand rate and permissible delay in payment. *Yugoslav Journal of Operational Research*, **27**(1), 109–24.

[38]. Singh, S., Sharma, S. and Singh, S. R. (2018). Two-warehouse inventory model for deteriorating items with time dependent demand and partial backlogging under inflation. *International Journal of Mathematical Modelling and Computations*, **8**(2), 73-88.

[39]. Chauhan, N. and Yadav, A. S. (2020). An inventory model for deteriorating items with two-warehouse and dependent demand using genetic algorithm. *International Journal of Advanced Science and Technology*, **29**(5), 1152-1162.

[40]. Choudhury, K. D., Karmakar, B., Das, M. and Datta, T. K. (2013). An inventory model for deteriorating items with stock dependent demand, time-varying holding cost and shortages. *Journal of the Operational Research Society*, **23**(1), 137–142.

[41]. Baraya, Y. M. and Sani, B. (2013). An EOQ model for delayed deteriorating items with inventory level dependent demand rate and partial backlogging. *Journal of the Nigerian Association of Mathematical Physics*, **25**(2), 295–308.

[42]. Sakar, B. and Sakar, S. (2013). An improved inventory model with partial backlogging, time-varying deterioration and stock-dependent demand. *Economic Modelling*, **30**(1), 924–932.

[43]. Dutta, D. and Kumar, P. (2015). A partial backlogging inventory model for deteriorating items with time-varying demand and holding cost. *Croatian Operational Research Review*, **6**(2), 321–334.

[44]. Yang, H. L., Teng, J. T. and Chern, M. S. (2010). An inventory model under inflation for deteriorating items with stock-dependent consumption rate and partial backlogging shortages. *International Journal of Production Economics*, **123**(1), 8–19.

[45]. Musa, A. and Anas, B. A. (2013). Ordering policies for delayed deteriorating items with unconstrained retailer's capital, linear trend demand and shortages. *Journal of the Nigerian Association of Mathematical Physics*, **24**(4), 393–398.