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ON THE UNIFORM STABILITY OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY USING VECTOR LYAPUNOV FUNCTIONS

JONAS OGAR ACHUOBI[1,2*] AND EDET PETER AKPAN[1]

1.Department of Mathematics, Akwa-Ibom State University, Ikot Akpaden.
2. Department of Mathematics, University of Calabar, Calabar.

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ABSTRACT

In this study, we explore the uniform stability properties of Caputo fractional delay differential equations using vector Lyapunov functions. By applying the Caputo fractional Dini derivative of Lyapunov-like functions, along with a new comparison theorem and differential inequalities, we offer novel insights into the uniform stability of these complex systems. An illustrative example is provided to demonstrate the method's applicability. Our results improve, extends and generalizes many existing results in the literature.

1. Introduction

Fractional calculus, which involves the theory of derivatives and integrals of arbitrary real or complex orders, plays a significant role in various fields of mathematical, physical, and engineering sciences. It extends the concepts of integer-order differentiation and n-fold integration. Fractional derivatives provide a powerful tool for describing the general properties of various materials and processes. This is the primary advantage of fractional derivatives over classical integer-order models, where such effects are often overlooked. The benefits of fractional derivatives are particularly evident in modeling the mechanical and electrical properties of real materials, as well as in describing the properties of gases, liquids, rocks, and many other areas [1, 2].

*Corresponding author: JONAS OGAR ACHUOBI
E-mail address: jonasogarachuobi@unical.edu.ng
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The class of fractional differential equations of various types plays a crucial role as a tool not only in mathematics but also in fields like physics, control systems, dynamical systems, and engineering for modeling many physical phenomena. Naturally, these equations need to be solved. Over the past three decades, numerous studies on fractional calculus and fractional differential equations, involving various operators such as Riemann-Liouville, Erdelyi-Kober, Weyl-Riesz, Caputo, and Grunwald-Letnikov, have been published [3, 4, 5, 6].
Fractional differential equations with delay have gained popularity due to their enhanced accuracy in modeling memory and hereditary behaviors. For example, fractional derivatives are used to describe the mechanical and electrical properties of real materials, as well as the characteristics of gases, liquids, and minerals across various disciplines (see [7, 8] and related references). The existence and uniform stability of solutions for fractional differential systems, both with and without delays, have been thoroughly investigated in [9, 10, 11, 12].

It is well known that there are two primary methods for analyzing the stability of ordinary differential equations (ODEs). The first method, known as the Lyapunov indirect method, focuses on studying local stability by linearizing the system around its steady state (equilibrium point). The second method, known as the Lyapunov direct method, involves finding or constructing an appropriate auxiliary function, called a Lyapunov candidate function. Additionally, the Lyapunov direct method is a powerful tool for stability analysis of nonlinear systems, enabling the determination of global dynamical behaviors without the need to explicitly solve the ODEs [13].

The stability of fractional differential equations (FDEs) has garnered significant attention from researchers. In 2010, Li et al. [14] investigated the stability of nonlinear systems of FDEs that involve the Caputo fractional derivative with a singular kernel [5]. They extended the Lyapunov direct method to apply to FDEs. In the same year, Sadati et al. [15] expanded the Mittag-Leffler stability theorem to fractional nonlinear systems of FDEs with delay. The stability of a class of nonlinear systems of FDEs involving the Hadamard fractional derivative [16] was studied in [13] using a fractional comparison principle.

When Lyapunov functions have been used to investigate the stability and uniform stability of fractional differential equations involving delay, it is important to note that scalar Lyapunov functions were employed. These may not fully capture the interactions between different dimensions. In contrast, vector Lyapunov functions offer greater flexibility in constructing Lyapunov functions for complex systems, providing more precise and tailored stability criteria that scalar functions may miss. Vector Lyapunov functions allow for the analysis of subsystems and their interactions, leading to a deeper understanding of the stability properties of individual components within a larger system. They are especially useful for examining nonlinear systems, where variable interactions are complex and nonlinear relationships are common (see [17, 18, 19, 20]).

Let $\mathbb{R}_+ = [0,\infty)$ and assume that $t_0 \geq 0 \in \mathbb{R}_+$. Let $J_0 = [-\gamma, 0]$, $J = [-\gamma, \infty)$, $\gamma > 0$ and $I = [t_0, T]$ be the intervals in $\mathbb{R}$. Let $D^N = C(J_0, \mathbb{R}^N)$ be the space of all continuous maps on J_0, where $\mathbb{R}^N$ is the N-dimensional Euclidean vector space endowed with norm $\|.\|$. For any $\phi \in D^N$, we define the norm of ϕ by

$$\|\phi\|_0 = \sup_{s \in J_0} \|\phi(s)\|.$$

In this paper, we consider the retarded Caputo fractional differential equation of the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t), x_t), t \geq t_0, \\ x_{t_0} = \omega_0, \end{cases} \tag{1.1}$$

Where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha, t \in J, x \in \mathbb{R}^N, \omega_0 \in D^N$, and $f \in C\left(\mathbb{R} \times B_\rho \times D^N, \mathbb{R}^N\right)$. Here, $x_t \in D^N$ represents the history of the state from time $t-\gamma$ to the present time t, defined by $x_t(s) = x(t+s), s \in J_0$.

We assume that the following conditions hold:

(1) The function f guarantees that for any initial condition $\left(t_0, \omega_0\right) \in \mathbb{R}_+ \times D^N$, then the system (1.1) possesses a solution $x(t_0, \omega_0)(t) \in C^{\alpha}\left(\left[t_0, T\right], \mathbb{R}^N\right)$.

(2) $f(t, 0, 0) = 0$ for $t \geq t_0$.

We will utilize comparison results for the Caputo fractional differential equation with delay of the form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = G(t, u, u_t), t \geq t_0, \\ u_{t_0} = \theta_0, \end{cases} \tag{1.2}$$

where $u \in \mathbb{R}^n, G \in C\left[\mathbb{R}_+ \times \mathbb{R}^n \times E^n, \mathbb{R}^n\right]$, and $G(t, 0, 0) \equiv 0$. The function G ensures that for any initial values $(t_0, \theta_0) \in \mathbb{R}_+ \times E^n$, the system (1.2) with the given initial condition has a solution $u(t_0, \theta_0)(t) \in C^{\alpha}\left(\left[t_0, T\right], \mathbb{R}^n\right)$.

The main objective of this paper is to analyze the uniform stability properties of Caputo fractional differential equations with delays (1.1) using vector Lyapunov function. This study employs the Caputo fractional Dini derivative definition for Lyapunov-like functions, as introduced in [21, 22], and applies the comparison theorem along with differential inequalities.

2. PRELIMINARIES

In this paper, we adopt the Caputo (C) definition for fractional derivative, which is expressed as follows:

$${}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} x(s) ds, t \geq t_0.$$

It is worth highlighting that the Caputo derivative offers a significant benefit: the initial conditions for fractional differential equations using the Caputo derivatives are presented in the same way as those for integer-order derivatives, which are already well understood in terms of their physical significance. There are several definitions for fractional derivatives, with one of the most commonly used being the Grunwald-Letnikov (GL) fractional derivative, which is defined as follows:

$${}^{GL}D^{\alpha}x(t) = \lim_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{l=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} x(t-lh), t \geq t_0.$$

The Riemann-Liouville (RL) fractional derivative of the form:

$${}^{RL}D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1}.$$

In all the definitions given above, we have that $n-1<\alpha<n, \alpha>0$, where n is a natural number and $\Gamma(.)$ represents the gamma function. In most applications, the order of α is often less than 1, so that $\alpha\in(0,1).$ For simplicity of notation, we will use ${}^{C}D^{\alpha}$ instead of ${}^{C}_{t_0}D^{\alpha}_{t}$ so that the Caputo fractional derivative of order α of the function $x(t)$ is given as

$$ {}^{C}D^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{n-\alpha-1}x(s)ds,\ t\geq t_0. \tag{2.1}$$

Definition 2.1. In this paper, we define the following sets:

$$B_\rho=\left\{x\in\mathbb{R}^N:\|x\|<\rho,\rho>0\right\},$$
$$S_\rho=\left\{x\in\mathbb{R}^n:\|x\|<\rho,\rho>0\right\},$$
$$C_\rho=\left\{\omega\in D^N:\|\omega\|_0<\rho,\rho>0\right\}.$$

Definition 2.2. A function $V(t,x_t):J\times C_\rho\to\mathbb{R}_+^N$ is considered a vector Lyapunov function for (1.1) if it is continuous on $J\times C_\rho,$ satisfies $V(t,0)=0$, and is locally Lipschitz continuous with respect to the second argument.

Definition 2.3. [21,22] Let $(t_0,\omega_0)\in\mathbb{R}_+\times C[J_0,B_\rho]$ represents the initial condition of the initial value problem (IVP) (1.1). The Caputo fractional Dini derivative of the Lyapunov function $V(t,x_t)$ is defined as

$$ {}^{C}D_+^{\alpha}V(t,\omega(0),\omega)=\limsup_{h\to0^+}\frac{1}{h^\alpha}\left\{V(t,\omega(0))+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}V(t-lh,\omega(0)-h^\alpha f(t,\omega(0)))\right\} \tag{2.2}$$
$$-\frac{V(t_0,\omega_0(0))}{(t-t_0)^\alpha\Gamma(1-\alpha)},$$

where it is understood that $\omega(0)=x(t_0,\omega_0)(t)$ is the state of the system (1.1) at the current time t. $\omega_0(0)$ is the initial condition of the system (1.1) at the beginning $t=0.$

Definition 2.4. A function $G\in C\left[\mathbb{R}^n,\mathbb{R}^n\right]$ is considered quasi-monotone non-decreasing in x if whenever $x\leq y$ and $x_i=y_i,$ for $1\leq i\leq n,$ it follows that $G_i(x)\leq G_i(y)$ for all i.

Definition 2.5. [18] A function $a(r)$ is considered to be in class K, if it is a continuous function on $[0,\rho)$ with values in $\mathbb{R}_+$, $a(0)=0,$ and $a(r)$ is strictly increasing in r.

Definition 2.6. [23] The zero solution of (1.1) is considered uniformly stable, if for every initial time $t_0\in\mathbb{R}_+$ and any $\varepsilon>0,$ there exists a $\delta=\delta(\varepsilon)>0,$ such that for any initial function $\omega_0\in D^N$ with $\|\omega_0\|_0<\delta,$ it follows that $\|x(t_0,\omega_0)(t)\|<\varepsilon$ for $t\geq t_0.$

Lemma 2.7. [24] Assume $p(t), r(t)\in C\left(\left[t_0,T\right],\mathbb{R}^N\right)$ and suppose there exists $\tau_*\in(t_0,T)$ such that $p(\tau_*)=r(\tau_*)$ and $p(t)<r(t)$ for $t\in[t_0,\tau_*).$ The inequality ${}^{C}D_+^{\alpha}p(\tau_*)>{}^{C}D_+^{\alpha}r(\tau_*)$ holds if the Caputo fractional Dini derivative of p and r exist at $t=\tau_*$.

Lemma 2.8. [24] Let $w, s : [t_0 - \gamma, T] \to \mathbb{R}^n$ be continuous on $[t_0, T]$, and let $G \in C\left(\left[t_0, T\right] \times \mathbb{R}^n \times C_q, \mathbb{R}^n\right)$ be quasi-monotone non-decreasing in w_t for each $(t, \omega) \in \mathbb{R}^n$. Additionally, for each t, we have

(i) ${}^C D_+^\alpha w(t) \le G(t, w, w_t)$,

(ii) ${}^C D_+^\alpha s(t) > G(t, s, s_t), t \in [t_0, T]$.

Then

$$w_{t_0} < s_{t_0}, \tag{2.3}$$

Implies

$$w(t) < s(t), t \in [t_0, T]. \tag{2.4}$$

Lemma 2.9.[24] Let $G \in C[R_c, \mathbb{R}^n]$, where $R_c \subset \mathbb{R}_+ \times \mathbb{R}^n \times C_q$ such that $R_c := \{(t, u, \xi) : t_0 \le t \le t_0 + a, \|u - \theta_0(0)\| \le b, \|\xi - \theta_0\|_0 \le b, u \in \mathbb{R}^n, \xi \in C_q := \{\xi \in E^n : \|\xi\| < q, q > 0\}$ $, \theta_0 \in E^n, a, b > 0\}$ and $\|G(t, u, u_t)\| \le H$ on R_c. Assume that $G(t, u, u_t)$ is quasi-monotone non-decreasing in u_t for every $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, the initial value problem (IVP) (1.2) has a maximal solution $h(t, (t_0, \theta_0))$ defined on the interval $\left[t_0, t_0 + q\right]$, where

$$q = \min\left\{a, \left(\frac{b\Gamma(\alpha + 1)}{2H + b}\right)^{\frac{1}{\alpha}}\right\}.$$

Lemma 2.10. [24] Assume that

(1) $V \in C\left[\left(-\gamma, \infty\right) \times C_\rho, \mathbb{R}_+^N\right]$, where $V(t, x_t)$ is locally Lipschitz continuous with respect to the second argument.

(2) $G \in C[\mathbb{R}_+ \times \mathbb{R}^n \times D_q, \mathbb{R}^n]$ and $G(t, u, u_t)$ is quasi-monotone non decreasing with respect to u_t.

(3) ${}^C D_+^\alpha V(t, \omega(0), \omega) \le G\left(t, V(t, \omega(0)), V_t\right)$ for all $t \in \mathbb{R}_+$, where $V_t = V\left(t + s, \omega(s)\right), s \in J_0$.

If $h(t_0, \theta_0)(t)$ is the maximal solution of (1.2) and $x(t_0, \omega_0)(t)$ is any solution of (1.1) defined in the future such that

$$\sup_{s \in J_0} V(t_0, \omega_0)(s) \le \theta_0, \tag{2.5}$$

then the inequality

$$V(t, x(t_0, \omega_0)(t) \le h(t_0, \theta_0)(t), t \ge t_0, \tag{2.6}$$

3 MAIN RESULT

Theorem 3.1. Assume that

(1) $G \in C\left(\mathbb{R}_+ \times \mathbb{R}^n \times E^n, \mathbb{R}^n\right), G(t, u, u_t)$ is quasi-monotone non decreasing in u_t with $G(t, 0, 0) = 0$

(2) $V \in C\left[\left(-\gamma, \infty\right) \times C_\rho, \mathbb{R}_+^N\right], V(t, 0) = 0, V(t, x_t)$ is locally Lipschitzian in x_t such that

$${}^C D_+^\alpha V(t, \omega(0), \omega) \le G(t, \omega(0), V_t) \tag{3.1}$$

holds for all $(t, x) \in \mathbb{R}_+ \times B_\rho$.

(3) $a\left(\|x(t_0,\omega_0)(t)\|\right) \le V_0(t,x_t) \le b\left(\|x(t_0,\omega_0)(t)\|\right)$, where $a,b \in k$ and $V_0(t,x_t) = \sum_{i=1}^{N} V_i\left(t,x_t\right)$. Then the uniform stability of the trivial solution $u=0$ of the FrDE (1.2) implies the uniform stability of the trivial solution $x=0$ of FrDDE (1.1).

Proof.

Let $\varepsilon \in (0,\rho)$. The uniform stability of the trivial solution of (1.2) implies that for any $a(\varepsilon)>0, t_0 \in \mathbb{R}_+$ and initial function $\theta_0 \in E^n$, there exist a $\delta = \delta(\varepsilon)$ (independent of t_0) such that

$$\theta_0 = \left\|\sum_{i=1}^{n} \theta_{i0}\right\| \le \delta \text{ implies } \sum_{i=1}^{n} u_i(t_0,\theta_0)(t) < a(\varepsilon),\ t \ge t_0, \tag{3.2}$$

where $u(t_0,\theta_0)(t)$ is any solution of (1.2), with initial function θ_0. Since V is continuous (condition 2) and V(t,0)=0, there exist a $\delta_1 > 0$ such that

$$\|\theta_0\| < \delta_1, \text{implies } V(t,\theta_0) < \delta. \tag{3.3}$$

Let $x(t_0,\omega_0)(t)$ is any solution of (1.1) with $\|\omega_0\| < \delta_1$.

Claim: $\|x(t_0,\omega_0)(t)\| < \varepsilon,$ for all $t \ge t_0$.

Assuming that this claim is not the case, then there exist a finite time t_*, such that $\|x(t_0,\omega_0)(t_*)\| = \varepsilon$ and $\|x(t_0,\omega_0)(t)\| < \varepsilon$ for $t \in [t_0,t_*)$. Let $\theta_0 = V_0(t,\omega_0)$, then from (3.2) we have that $V_0(t,\omega_0) < \delta < \varepsilon < \rho$. Let $h_m(t_0,\theta_0)(t) = \sum_{i=1}^{n} h_i(t_0,\theta_0)(t)$ be the maximal solution of (1.2) with $h_0(t_0,\theta_0) < \delta$ such that

$$V_0(t_0,\omega_0)(t) \le h_m(t_0,\theta_0)(t) \quad . \tag{3.4}$$

At $t = t_*$, $\|x(t_0,\omega_0)(t_*)\| = \varepsilon$ and by condition (3) and (3.2), we have

$$a(\varepsilon) \le V_0(t_0,x_t) \le h_m(t_0,\theta_0)(t_*) < a(\varepsilon)\ . \tag{3.5}$$

This is a contradiction. Therefore for arbitrary $\varepsilon \in (0,\rho), t_0 \in \mathbb{R}_+$ there exist a $\delta(\varepsilon) > 0$ such that $\|\omega_0\| < \delta$ implies $\|x(t_0,\omega_0)(t)\| < \varepsilon$.

4. EXAMPLE

Through this example, we exhibit the advantage of using the vector Lyapunov function over the scalar Lyapunov function. We Consider the system of Caputo Fractional differential equations

$$\begin{aligned}
{}^{C}D^{\alpha}x(t) &= \frac{-x^2(t-1)}{4} + y^2(t-1)\sin x(t-1) + 3x^2(t-1) \\
{}^{C}D^{\alpha}y(t) &= -x^2(t-1)\cos^2 z(t-1) - y^2(t-1) + \frac{z^2(t-1)}{2} \\
{}^{C}D^{\alpha}z(t) &= x^2(t-1)\sin y(t-1) + 2y^2(t-1) - z^2(t-1)\cos^2 x(t-1)
\end{aligned} \tag{4.1}$$

for $t \ge t_0$, with initial functions

$x(s) = \omega_1(s),\ y(s) = \omega_2(s),\ z(s) = \omega_3(s)$ for $s \in \in [-1,0]$, where $\omega_1(s), \omega_2(s)$ and $\omega_3(s)$ are the initial functions defined on $-1 \le s \le 0$.

We recall that the initial functions capture the state of the system at time $t+s.$ In this example, $x_1(s)=x(t+s)=\omega_1(s),$ so that at $s=-1$ we have $x_t(s)=x(t-1)=w_1(-1).$ Similarly, $y_t(s)=y(t-1)=w_2(-1)$ and $z_t(s)=z(t-1)=w_3(-1).$ With these, the system (4.1) can therefore be written as

$$
\begin{aligned}
&{}^{C}D^{\alpha}x(t)=\frac{-\omega_1^2(-1)}{4}+\omega_2^2(-1)\sin\omega_1(-1)+3\omega_3^2(-1),\\
&{}^{C}D^{\alpha}y(t)=-\omega_1^2(-1)\cos^2\omega_3(-1)-\omega_2^2(-1)+\frac{\omega_3^2(-1)}{2},\\
&{}^{C}D^{\alpha}z(t)=\omega_1^2(-1)\sin\omega_2(-1)+2\omega_2^2(-1)-\omega_3^2(-1)\cos^2\omega_1(-1).
\end{aligned}
\tag{4.2}
$$

Now, we consider a scalar Lyapunov function given by

$$
V(t,\omega)=\frac{\omega_1^2(-1)+\omega_2^2(-1)+\omega_3^2(-1)}{2}
\tag{4.3}
$$

According to equation (2.2), we obtain

$$
\begin{aligned}
{}^{C}D_+^{\alpha}V&=\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\left\{\frac{1}{2}(\omega_1^2(-1)+\omega_2^2(-1)+\omega_3^2(-1))+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\left[\frac{1}{2}\Big(\omega_1^2(-1)-h^{\alpha}f_1+\omega_2^2(-1)\right.\right.\\
&\quad\left.\left.+\omega_3^2(-1)-h^{\alpha}f_3\Big)\right]\right\}-\frac{\left(\omega_{10}^2(-1)+\omega_{20}^2(-1)+\omega_{30}^2(-1)\right)}{2t^{\alpha}\Gamma(1-\alpha)},\\
&=\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\left\{\frac{\omega_1^2(-1)}{2}+\frac{\omega_2^2(-1)}{2}+\frac{\omega_3^2(-1)}{2}+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_1^2(-1)}{2}+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_2^2(-1)}{2}\right.\\
&\quad\left.+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_3^2(-1)}{2}-\frac{1}{2}h^{\alpha}(f_1+f_2+f_3)\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\right\}-\frac{\left(\omega_{10}^2(-1)+\omega_{20}^2(-1)+\omega_{30}^2(-1)\right)}{2t^{\alpha}\Gamma(1-\alpha)},\\
&=\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\sum_{l=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_1^2(-1)}{2}+\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\sum_{l=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_2^2(-1)}{2}\\
&\quad+\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\sum_{l=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{\omega_3^2(-1)}{2}-\frac{1}{2}(f_1+f_2+f_3)\limsup_{h\to0^+}\frac{1}{h^{\alpha}}\sum_{l=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\\
&\quad-\frac{\left(\omega_{10}^2(-1)+\omega_{20}^2(-1)+\omega_{30}^2(-1)\right)}{2t^{\alpha}\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying equation (3.7) and (3.8) in [8], we obtain

$$
\begin{aligned}
{}^{C}D_+^{\alpha}V&=\frac{\omega_1^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{\omega_2^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{\omega_3^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{1}{2}(f_1+f_2+f_3)\\
&\quad-\frac{\left(\omega_{10}^2(-1)+\omega_{20}^2(-1)+\omega_{30}^2(-1)\right)}{2t^{\alpha}\Gamma(1-\alpha)},\\
&\le\frac{\omega_1^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{\omega_2^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{\omega_3^2(-1)}{2t^{\alpha}\Gamma(1-\alpha)}+\frac{1}{2}(f_1+f_2+f_3).
\end{aligned}
$$

As $t \to \infty$, the first three terms tend to zero and using (4.2), we obtain

$$
\begin{aligned}
{}^{C}D_{+}^{\alpha}V &\le \frac{1}{2}(f_1 + f_2 + f_3),\\
&= \frac{1}{2}\Big[-\frac{\omega_1^2(-1)}{4} + \omega_2^2(-1)\sin\omega_1(-1) + 3\omega_3^2(-1) - \omega_1^2(-1)\cos^2\omega_3(-1) - \omega_2^2(-1)\\
&\quad + \frac{\omega_3^2(-1)}{2} + \omega_1^2(-1)\sin\omega_2(-1) + 2\omega_3^2(-1) - \omega_3^2(-1)\cos^2\omega_1(-1)\Big],\\
&\le -\frac{\omega_1^2(-1)}{8} + \frac{\omega_2^2(-1)|\sin\omega_1(-1)|}{2} + \frac{3\omega_3^2(-1)}{2} - \frac{\omega_1^2(-1)|\cos^2\omega_3(-1)|}{2} - \frac{\omega_2^2(-1)}{8}\\
&\quad + \frac{\omega_3^2(-1)}{8} + \frac{\omega_1^2(-1)|\sin\omega_2(-1)|}{2} + \omega_3^2(-1) - \frac{\omega_3^2(-1)|\cos^2\omega_1(-1)|}{2},\\
&= -\frac{\omega_1^2(-1)}{8} + \frac{\omega_2^2(-1)}{2} + \frac{6\omega_3^2(-1) + \omega_3^2(-1) - 2\omega_3^2(-1)}{4},\\
&= -\frac{\omega_1^2(-1)}{8} + \frac{\omega_2^2(-1)}{2} + \frac{5\omega_3^2(-1)}{4},\\
&\le \frac{7\omega_1^2(-1)}{2} + \frac{7\omega_2^2(-1)}{2} + \frac{7\omega_3^2(-1)}{2},\\
&= 7\left(\frac{\omega_1^2(-1) + \omega_2^2(-1) + \omega_3^2(-1)}{2}\right),\\
&= 7V(t,\omega).
\end{aligned}
$$

Therefore, we have

$$ {}^{C}D_{+}^{\alpha}V \le 7V(t,\omega) = G(t, V(t,\omega)). \tag{4.4} $$

Now consider the scalar comparison equation

$$
\begin{aligned}
&{}^{C}D^{\alpha}u = G(t, u(t), u(t-1)) = 7u(t-1)\\
&u(s) = \theta(s) = \theta_0,\ s \in [-1, 0],
\end{aligned} \tag{4.5}
$$

where $\theta_0 = 1$ remains constant throughout the given interval. Solving (4.5) by Laplace transform method and noting that $u(t-1)$ is a Heaviside step function, we obtain the following

$$ \ell\left({}^{C}D^{\alpha}u\right) = 7\ell\left(u(t-1)\right). $$

This implies that

$$ s^{\alpha}U(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}U^{k}(0) = 7\frac{e^{-s}}{s}, $$

So that

$$s^{\alpha}U(s)-s^{\alpha-1}=\frac{7e^{-s}}{s},$$

$$s^{\alpha}U(s)=s^{\alpha-1}+7\frac{e^{-s}}{s},$$

$$U(s)=\frac{1}{s}+7\frac{e^{-s}}{s^{\alpha+1}}.$$

Taking the inverse Laplace transform we obtain

$$\ell^{-1}U(s)=\ell^{-1}\left(\frac{1}{s}\right)+7\ell^{-1}\left(\frac{e^{-s}}{s^{\alpha+1}}\right),$$

So that

$$u(t)=1+7\ell^{-1}\left(\frac{e^{-s}}{s^{\alpha+1}}\right).$$

Utilizing the fact that $\ell\left(t^{\alpha}\right)=\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, we have

$$u(t)=1+7(t-1)^{\frac{1}{\Gamma(\alpha)}}u(t-1), \tag{4.6}$$

We observe that

$$|u(t)|=\left|1+7(t-1)^{\frac{1}{\Gamma(\alpha)}}u(t-1)\right| \text{ with} |\theta_0|=1.$$

As t increases, the term $7(t-1)^{\frac{1}{\Gamma(\alpha)}}$ grows causing u(t) to be unbounded. This indicates that for any nonzero initial condition $\theta_0, u(t)$ will eventually grow without bound as t increases. Hence for any small $\delta>0$ such that $\|\theta_0\|_0<\delta$, there exist some t>0 at which u(t) becomes unbounded. This means that no matter how small we chose δ, u(t) will eventually exceed any prescribed ε.

All conditions of Theorem 3.1 are satisfied, except that the trivial solution u=0 of (4.6) is not stable. Therefore, Theorem 3.1 cannot yield any information on uniform stability for the zero solution of (4.5).

Now, we consider a vector Lyapunov function of the form

$$V(t,\omega(0))=\left(V_1,V_2,V_3\right)^T=\left(\frac{\omega_1^2(-1)}{2},\frac{\omega_2^2(-1)}{2},\frac{\omega_3^2(-1)}{2}\right)^T \tag{4.7}$$

where $V_1=\dfrac{\omega_1^2(-1)}{2}, V_2=\dfrac{\omega_2^2(-1)}{2}$ and $V_3=\dfrac{\omega_1^2(-1)}{2}$ with $\omega=(\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$, so that the associated norm $\|\omega\|=\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$.

now,

$$V_0=\sum_{i=1}^{3}V_i=\frac{\omega_1^2(-1)}{2}+\frac{\omega_2^2(-1)}{2}+\frac{\omega_3^2(-1)}{2}=\frac{\omega_1^2(-1)+\omega_2^2(-1)+\omega_3^2(-1)}{2},$$

and so $a(\|\omega\|)\le V_0(t,x_t)\le b\|\omega\|$ with $a(r)=r$ and $b(r)=2r$, implying that $a,b\in K$. We compute the Caputo fractional Dini derivative for V_1 using (2.2) as follows

$$ {}^{C}D_{+}^{\alpha}V_1 = \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\left\{\frac{\omega_1^2(-1)}{2}+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{1}{2}\left[\omega_1(-1)-h^{\alpha}f_1(t,\omega_1(-1))\right]^2\right\}-\frac{\omega_{10}^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)} $$

$$ =\limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\left\{\frac{\omega_1^2(-1)}{2}-2\omega_1(-1)h^{\alpha}f_1(t,\omega_1(-1))\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}h^{2\alpha}f_1(t,\omega_1(-1))\right\} $$

$$ -\frac{\omega_{10}^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}, $$

$$ =\frac{\omega_1^2(-1)}{2}\limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}-2\omega_1(-1)f_1(t,\omega_1(-1))\limsup_{h\to 0^+}\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l} $$

$$ \times\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}h^{\alpha}f_1(t,\omega_1(-1))-\frac{\omega_{10}^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}. $$

Applying equation (3.7) and (3.8) in [8], we obtain

$$ {}^{C}D_{+}^{\alpha}V_1=\frac{\omega_1^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}+2f_1(t,\omega_1(-1))-\frac{\omega_{10}^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}, $$

$$ =\frac{\left[\omega_1^2(-1)-\omega_{10}^2(-1)\right]}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}+2f_1(t,\omega_1(-1)) $$

As $t\to\infty$, the first term tends to zero and using (4.2) we obtain

$$ {}^{C}D_{+}^{\alpha}V_1=2\left[\frac{-\omega_1^2(-1)}{4}+\omega_2^2(-1)\sin\omega_1(-1)+3\omega_3^2(-1)\right], $$

$$ =\frac{-\omega_1^2(-1)}{2}+2\omega_2^2(-1)\sin\omega_1(-1)+6\omega_3^2(-1), $$

$$ \le\frac{-\omega_1^2(-1)}{2}+4\frac{-\omega_2^2(-1)}{2}+12\frac{\omega_3^2(-1)}{2}, $$

$$ =-V_1+4V_2+12V_3. $$

Therefore,

$$ {}^{C}D_{+}^{\alpha}V_1\le -V_1+4V_2+12V_3 \tag{4.8} $$

Similarly, we compute the Caputo fractional derivative for V_2 using (2.3) as follows

$$ {}^{C}D_{+}^{\alpha}V_2 = \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\left\{\frac{\omega_2^2(-1)}{2}+\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}\frac{1}{2}\left[\omega_2(-1)-h^{\alpha}f_2(t,\omega_2(-1))\right]^2\right\}-\frac{\omega_{20}^2(-1)}{2(t-t_0)^{\alpha}\Gamma(1-\alpha)}, $$

Applying equation (3.7) and (3.8) in [8], we obtain

$$= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\left\{ \frac{\omega_2^2(-1)}{2} - 2\omega_2(-1) h^\alpha f_2(t,\omega_2(-1)) \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} + \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} h^{2\alpha} f_2(t,\omega_2(-1)) \right\}$$

$$-\frac{\omega_{20}^2(-1)}{2(t-t_0)^\alpha \Gamma(1-\alpha)},$$

$$= \frac{\omega_2^2(-1)}{2} \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} - 2\omega_2(-1) f_2(t,\omega_2(-1)) \limsup_{h\to 0^+} \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l}$$

$$\times \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} h^\alpha f_2(t,\omega_2(-1)) - \frac{\omega_{20}^2(-1)}{2(t-t_0)^\alpha \Gamma(1-\alpha)}.$$

$${}^{C}D_+^\alpha V_2 = \frac{\omega_2^2(-1)}{2(t-t_0)^\alpha \Gamma(1-\alpha)} + 2 f_2(t,\omega_2(-1)) - \frac{\omega_{20}^2(-1)}{2(t-t_0)^\alpha \Gamma(1-\alpha)},$$

$$= \frac{\left[\omega_2^2(-1) - \omega_{20}^2(-1)\right]}{2(t-t_0)^\alpha \Gamma(1-\alpha)} + 2 f_2(t,\omega_2(-1))$$

As $t \to \infty$, the first term tends to infinity and using (4.2) we obtain

$$\begin{aligned} {}^{C}D_+^\alpha V_2 &= 2\left[-\omega_1(-1)\cos^2 \omega_3(-1) - \omega_2^2(-1) + \frac{\omega_3^2(-1)}{2} \right], \\ &= -2\omega_1(-1)\cos^2 \omega_3(-1) - 2\omega_2^2(-1) + \omega_3^2(-1), \\ &\le \frac{-4\omega_1^2(-1)}{2} - 4\frac{\omega_2^2(-1)}{2} + 2\frac{\omega_3^2(-1)}{2}, \\ &= -4V_1 - 4V_2 + 2V_3. \end{aligned}$$

Therefore

$${}^{C}D_+^\alpha V_2 \le V_1 - 4V_2 + 2V_3. \tag{4.9}$$

Similarly, we compute the Caputo fractional Dini derivative for V_3 derivative using (2.2) as follows

$${}^{C}D_+^\alpha V_3 = \limsup_{h\to 0^+} \frac{1}{h^\alpha}\left\{ \frac{\omega_3^2(-1)}{2} + \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^l \binom{\alpha}{l} \frac{1}{2}\left[\omega_3(-1) - h^\alpha f_3(t,\omega_3(-1))\right]^2 \right\} - \frac{\omega_{30}^2(-1)}{2(t-t_0)^\alpha \Gamma(1-\alpha)},$$

Applying equation (3.7) and (3.8) in [8], we obtain

$$= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\left\{\frac{\omega_3^2(-1)}{2} - 2\omega_3(-1)h^\alpha f_3(t,\omega_3(-1))\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l} + \sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}h^{2\alpha}f_3(t,\omega_3(-1))\right\}$$

$$-\frac{\omega_{30}^2(-1)}{2(t-t_0)^\alpha\Gamma(1-\alpha)},$$

$$= \frac{\omega_3^2(-1)}{2}\limsup_{h\to 0^+}\frac{1}{h^\alpha}\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l} - 2\omega_3(-1)f_3(t,\omega_3(-1))\limsup_{h\to 0^+}\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}$$

$$\times\sum_{l=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^l\binom{\alpha}{l}h^\alpha f_3(t,\omega_3(-1)) - \frac{\omega_{30}^2(-1)}{2(t-t_0)^\alpha\Gamma(1-\alpha)}.$$

$$^C D_+^\alpha V_3 = \frac{\omega_3^2(-1)}{2(t-t_0)^\alpha\Gamma(1-\alpha)} + 2f_3(t,\omega_3(-1)) - \frac{\omega_{30}^2(-1)}{2(t-t_0)^\alpha\Gamma(1-\alpha)},$$

$$= \frac{\left[\omega_3^2(-1)-\omega_{30}^2(-1)\right]}{2(t-t_0)^\alpha\Gamma(1-\alpha)} + 2f_3(t,\omega_3(-1))$$

As $t\to\infty$, the first term tends to zero and using (4.2), we obtain

$$\begin{aligned} ^C D_+^\alpha V_3 &= 2\left[-\omega_1^2(-1)\sin\omega_2(-1) - 2\omega_2^2(-1) - \omega_3^2(-1)\right],\\ &= -2\omega_1(-1)\sin\omega_2(-1) - 4\omega_2^2(-1) - 2\omega_3^2(-1)\cos^2\omega_1(-1),\\ &\le \frac{-4\omega_1^2(-1)}{2} - 8\frac{\omega_2^2(-1)}{2} - 4\frac{\omega_3^2(-1)}{2},\\ &= V_1 + V_2 - 4V_3. \end{aligned}$$

Therefore

$$^C D_+^\alpha V_3 \le V_1 + V_2 - 4V_3. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) we obtain

$$^C D_+^\alpha V \le \begin{pmatrix} -1 & 4 & 12\\ 1 & -4 & 2\\ 1 & 1 & -4\end{pmatrix}\begin{pmatrix} V_1\\ V_2\\ V_3\end{pmatrix} = G(t,V). \tag{4.11}$$

Now consider the comparison system

$$^C D^\alpha u = G(t,u(t-1)) = Au(t-1),$$

where $A = \begin{pmatrix} -1 & 4 & 12\\ 1 & -4 & 2\\ 1 & 1 & -4\end{pmatrix}, u(s) = \theta_0,$ for $s\in[-1,0]$ where $\theta_0 = (1,1,1)^T$ is a constant function defined on the given interval.

The vectorial inequality (4.11) and all other conditions of Theorem 3.1 are satisfied by (4.7). Therefore, we conclude that the zero solution x=0 of the system (4.1) is uniformly stable according to Theorem 3.1.

CONCLUSION

With the growing academic focus on fractional differential equations with delays, which are recognized for their enhanced precision in modeling hereditary and memory-related behaviors, this paper explores the uniform stability of Caputo fractional differential equations with delays through the use of vector Lyapunov function. By utilizing the Caputo fractional Dini derivative, we have derived strong conditions for the uniform stability for these systems using vector Lyapunov functions. Our approach surpasses traditional methods that rely on scalar Lyapunov functions, offering a significant improvement in stability analysis. The included example demonstrates the practical advantages and increased accuracy of our method, marking a noteworthy contribution to the field

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