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ON THE UNIFORM STABILITY OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS USING VECTOR LYAPUNOV FUNCTIONS

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ABSTRACT

This paper investigates the uniform stability of the trivial solution for nonlinear Caputo fractional differential equations (FrDEs). Unlike traditional approaches that rely on scalar Lyapunov functions (SLFs), this study employs vector Lyapunov functions (VLFs) to analyze the stability properties of these equations. By utilizing comparison results specific to vector FrDEs, the paper establishes sufficient conditions under which uniform stability can be guaranteed. The theoretical findings are further substantiated through two illustrative examples, demonstrating the practical applicability of the derived stability criteria. The results contribute to a deeper understanding of stability in the context of FrDEs and provide a novel methodological framework for addressing complex nonlinear systems in this domain.

1. Introduction

The study of the qualitative properties of FrDEs has garnered significant attention in recent years (see [1], [4], [5], [11], and [15]). Recently, fractional order systems which are founded on the idea of non-integer derivatives have seen significant advancements in modern control theory. A noteworthy feature of fractional order systems is their capacity to simulate the behaviour of real systems that are not amenable to typical integer-order models, such as those with long memory and hereditary features. Due to these capabilities, fractional order systems are widely used in many different scientific and engineering domains.

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The employment of non-integer derivatives and integrals has proven to be effective, leading in important breakthroughs in control theory [3].

For a long time, the qualitative properties of differential equations have been studied using Lyapunov's second method. This approach involves employing a system of comparison equations, whose solutions are then used to deduce the properties of the original system. This method has been shown to be effective in a wide range of problems and has provided a unified framework for investigating the properties of differential equations. When examining the characteristics of differential equations, the comparison principle employed in this approach has shown to be an effective tool [2].

Agarwal et al. [1] provided sufficient conditions for the uniform stability of the trivial solutions of systems of comparison equations. In their investigation, they considered the uniform stability of the comparison system using the method of SLFs.

Consider the system of FrDE with the derivative in the sense of Caputo for $0 < q < 1$

$$ {}^{C}D^{q}\chi = f(t,\chi), \qquad \chi(t_0) = \chi_0, \ \ t \geq t_0 \tag{1.1} $$

where $\chi \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$. Let the function f be such that for any initial data $(t_0, \chi_0) \in \mathbb{R}_+ \times \mathbb{R}^N$, the system (1.1) has a solution $\chi(t; t_0, \chi_0) \in C^q([t_0, \infty), \mathbb{R}^N)$ for initial condition $\chi(t_0) = \chi_0$. Sufficient conditions for the existence of solutions of (1.1) can be found in [3],[5] and [10].

In this paper, we examine the uniform stability of the zero solution of nonlinear FrDE (1.1). The stability of FrDEs using SLFs was previously investigated by Lakshmikantham and Vatsala [9]. Their paper was one of the first to address this topic and its findings laid the foundation for further research in this area. Due to some difficulties encountered in the application of this definition, as highlighted in [6], a new definition was proposed in [1], and sufficient conditions for the stability of nonlinear systems using a scalar Lyapunov-like function were obtained. In [4], qualitative results for scalar FrDEs were derived using the Lyapunov functional and matrix inequality. However, other methods for achieving the stability of fractional order systems using Lyapunov-like functions exist, though they often come with several challenges and limitations (see [5]).

This paper is organized into six sections. In the next section (Section 2), we present some important definitions and lemmas that assist in introducing the main result. Section 3 provides the comparison results, where we compare the solution of the comparison system with the LF along the solution path of (1.1). In Section 4, we present the main results. Section 5 includes two examples demonstrating the application of our VLF method in determining the stability of the system. Finally, Section 6 offers the conclusion.

Preliminary Notes and Definitions

This section highlights the significance of fractional calculus, which is a powerful tool for describing the behavior of materials and processes. The main advantage of fractional calculus over classical calculus is its ability to handle systems with non-integer orders. This makes it a valuable tool for modeling systems with complex or irregular behavior. The applications of fractional calculus are diverse, ranging from mathematical analysis to engineering and physical sciences [7]. There is no single definition of fractional derivatives and integrals that is universally applicable to all situations. Instead, there are multiple approaches, each of which has its own advantages and disadvantages. The choice of which definition to use in a particular application depends on the specific requirements and constraints of the problem. Some of the most common definitions include the Riemann-Liouville (R-L), Caputo, and Grunwald-Letnikov (G-L) definitions (see [11], [12], [13], and [14]).

General case. Let the number $n-1<q<n$, $q>0$ be given, where $n \in \mathbb{N}$ and $\Gamma(.)$ is the gamma function.

According to [13], the R-L fractional derivative of $\chi(t)$ of order q is:

$$ {}_{t_0}^{RL}D_t^q \chi(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} \chi(s) ds, \qquad t \geq t_0, $$

and Caputo fractional derivative (CFrDe) of $\chi(t)$ of order q is defined by:

$$ {}_{t_0}^{C}D_t^q \chi(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} \chi^n(s) ds, \qquad t \geq t_0. $$

Due to their numerous shared characteristics with standard derivatives, the CFrDe are simpler to comprehend and utilize. The CFrDe is frequently employed in applications of FrDEs because it makes the initial conditions of FrDEs easier to understand in a practical setting.

In [1], the G-L fractional derivative of $\chi(t)$ of order q is:

$$ {}^{GL}D_0^q \chi(t) = \lim_{k \to 0^+} \frac{1}{k^q} \sum_{r=0}^{\left[\frac{(t-t_0)}{\kappa}\right]} (-1)^r ({}^{\alpha}C_r) \chi(t - r\kappa), \quad t \geq t_0, $$

and the G-L fractional Dini derivative of $\chi(t)$ of order q is:

$$ {}^{GL}D_0^q \chi(t) = \lim_{k \to 0^+} \sup \frac{1}{k^q} \sum_{r=0}^{\left[\frac{(t-t_0)}{\kappa}\right]} (-1)^r ({}^{\alpha}C_r) \chi(t - r\kappa), \quad t \geq t_0, $$

where ${}^{\alpha}C_r = \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!}$ and $\left[\frac{(t-t_0)}{k}\right]$ is the integer part of $\frac{(t-t_0)}{k}$.

Particular case. In most applications, the order of q is often less than 1, so that $q \in (0,1)$. For simplicity of notation, we will use ${}^{C}D^q$ instead of ${}_{t_0}^{C}D_t^q$ and the CFrDe of $\chi(t)$ is:

$$ {}^{C}D^q \chi(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{n-q-1} \chi' ds, \qquad t \geq t_0. \tag{2.1} $$

Definition 2.1. Let $[t_0, \infty) \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^N$ We say that the function$V(t,\chi) \in C([t_0,\infty) \times \xi, \mathbb{R}_+^N)$ belongs to class $\Omega([t_0,\infty),\xi)$ if it is locally Lipschitz in χ and $V(t,0) \equiv 0$.

Now, for any function $V(t,\chi))$ we define the Caputo fractional Dini derivative (CFrDiDe) as:

$$ {}^{C}D_+^q V(t,) = \lim_{k \to 0^+} \sup \frac{1}{k^q} \left\{ V(t,\chi) - V(t_0\chi_0) - \sum_{r=0}^{\left[\frac{(t-t_0)}{\kappa}\right]} (-1)^{r+1} ({}^{\alpha}C_r) \left[V\left(t - r\kappa, \chi - \kappa^q \Psi(t,\chi)\right) - V(t_0\chi_0) \right] \right\}, \tag{2.2} $$

$t \geq t_0$, where $t \in [t_0, \infty)$, $\chi, \chi_0 \in \xi$, and $\exists\ \kappa > 0 : t - r\kappa \in [t_0, T)$.

Definition 2.2. A function $g \in C[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in x, if $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$ implies $g_i(x) \leq g_i(y)$, $\forall i$.

Definition 2.3. The steady state, $\chi = 0$ of (1.1) is said to be uniformly stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, $\exists\ \delta = \delta(\epsilon) > 0$: for any $\chi_0 \in \mathbb{R}^N$, the inequality $\|\chi_0\| < \delta \Longrightarrow \|\chi(t; t_0, \chi_0)\| < \epsilon$ for $t \geq t_0$.

Definition 2.4. If a function $a(r)$ is strictly monotone increasing in r whenever $a \in C([0,\rho), \mathbb{R}_+]$ and $a(0) = 0$, then $a(r)$ is said to be a class K function.

In this work, we define the sets listed below:

$$\overline{S_\rho} = \{\chi \in \mathbb{R}^N : \|\chi\| \le \rho\}$$

$$S_\rho = \{\chi \in \mathbb{R}^N : \|\chi\| < \rho\}$$

It suffices to indicate that the inequalities between vectors are considered to be component-wise inequalities.
We will use the comparison results for the FrDE of the type

$$^{C}D^{q}u(t) = \oplus\,(t,u), u(t_0) = u_0, \quad t_0 \in \mathbb{R}_+ \tag{2.3}$$

existing for $t \ge t_0$, $t, t_0 \in \mathbb{R}_+ = [t_0, \infty)$ and $u \in \mathbb{R}^n$, $\oplus: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\oplus\,(t,0) \equiv 0$.

With respect to any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the function $\oplus\,(t,u)$ is such that the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in C^q([t_0, \infty), \mathbb{R}^n)$.

Lemma 2.5. [15] Assume $\Psi \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}^N]$. If $\chi(t)$ is a solution of (1.1) on $[t_0, T)$, then $(t, x(t))$ can be extended over a maximal interval of existence $[t_0, \infty)$.

Lemma 2.6. Assume $m \in C([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ and suppose there exists $t^* \in (t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \le t < t^*$. Then if the CFrDiDe of m exists at t^*, then the inequality $^{C}D^q_+ m(t^*, \alpha_1) - {}^{C}D^q_+ m(t^*, \alpha_2) > 0$ holds.

Proof. Applying (2.2), we have

$$\begin{aligned}
{}^{C}D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{[m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r+1}\, {}^{q}C_r [m(t^* - r\kappa, \alpha_1) - m(t^* - r\kappa, \alpha_2)] \\
&\quad - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\}. \\
{}^{C}D^q_+ m(t^*, \alpha_1) &= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{[m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r+1}\, {}^{q}C_r [m(t^* - r\kappa, \alpha_1) - m(t^* - r\kappa, \alpha_2)] \\
&\quad - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\} + {}^{C}D^q_+ m(t^*, \alpha_2).
\end{aligned}$$

When $m(t^*, \alpha_1) = m(t^*, \alpha_2)$, we have

$$\begin{aligned}
{}^{C}D^q_+ m(t^*, \alpha_1) &= -\limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad + \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q} \sum_{r=1}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r}\, {}^{q}C_r [m(t^* - r\kappa, \alpha_1) - m(t^* - r\kappa, \alpha_2)] \\
&\quad - \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q} \sum_{r=1}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r}\, {}^{q}C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] + {}^{C}D^q_+ m(t^*, \alpha_2), \\
&= -\limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q} \sum_{r=1}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r}\, {}^{q}C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] + {}^{C}D^q_+ m(t^*, \alpha_2), \\
&= -\limsup_{\kappa \to 0^+} \frac{1}{\kappa^q} \sum_{r=0}^{[\frac{t^*-t_0}{\kappa}]} (-1)^{r}\, {}^{q}C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] + {}^{C}D^q_+ m(t^*, \alpha_2)
\end{aligned}$$

.

Applying equation 3.8 in [1], we have

$$^{C}D_{+}^{q}m(t^{*},\alpha_1) = -\frac{(t^{*}-t_0)^{-q}}{\Gamma(1-q)}[m(t_0,\alpha_1)-m(t_0,\alpha_2)] + {}^{C}D_{+}^{q}m(t^{*},\alpha_2).$$

By the lemma, we have

$$m(t,\alpha_1) - m(t,\alpha_2) < 0, \text{ for } t_0 \le t < t^{*}$$

And so it follows that

$$^{C}D_{+}^{q}m(t^{*},\alpha_1) > {}^{C}D_{+}^{q}m(t^{*},\alpha_2).$$

Fractional Differential Inequalities and Comparison results for vector FrDEs

In this section, we assume that $0 < q < 1$.

Theorem 3.1.

Assume the following conditions:

(i) $\oplus \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ and $\oplus\,(t,u)$ is quasi-monotone non-decreasing with respect to u.

(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}_+^N]$ is locally Lipschitz continuous in χ such that
$$^{C}D_{+}^{q}V(t,\chi) \le \oplus(t,V(t,\chi)), \quad (t,\chi) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

(iii) $r(t) = r(t; t_0, u_0)$ is the maximal solution of the system (2.3)

Then,

$$V(t,\chi(t)) \le r(t), t \ge t0, \tag{3.1}$$

where $\chi(t) = \chi(t; t_0, \chi_0)$ is any solution of (1.1) that exists on $[t_0, \infty)$, provided that

$$V\,(t_0, \chi_0) \le u_0. \tag{3.2}$$

Proof. Let $\eta \in \bar{S}_\zeta = \{\eta \in \mathbb{R}^n : \| \eta \| \le \zeta\}$ be a small enough arbitrary vector and consider the following system of FrDE

$$^{C}Du = \oplus\,(t,u) + \eta, u(t_0) = u_0 + \eta, \tag{3.3}$$

for $t \in [t_0, \infty)$.

If the Volterra Integral equation

$$u_\eta(t,\alpha) = u_0 + \eta + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left(\oplus\,(s,u_\eta(s,\alpha)) + \eta\right)ds, \qquad t \in [t_0,\infty) \tag{3.4}$$

is satisfied, then the function $u(t,\alpha)$ is a solution of (2.3).

$$^{C}D_{+}^{q}V(t_1,\chi(t_1)) > {}^{C}D_{+}^{q}u(t_1,\alpha),$$

and using (3.3) we arrive at

$$^{C}D_{+}^{q}V(t_1,\chi(t_1)) > \oplus(t_1, u_\eta(t_1,\alpha)+\eta) > \oplus(t_1, u(t_1,\alpha)).$$

Therefore,

$$^{C}D_{+}^{q}m(t_1,\alpha) > \oplus(t_1,u(t_1,\alpha)). \tag{3.6}$$

For $t \in [t_0, T]$, we have

$$
\begin{aligned}
&{}^{C}D_{+}^{q}m(t,\alpha)\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{m(t,\alpha)-m(t_0,\alpha)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[m(t-r\kappa,\alpha)-m(t_0,\alpha)]\right\},\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{V(t,\chi^*(t))-V(t_0,\chi_0)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t-r\kappa))\right.\\
&\left.-V(t_0,\chi_0)]\right\},\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{V(t,\chi^*(t))-V(t_0,\chi_0)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t))-V(t_0,\chi_0)\right.\\
&\left.-[V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))-V(t_0,\chi_0)]+[V(t-r\kappa,\chi^*(t-r\kappa))-V(t_0,\chi_0)]]\right\},\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{V(t,\chi^*(t))-V(t_0,\chi_0)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t))-V(t_0,\chi_0)\right.\\
&\left.-[V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))-V(t_0,\chi_0)]+[V(t-r\kappa,\chi^*(t-r\kappa))-V(t_0,\chi_0)]]\right\},\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{V(t,\chi^*(t))-V(t_0,\chi_0)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t))-V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))\right.\\
&\left.-[V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))-V(t_0,\chi_0)]+[V(t-r\kappa,\chi^*(t-r\kappa))-V(t_0,\chi_0)]]\right\},\\
&=\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\left\{V(t,\chi^*(t))-V(t_0,\chi_0)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t))-V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))\right\}\\
&-\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}\binom{q}{r}[V(t-r\kappa,\chi^*(t-r\kappa))-V(t-r\kappa,\chi^*(t)-\kappa^q\Omega(t,\chi^*(t))]
\end{aligned}
$$

Since $V(t,\chi)$ is locally Lipschitz in the second variable, then

$$
{}^{C}D_{+}^{q}m(t,\alpha)\ \le\ {}^{C}D_{+}^{q}V(t,\chi^*(t))+L\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r}({}^{q}C_r)\|\chi^*(t-r\kappa)-(\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))\|
$$

,

where $L > 0$ is a Lipschitz constant.

$$
{}^{C}D_{+}^{q}m(t,\alpha)\le {}^{C}D_{+}^{q}V(t,\chi^*(t))-L\limsup_{\kappa\to0^{+}}\frac{1}{\kappa^{q}}\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r}({}^{q}C_r)\|\chi^*(t-r\kappa)-(\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))\|.
$$

As $\kappa\to 0$, $\|\chi^*(t-r\kappa)-(\chi^*(t)-\kappa^q\Omega(t,\chi^*(t)))\|\to 0$, so that

$$
{}^{C}D_{+}^{q}m(t,\alpha)={}^{C}D_{+}^{q}V(t,\chi^*(t))\le\oplus(t,V(t,\chi^*(t)))=\oplus(t,m(t,\alpha)). \tag{3.7}
$$

Now (3.7) with $t = t_1$ contradicts (3.6), hence (3.5) is true. □

For $t \in [t_0, T]$, we now show that whenever $\eta_1 < \eta_2$, then

$$
u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha). \tag{3.8}
$$

It is obvious that (3.8) holds for $t = t_0$. If the inequality (3.8) is false, then there would exists a point t_1 where $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_1,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha)$ for $t \in [t_0, t_1)$.

By lemma (2.6), we have that

$$
{}^{C}D_{+}^{q}u_{\eta_1}(t_1,\alpha)>{}^{C}D_{+}^{q}u_{\eta_2}(t_1,\alpha).
$$

However,

$$
{}^{C}D_{+}^{q}u_{\eta_1}(t_1,\alpha)-{}^{C}D_{+}^{q}u_{\eta_2}(t_1,\alpha)=\oplus(t_1,u_{\eta_1}(t_1,\alpha)+\eta_1)-[\oplus(t_1,u_{\eta_2}(t_1,\alpha)+\eta_2)]=\eta_1-\eta_2<0,
$$

which is a contradiction and so the family of solutions $\{u_{\eta_i}(t,\alpha)\}$ is uniformly bounded with bound L on $[t_0, T]$. We now show that $\{u_{\eta_i}(t,\alpha)\}$ is equicontinuous on $[t_0, T]$. Assume $K =$

$\sup\{\bigoplus(t,\chi):(t,\chi)\in[t_0,T]\times[-L,L]\}$. Also, fix a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}$), such that $\lim_{i\to\infty}\eta_i=0$ and consider a sequence of functions $u_{\eta_i}(t,\alpha)$. Again let $t_1,t_2\in[t_0,T]$ with $t_1<t_2$, then we have the following estimate

$$\begin{aligned}
\|u_{\eta_i}(t_2,\alpha) \;-\; u_{\eta_i}(t_1,\alpha)\| &= \Big\|u_0+\eta_i+\frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-s)^{q-1}(\oplus(s,u_{\eta_i}(s,\alpha))+\eta_i)\\
&- (u_0+\eta_i+\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}(\oplus(s,u_{\eta_i}(s,\alpha))+\eta_i))\Big\|,\\
&= \frac{1}{\Gamma(q)}\Big\|\int_{t_0}^{t_2}(t_2-s)^{q-1}(\oplus(s,u_{\eta_i}(s,\alpha)))ds-\int_{t_0}^{t_1}(t_1-s)^{q-1}(\oplus(s,u_{\eta_i}(s,\alpha)))ds\Big\|,\\
&\le \frac{M}{\Gamma(q)}\left[\left|\int_{t_0}^{t_2}(t_2-s)^{q-1}\right|\|\oplus(s,u_{\eta_i}(s,\alpha))\|+\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}\right|ds\|\oplus(s,u_{\eta_i}(s,\alpha))\|\right]\\
&\le \frac{M}{\Gamma(q)}\left[\left|\int_{t_0}^{t_2}(t_2-s)^{q-1}\right|+\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}\right|ds\right],\\
&= \frac{M}{\Gamma(q)}\left[\left|\int_{t_0}^{t_1}(t_2-s)^{q-1}ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right|+\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}ds\right|\right],\\
&= \frac{M}{\Gamma(q)}\left[\left|-\left[\frac{(t_2-t_1)^q}{q}-\frac{(t_2-t_0)^q}{q}\right]+\frac{(t_2-t_1)^q}{q}\right|+\left|\frac{(t_1-t_0)^q}{q}\right|\right],\\
&= \frac{M}{\Gamma(q)}\left[\left|-\frac{(t_2-t_1)^q}{q}+\frac{(t_2-t_0)^q}{q}+\frac{(t_2-t_1)^q}{q}\right|+\left|\frac{(t_1-t_0)^q}{q}\right|\right],\\
&= \frac{M}{\Gamma(q)}\left[\left|\frac{(t_2-t_0)^q}{q}\right|+\left|\frac{(t_1-t_0)^q}{q}\right|\right],\\
&= \frac{M}{\Gamma(q+1)}\left[(t_2-t_0)^q+(t_1-t_0)^q\right]\\
&\le \frac{2M}{\Gamma(q+1)}\left[(t_2-t_0)^q\right]<\epsilon,
\end{aligned}$$

provided $|t_2-t_1|<\delta=\left(\frac{\epsilon\Gamma(q+1)}{2M}\right)^{\frac{1}{q}}$. This shows that the family of solutions $\{u_{\eta_i}(t,\alpha)\}$ is equicontinuous. By the

Arzela-Ascoli theorem, $\{u_{\eta_i}(t,\alpha)\}$ has a subsequence $\{u_{\eta_{i_j}}(t,\alpha)\}$ which converges uniformly to a function $r(t)$ on $[t_0,T]$. We then show that $r(t)$ is a solution of (2.3). Equation (3.4) becomes

$$u_{\eta_{i_j}}(t,\alpha)=u_{0_{i_j}}+\eta_{i_j}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left(\bigoplus_{\eta_{i_j}}(s,u_{i_j}\left(s,\eta_{i_j}\right))+\eta_{i_j}\right)ds \tag{3.9}$$

Taking the limit as $i_j\to\infty$ in (3.9), yields

$$r(t)=u_0+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(\bigoplus(s,r(t)))ds \tag{3.10}$$

Thus, $r(t)$ is a solution of (2.3) on $[t_0,T]$. We claim that $r(t)$ is the maximal solution of (2.3). To prove this, assume that $p(t)$ is another solution of (2.3), then from (3.5), we have that $p(t)<u(t,\alpha)\le r(t)$ on $[t0,T]$.

1. Main Results

In this section, we will obtain sufficient conditions for the uniform stability of the system (1.1)

Theorem 4.1 (Uniform Stability)**.** Consider the following assumptions:

(1) Let $\bigoplus\in C[\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n]$ be a function such that $\bigoplus(t,u)$ is quasi-monotone non-decreasing in u and satisfies $\bigoplus(t,u)\equiv 0$.

(2) Let $V\in C[\mathbb{R}_+\times\mathbb{R}^N,\mathbb{R}_+^N]$ be a function with the following properties:

(i) V is locally Lipschitz continuous in x and $V(t,0)\equiv 0$.

(ii) $\exists$ functions $\phi(\|\chi\|)$ and $\beta(\|\chi\|)$ such that $\beta(\|\chi\|) \leq V_0(t,\chi) \leq \phi(\|\chi\|)$, where $V_0(t,\chi) = \sum_{i=1}^{N} V_i(t,\chi)$, and $\phi, \beta \in K$

(iii) For any $t, t_0 \geq 0$ and $\chi, \chi_0 \in \mathbb{R}^N$, the inequality

$$ {}^{C}D_{+}^{q} \leq \oplus\ (t, V(t,\chi)) \tag{4.1}$$

holds for all $(t,\chi) \in \mathbb{R}_+ \times \mathbb{R}^N$.

(3) The steady state of the FrDE (2.3) is uniformly stable.

Then, under these assumptions, the steady state of the FrDE (1.1) is also uniformly stable.

Proof. Given $\epsilon \in (0,\rho)$ and $t_0 \in \mathbb{R}_+$. Assume that the steady state $\chi = 0$ of (2.3) is uniformly stable. Then given $\beta(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, $\exists$ a $\delta = \delta(\epsilon) > 0$ (independent of t_0):

$$\chi_0 = \sum_{i=1}^{n} \chi_{i_0} < \delta \quad \Longrightarrow \quad \sum_{i=1}^{n} \chi_i(t; t_0, \chi_0) \leq \beta(\epsilon), \quad t \geq t_0$$
(4.2)

where $\chi(t; t_0, \chi_0)$ is any solution of (2.3).

Let $\delta_1 \in (0,\delta)$ be a number : $\phi(\delta_1) < \delta$, then since $V(t,0) = 0$ and by the continuity of V, $\exists$ a $\delta_1 = \delta_1(\delta(\epsilon)) > 0$ (independent of t_0) :

$$\|\chi\| < \delta_1 \Longrightarrow V_0(t,0) < \delta. \tag{4.3}$$

Let $\chi(t) = \chi(t; t_0, \chi_0)$ be any solution of (1.1), with $\|\chi_0\| < \delta_1$. Then it follows from condition (ii) that $V_0(t_0,\chi_0) \leq \phi(\|\chi_0\|) \leq \phi(\delta_1) < \delta$.

Claim:

$$\|\chi(t)\| < \epsilon, \quad t \geq t_0. \tag{4.4}$$

Considering (4.4) to be untrue, then $\exists$ a $t_1 \geq t_0$, : $\| \chi(t_1) \| = \epsilon$ and $\| \chi(t) \| < \epsilon$, $\forall\ t \in [t_0, t_1)$.

Let $\chi_0 = V_0(t_0,\chi_0)$ then it follows from (4.2) that $\chi_0 < \delta$ and $\sum_{i=1}^{n} \chi_i(t; t_0, \chi_0) \leq \beta(\epsilon)$, $\forall\ t \geq t_0$.

Let $r_0 = \sum_{i=0}^{n} r_i(t; t_0, \chi_0)$ be the maximal solution of (2.3) :

$$V_0(t,\chi) \leq r_0(t). \tag{4.5}$$

Then at $t = t_1$, $\|\chi(t_1)\| = \epsilon$ and from condition (ii), (4.2) and (4.5) we have that

$$\beta(\| \chi(t_1) \|) \leq V_0(t_1, \chi(t_1)) \leq r_0(t_1) < \beta(\epsilon).$$

So that,

$$\beta(\epsilon) \leq V_0(t_1, \chi(t_1)) \leq r_0(t_1) < \beta(\epsilon).$$

This contradiction proves that (4.4) is true. i.e for arbitrary $\epsilon \in (0,\rho)$, $t_0 \in \mathbb{R}_+$, $\exists\ \delta_1(\epsilon)$ (independent of t_0) : $\| \chi_0 \| < \delta_1 \Longrightarrow \| \chi(t) \| < \epsilon$, $\forall\ t \geq t_0$. Thus, we draw the conclusion that the steady state $\chi = 0$ of

(1.1) is uniformly stable.

2. Application

Example 1.: Consider the system of FrDEs

$$\begin{aligned} {}^{C}D^{q}\chi_1(t) &= -4\chi_1 + \frac{\chi_2^2 \sin\chi_1}{\chi_1} + \chi_1 \sec\chi_2, \\ {}^{C}D^{q}\chi_2(t) &= \frac{\chi_1^2 \cos\chi_2}{\chi_2} - 2\chi_2 \sec\chi_1 - \chi_2 \sin\chi_1 \end{aligned} \tag{5.1}$$
.

for $t \geq t_0$, with initial conditions

$$\chi_1(t_0) = \chi_{10} \text{ and } \chi_2(t_0) = \chi_{20}.$$

Consider the functions of the form $V = (V_1, V_2)^T$, where $V_1(t, \chi_1, \chi_2) = \chi_1^2$ and $V_2(t, \chi_1, \chi_2) = \chi_2^2$, $\chi = (\chi_1, \chi_2) \in \mathbb{R}^2$, with the associated norm $\|\chi\| = \sqrt{\chi_1^2 + \chi_2^2}$.

Now, the Lyapunov function

$$V_0(t, \chi) = \sum_{i=1}^{2} V_i(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$$

and so $b(\|\chi\|) \leq V_0(t, \chi) \leq a(\|\chi\|)$ with $b(r) = r$ and $a(r) = 2r^2$, implying that $a, b \in K$.

From (2.2), we compute the CFrDiDe for $V_1(t, \chi_1, \chi_2) = \chi_1^2$ as follows

$$^{C}D_{+}^{q}V(t,) = \limsup_{k \to 0^+} \frac{1}{k^q}\left\{V(t,\chi) - V(t_0\chi_0) + \sum_{r=0}^{\left[\frac{(t-t_0)}{\kappa}\right]}(-1)^r \binom{q}{r}\left[V\big(t - r\kappa, \chi - \kappa^q\Psi(t,\chi)\big) - V(t_0\chi_0)\right]\right\}$$

$$= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[(\chi_1 - \kappa^q\Psi_1(t;\chi_1,\chi_2))^2 - \chi_{10}^2]\},$$

$$= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[\chi_1^2 - 2\chi_1\kappa^q\Psi_1(t,\chi_1,\chi_2) + \kappa^{2q}\Psi_1^2(t;\chi_1,\chi_2) - \chi_{10}^2]\}$$

$$= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - 2\chi_1\Psi_1(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_1^2(t,\chi_1,\chi_2) - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2,$$

$$= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\{\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - \sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_1^2(t,\chi_1,\chi_2) - 2\chi_1\Psi_1(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q,$$

$$= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - \limsup_{\kappa \to 0^+} \frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2 - 2\chi_1\Psi_1(t,\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)$$
$$+ \limsup_{\kappa \to 0^+} \kappa^q \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\Psi_1^2(t,\chi_1,\chi_2).$$

Applying equation (3.7) and (3.8) in [1], we have

$$^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) = \frac{\chi_1^2}{t^q\Gamma(1-q)} - \frac{\chi_{10}^2}{t^q\Gamma(1-q)} + 2\chi_1\Psi_1(t;\chi_1,\chi_2).$$

As $t \to \infty$, $\frac{\chi_1^2}{t^q\Gamma(1-q)} \to 0$, and $\frac{\chi_{10}^2}{t^q\Gamma(1-q)} \to 0$, so that we have

$$^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) = 2\chi_1\Psi_1(t;\chi_1,\chi_2).$$

Substituting for $\Psi_1(t, \chi_1, \chi_2)$ we have

$$\begin{aligned}{}^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) &= 2\chi_1\left(-4\chi_1+\frac{\chi_2^2\sin\chi_1}{\chi_1}+\chi_1\sec\chi_2\right)\\ &= -8\chi_1^2+2\chi_2^2\sin\chi_1+2\chi_1^2\sec\chi_2,\\ &= \chi_1^2\left(-8+2\sec\chi_2\right)+\chi_2^2\left(2\sin\chi_1\right),\\ &= \chi_1^2\left(-8+\frac{2}{\cos\chi_2}\right)+2\chi_2^2\sin\chi_1,\\ &\leq -6\chi_1^2+2\chi_2^2. \end{aligned},$$

Therefore

$$^{C}D_{+}^{q}V_1(t,\chi_1,\chi_2) \leq -6V_1+2V_2. \qquad (5.2)$$

Similarly, using (2.2), we compute the CFrDiDe for $V_2(t,\chi_1,\chi_2)=\chi_2^2$ as follows

$$\begin{aligned}{}^{C}D_{+}^{q}V_2(t;\chi_1,\chi_2) &= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\Big\{V(t,\chi)-V(t_0,\chi_0)+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[V(t-r\kappa,\chi-\kappa^q\Psi(t,\chi))-V(t_0,\chi_0)]\Big\}\\
&= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\Big\{\chi_2^2-\chi_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[(\chi_2-\kappa^q\Psi_2(t;\chi_1,\chi_2))^2-\chi_{20}^2]\Big\},\\
&= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\Big\{\chi_2^2-\chi_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[\chi_2^2-2\chi_2\kappa^q\Psi_2(t;\chi_1,\chi_2)+\kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2)-\chi_{20}^2]\Big\}\\
&= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\Big\{\chi_2^2-\chi_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2-2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2)-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2,\\
&= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\Big\{\chi_2^2+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2-\chi_{20}^2-\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2,\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2)-2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q,\\
&= \limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2-\limsup_{\kappa\to0^+}\frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2-2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\\
&\quad+\limsup_{\kappa\to0^+}\kappa^q\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\Psi_2^2(t;\chi_1,\chi_2).\end{aligned}$$

Applying equation (3.7) and (3.8) in [1], we have

$$^{C}D_{+}^{q}V_2(t;\chi_1,\chi_2)=\frac{\chi_2^2}{t^q\Gamma(1-q)}-\frac{\chi_{20}^2}{t^q\Gamma(1-q)}+2\chi_2\Psi_2(t;\chi_1,\chi_2).$$

As $t\to\infty$, $\frac{\chi_2^2}{t^q\Gamma(1-q)}\to 0$, and $\frac{\chi_{20}^2}{t^q\Gamma(1-q)}\to 0$, so that we have

$$^{C}D_{+}^{q}V_2(t;\chi_1,\chi_2)=2\chi_2\Psi_2(t;\chi_1,\chi_2).$$

Substituting for $\Psi_2(t,\chi_1,\chi_2)$ we have

$$\begin{aligned}{}^{C}D_{+}^{q}V_2(t;\chi_1,\chi_2) &= 2\chi_2\left(\frac{\chi_1^2\cos\chi_2}{\chi_2}-2\chi_2\sec\chi_1-\chi_2\sin\chi_1\right)\\ &= 2\chi_1^2\cos\chi_2-4\chi_2^2\sec\chi_1-2\chi_2^2\sin\chi_1,\\ &= 2\chi_1^2\cos\chi_2-\chi_2^2\left(4\sec\chi_1-2\sin\chi_1\right),\\ &\leq 2\chi_1^2|\cos\chi_2|-\chi_2^2\left(4|\sec\chi_1|-2|\sin\chi_1|\right),\\ &= 2\chi_1^2-2\chi_2^2. \end{aligned},$$

Therefore

$$ {}^{C}D_{+}^{q}V_2(t;\chi_1,\chi_2) \le 2V_1 - 2V_2. \quad (5.3)$$

Combining (5.7) and (5.8), we have

$$ {}^{C}D_{+}^{q}V \le \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \oplus\,(t,V). \quad (5.4)$$

Now consider the comparison system

$$ {}^{C}D^{q} \oplus (t,u) = Au \quad (5.5)$$

where $A = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}$.

The vector inequality (5.2) and all other conditions of theorem (4.1) are fulfilled if the matrix A is Metzler. Therefore, the steady state u = 0 of the system (5.5) is uniformly stable. Consequently, we can therefore draw the conclusion that the steady state $x = 0$ of the system (5.1) is also uniformly stable.

Example 2.: Consider the system of FrDEs

$$\begin{aligned} {}^{C}D^{q}\chi_1(t) &= 2\chi_1\cos^2\chi_2 + \frac{\chi_2^2\csc(\chi_1)}{\chi_1\cos\chi_2} - 3\chi_1\sin\chi_1 \quad , \\ {}^{C}D^{q}\chi_2(t) &= \chi_2\sin^2\chi_1 - 2\chi_2 - \chi_2\cos\chi_1, \end{aligned} \quad (5.6)$$

for $t \ge t_0$, with initial conditions

$$\chi_1(t_0) = \chi_{10} \text{ and } \chi_2(t_0) = \chi_{20}.$$

Consider the functions of the form $V = (V_1, V_2)^T$, where $V_1(t,\chi_1,\chi_2) = \chi_1^2$ and $V_2(t,\chi_1,\chi_2) = \chi_2^2$, $\chi = (\chi_1,\chi_2) \in \mathbb{R}^2$, with the associated norm $\|\chi\| = \sqrt{\chi_1^2 + \chi_2^2}$.

Now, the Lyapunov function

$$V_0(t,\chi) = \sum_{i=1}^{2} V_i(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2$$

and so $b(\|\chi\|) \le V_0(t,\chi) \le a(\|\chi\|)$ with $b(r) = r$ and $a(r) = 2r^2$, implying that $a, b \in K$.

From (2.2), we compute the CFrDiDe for $V_1(t,\chi_1,\chi_2) = \chi_1^2$ as follows ${}^{C}D_{+}^{q}V(t,) = \limsup_{k\to 0^+} \frac{1}{k^q}\left\{V(t,\chi) - V(t_0\chi_0) - \sum_{r=0}^{\left[\frac{(t-t_0)}{\kappa}\right]} (-1)^{r+1}\binom{q}{r}\left[V\left(t - r\kappa, \chi - \kappa^q\Psi(t,\chi)\right) - V(t_0\chi_0)\right]\right\}$,

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[(\chi_1 - \kappa^q\Psi_1(t;\chi_1,\chi_2))^2 - \chi_{10}^2]\},$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[\chi_1^2 - 2\chi_1\kappa^q\Psi_1(t;\chi_1,\chi_2) + \kappa^{2q}\Psi_1^2(t;\chi_1,\chi_2) - \chi_{10}^2]\}$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_1^2 - \chi_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - 2\chi_1\Psi_1(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_1^2(t;\chi_1,\chi_2) - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2,$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - \sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_1^2(t;\chi_1,\chi_2) - 2\chi_1\Psi_1(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q,$$

,

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_1^2 - \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{10}^2 - 2\chi_1\Psi_1(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)$$
$$+ \limsup_{\kappa\to 0^+} \kappa^q\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\Psi_1^2(t;\chi_1,\chi_2).$$

Applying equation (3.7) and (3.8) in [1], we have

$${}^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) = \frac{\chi_1^2}{t^q\Gamma(1-q)} - \frac{\chi_{10}^2}{t^q\Gamma(1-q)} + 2\chi_1\Psi_1(t;\chi_1,\chi_2).$$

As $t \to \infty$, $\frac{\chi_1^2}{t^q\Gamma(1-q)} \to 0$, and $\frac{\chi_{10}^2}{t^q\Gamma(1-q)} \to 0$, so that we have

$${}^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) = 2\chi_1\Psi_1(t;\chi_1,\chi_2).$$

Substituting for $\Psi_1(t,\chi_1,\chi_2)$ we have

$${}^{C}D_{+}^{q}\tilde{V}_1(t;\chi_1,\chi_2) = 2\left[\chi_1\left(2\chi_1\cos^2\chi_2 + \frac{\chi_2^2\csc(\chi_1)}{\chi_1\cos\chi_2} - 3\chi_1\sin\chi_1\right)\right]$$
$$= 2\left(2\chi_1^2\cos^2\chi_2 + \frac{\chi_2^2\csc(\chi_1)}{\cos\chi_2} - 3\chi_1^2\sin\chi_1\right),$$

,

$$= 4\chi_1^2\cos^2\chi_2 + 2\frac{\chi_2^2\csc(\chi_1)}{\cos\chi_2} - 6\chi_1^2\sin\chi_1,$$
$$\leq 4\chi_1^2|\cos^2\chi_2| + 2\frac{\chi_2^2|\csc(\chi_1)|}{|\cos\chi_2|} - 6\chi_1^2|\sin\chi_1|$$
$$= 4\chi_1^2 + 2\chi_2^2 - 6\chi_1^2 = -2\chi_1^2 + 2\chi_2^2$$

,

Therefore

$${}^{C}D_{+}^{q}V_1(t;\chi_1,\chi_2) \leq -2V_1 + 2V_2. \qquad (5.7)$$

Similarly, using (2.2), we compute the CFrDiDe for $V_2(t,\chi_1,\chi_2) = \chi_2^2$ as follows

$${}^{C}D_{+}^{q}\tilde{V}_2(t;\chi_1,\chi_2) = \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\tilde{V}(t,\chi) - \tilde{V}(t_0,\chi_0) + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[\tilde{V}(t-r\kappa,\chi-\kappa^q\Psi(t,\chi)) - \tilde{V}(t_0,\chi_0)]\}$$

,

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_2^2 - \chi_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[(\chi_2 - \kappa^q\Psi_2(t;\chi_1,\chi_2))^2 - \chi_{20}^2]\},$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_2^2 - \chi_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)[\chi_2^2 - 2\chi_2\kappa^q\Psi_2(t;\chi_1,\chi_2) + \kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2) - \chi_{20}^2]\}$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\chi_2^2 - \chi_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2 - 2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2) - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2\},$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\{\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2 - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^{2q}\Psi_2^2(t;\chi_1,\chi_2)$$
$$-2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\kappa^q\},$$

$$= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_2^2 - \limsup_{\kappa\to 0^+}\frac{1}{\kappa^q}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\chi_{20}^2$$
$$-2\chi_2\Psi_2(t;\chi_1,\chi_2)\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r) + \limsup_{\kappa\to 0^+}\kappa^q\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^r({}^qC_r)\Psi_2^2(t;\chi_1,\chi_2).$$
,

Applying equation (3.7) and (3.8) in [1], we have

$${}^{C}D_+^q V_2(t;\chi_1,\chi_2) = \frac{\chi_2^2}{t^q\Gamma(1-q)} - \frac{\chi_{20}^2}{t^q\Gamma(1-q)} + 2\chi_2\Psi_2(t;\chi_1,\chi_2).$$

As $t\to\infty$, $\frac{\chi_2^2}{t^q\Gamma(1-q)}\to 0$, and $\frac{\chi_{20}^2}{t^q\Gamma(1-q)}\to 0$, so that we have

$${}^{C}D_+^q V_2(t;\chi_1,\chi_2) \le 2\chi_2\Psi_2(t;\chi_1,\chi_2).$$

Substituting for $\Psi_2(t,\chi_1,\chi_2)$ we have

$$\begin{aligned} {}^{C}D_+^q\tilde{V}_2(t;\chi_1,\chi_2) &= 2\left[\chi_2\left(\chi_2\sin^2\chi_1 - 2\chi_2 - \chi_2\cos\chi_1\right)\right] \\ &= 2\left(\chi_2^2\sin^2\chi_1 - 2\chi_2^2 - \chi_2^2\cos\chi_1\right), \\ &= 2\chi_2^2\sin^2\chi_1 - 4\chi_2^2 - 2\chi_2^2\cos\chi_1, \\ &\le 2\chi_2^2|\sin^2\chi_1| - 4\chi_2^2 - 2\chi_2^2|\cos\chi_1|, \end{aligned},$$
$$= 2\chi_2^2 - 4\chi_2^2 - 2\chi_2^2 = -4\chi_2^2.$$

Therefore

$${}^{C}D_+^q V_2(t;\chi_1,\chi_2) \le -4V_2. \qquad (5.8)$$

Combining (5.7) and (5.8), we have

$${}^{C}D_+V \le \begin{pmatrix}-2 & 2\\ 0 & -4\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix} = \oplus(t,V). \qquad (5.9)$$

Now consider the comparison system

$${}^{C}D^q \oplus (t,u) = Au \qquad (5.10)$$

Where $A = \begin{pmatrix}-2 & 2\\ 0 & -4\end{pmatrix}$.

The vector inequality (5.7) and all other conditions of theorem (4.1) are met if the matrix A has eigenvalues with negative real parts. Consequently, the steady-state solution u = 0 of the system (5.10) is uniformly stable. Therefore, it follows that the steady-state solution $x = 0$ of the system (5.6) is uniformly stable as well.

CONCLUSION

In conclusion, the investigation into the uniform stability of Caputo FrDEs using VLFs has yielded significant insights. This study successfully demonstrates that VLFs offer a robust framework for analyzing the stability of nonlinear FrDEs expanding beyond the limitations of traditional scalar approaches. The derived sufficient conditions for uniform stability not only enhance our theoretical understanding but also provide practical tools for addressing complex stability challenges in fractional systems. The illustrative examples further validate the applicability of the proposed methodology, reinforcing its potential for broader use in the study of nonlinear dynamics within the fractional calculus domain. Overall, this work contributes to the ongoing development of FrDEs by offering a novel and effective approach to stability analysis.

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