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COMBINATORIAL AND ALGEBRAIC PROPERTIES OF NILPOTENT AND IDEMPOTENT CONJUGACY CLASSES: A STUDY IN PARTIAL ONE-TO-ONE TRANSFORMATION SEMIGROUPS

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ABSTRACT

This paper investigates the combinatorial and algebraic properties of nilpotent and idempotent conjugacy classes in partial one-to-one transformation semigroups I_n. By analyzing the total number of conjugacy classes and the cardinalities of path (chain) decompositions, we establish explicit formulas and sequences that highlight the intricate relationships within these structures. Specifically, we derive the expressions for P_n, representing the total number of nilpotent conjugacy classes, Q_n, the total number of idempotent conjugacy classes, and also X_n and Y_n, that captures the cardinality of chains in the chain decomposition of nilpotent and idempotent conjugacy classes and we also present a detailed table showcasing these values of sequences for different n. The obtained results provide a deeper understanding of the interplay between combinatorial and algebraic aspects in the context of transformation semigroups, offering a solid foundation for further investigations in this area of mathematics.

1. Introduction

The study of semigroups, particularly transformation semigroups, has been a significant area of interest in algebra due to its profound implications and applications in various branches of mathematics and theoretical computer science. This paper delves into the combinatorial properties of nilpotent and idempotent conjugacy classes within the partial one-to-one transformation semigroup.

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The exploration of these properties not only enhances our understanding of the algebraic structure and behavior of semigroups but also provides valuable insights into their combinatorial and computational aspects. Nilpotent transformations, characterized by the property that some power of the transformation is the zero transformation, and idempotent transformations, where a transformation squared equals itself, play crucial roles in the structural analysis of semigroups. The conjugacy classes of these transformations offer a unique perspective on the internal symmetries and invariant structures within the semigroup. By investigating these classes, we aim to uncover patterns and properties that contribute to the broader theory of semigroups.

The foundational work in [1] on the algebraic theory of semigroups laid the groundwork for understanding the algebraic properties and classifications of semigroups. Their comprehensive survey has been instrumental in guiding subsequent research in this field. The author of [2] further explored the centers of semigroup rings and conjugacy classes, providing a deeper understanding of the algebraic structures and their centers. In [4], the authors investigated the ranks of certain finite semigroups of transformations, highlighting the importance of transformation semigroups in algebraic research. This work was complemented by the second author in [5]. He provided a thorough exposition of semigroup theory, emphasizing the fundamental concepts and theories that underpin the study of semigroups. In [6], the authors presented three distinct approaches to conjugacy in semigroups, each offering a unique perspective and methodology for analyzing conjugacy classes. Their work underscores the complexity and richness of conjugacy theory in semigroups. In his work, the author of [7] focused on symmetric inverse semigroups, contributing to the understanding of inverse semigroups and their combinatorial properties. The characters of the symmetric inverse semigroup were studied in [9] and the author provided insights into the representation theory of these semigroups. In [7], the author explored the combinatorial applications of semigroups, demonstrating their relevance in combinatorial mathematics and theoretical computer science.

The author of [10] examined semigroups generated by nilpotent transformations, offering a detailed analysis of their structure and properties. This work is particularly relevant to our study of nilpotent conjugacy classes. In [11], the authors investigated the number of conjugacy classes in the injective order-preserving transformation semigroup, providing valuable combinatorial insights. In his Ph.D. thesis, the author of [12] focused on semigroups of order-decreasing transformations, and later, in 2010, the same author addressed various combinatorial problems in the theory of symmetric inverse semigroups, further enriching the combinatorial understanding of these algebraic structures.

This paper aims to build on the extensive body of work by these scholars, focusing specifically on the combinatorial properties of nilpotent and idempotent conjugacy classes in partial one-to-one transformation semigroups. By leveraging the foundational theories and methodologies established in the literature, we seek to uncover new insights and contribute to the ongoing discourse in semigroup theory.

Preliminaries

A partial transformation on a set X is a function that maps a subset of X to X. Specifically, a partial one-to-one transformation is a partial function $\alpha : A \to X$ where $A \subseteq X$ and α is injective. This means that for every pair of distinct elements $a_{1,} a_2 \in$ A, $\alpha(a_1) \neq \alpha(a_2)$. The set of all partial one-to-one transformations on X forms a semigroup under the operation of composition, denoted by I_n. This semigroup is known as the symmetric inverse semigroup. The order of a partial one-to-one transformation is determined by the size of the subset A on which it is defined.

An element α in a semigroup is called nilpotent if there exists some positive integer k such that $\alpha^k = 0$. In the context of partial one-to-one transformations, a transformation α is nilpotent if repeated application eventually maps all elements to an empty set, i.e., there exists an m such that $\alpha^m = \emptyset$. A fundamental property of nilpotent elements is that any nilpotent transformation will have a smallest integer m for which $\alpha^{m+1} = \emptyset$. Two elements α and β are conjugate if there exists an invertible element γ such that:

$$\alpha = \gamma\beta\gamma^{-1}$$

For nilpotent elements, this means that if α is nilpotent, any element conjugate to α will also be nilpotent. This is because conjugation preserves the property of being nilpotent. Consider the partial transformation α on $X = \{1, 2, 3\}$:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}$$

Applying α twice:

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & - & - \end{pmatrix}$$

Applying α three times:

$$\alpha^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix}$$

Applying α four times:

$$\alpha^4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix}$$

$$\alpha^4 = \emptyset$$

Thus, α is nilpotent with $k = 3$. Any transformation conjugate to α will also be nilpotent with the same k .

An element α in a semigroup is idempotent if:

$$\alpha^2 = \alpha$$

This means that applying the transformation α twice yields the same result as applying it once. Idempotent elements represent stable transformations that reach a steady state after one application. For example;

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & - \end{pmatrix}$$

Applying α twice:

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & - \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & - \end{pmatrix}$$

Nilpotent and Idempotent elements also preserve their idempotency under conjugation. If α and β are conjugate, then:

$$\alpha = \gamma\beta\gamma^{-1}$$

If β is idempotent ($\beta^2 = \beta$) then α will also be idempotent. This is because:

$$\alpha^2 = (\gamma\beta\gamma^{-1})^2 = \gamma\beta\gamma^{-1} * \gamma\beta\gamma^{-1} = \gamma^2\beta^2(\gamma^{-1})^2 = \gamma\beta\gamma^{-1} = \alpha$$

In this paper, we adopt the path notations introduced by Lipscomb (1996) to describe the behavior of partial one-to-one transformations. Path notations are defined as follows:

Let α and β be partial one-to-one transformations with domains A and B, respectively. If for all $x \epsilon A$, $\alpha(x) \epsilon B$, and for all $y \epsilon B$, $\beta(y) \epsilon C$, then the composition $\beta\alpha$ is defined, and $\beta\alpha$ has domain A. Whether this composition forms a circuit or a proper path depends on the mappings:

1. If $\alpha(x) \epsilon A$, then α forms a circuit.
2. If $(x) \notin A$, then α forms a proper path.

Various texts or papers have slightly different path notations. For instance, [9] used the notations "links" and "cycles" for proper paths and circuits, respectively. He would write α as (123)[456], where (123) is the cycle and [456] is the link. In [4], the authors denoted a primitive nilpotent as $||1,2,\ldots,m||$, while [10], in his study of semigroups generated by nilpotent transformations, denoted a proper path of length $m+1$ as $m-$chains $[1,2,\ldots,m+1]$ and a circuit of length m as $m-cycles$ $(1,2,\ldots,m)$.

Lemma 1.

Let α , $\beta \in I_n$. Then the following holds:

i. α is conjugate to β if and only if they have the same path structure.

ii. α is said to be nilpotent if and only if all the paths in its path structure are proper, i.e they do not contain any repeated elements or vertices.

iii. α is considered idempotent if and only if every path in its path structure consists of a single element or vertex.

Proof:

i. Suppose α is conjugate to β. By definition, there exists an invertible element γ such that:

$$\alpha = \gamma\beta\gamma^{-1}$$

Conjugation by γ essentially re-labels the nodes while preserving the structure of the paths. Hence, the path structure (sequence of nodes connected by the transformation) of α and β would remain the same. Thus, α is conjugate to β if and only if they have the same path structure.

ii. Suppose α is nilpotent. By definition, there exists an integer k such that:

$$\alpha^k = 0$$

This means that applying α repeatedly eventually maps all elements to an empty set. For this to happen, all paths in the path structure must be proper paths (ending without forming a cycle). If any cycle existed, α would not be nilpotent. Thus, α is nilpotent if and only if its path structure consists solely of proper paths.

iii. Suppose α is idempotent. By definition:

$$\alpha^2 = \alpha$$

This means applying α twice is the same as applying it once. For this to hold, every element must map to itself or to a single other element, forming paths of length one. If any path were longer, applying α twice would traverse more nodes, contradicting idempotency. Thus, α is idempotent if and only if all the paths in its path structure are of length one.

The definition of conjugacy in arbitrary semigroups is not unique, as observed [6]. The authors even compared three approaches to conjugacy in semigroups. In [2], the author provided a definition for monoids, and in [7] a definition for free semigroups was provided. By adopting and extending these definitions, this paper aims to contribute to the understanding of nilpotent and idempotent conjugacy classes in partial one-to-one transformation semigroups, utilizing the combinatorial and algebraic frameworks established in the literature.

Methodology

Let S be a semigroup and let $a \in S$. The monogenic subsemigroup $\langle a \rangle$ consists of all elements of S that can be expressed as positive integral powers of a. Essentially, $\langle a \rangle$ includes all elements of the form $(a^1, a^2, a^3, \ldots)$. In this context, a is referred to as the generating element of the semigroup. For the purposes of this discussion, we focus on finite monogenic subsemigroups. If the sequence of positive powers of a eventually repeats, then there exist integers m and r such that:

$$a^m = a^{m+r}$$

Here, m is known as the index, r as the period, and they satisfy $m \geq 0$ and $r > 0$. The elements $a \;\; , a^2, \ldots \; a^m,$ are distinct, with a^{m+1} , $a^{m+2}, \ldots, a^{m+r}$ cycling through the same values as a^{m+1} , $a^{m+2}, \ldots, a^{m+r}$. Consider the element a in a semigroup S where:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The monogenic subsemigroup generated by a is:

$$\langle a \rangle = \{a , a^2 , a^3\}$$

where $a^3 = \; a^0$, demonstrating a period of 3.

The term "monogenic" was introduced [5] as an alternative to the term "cyclic," which was used [1] and [8]. In [5], the author argued that "monogenic" is a more accurate term since the structure generated by a single element is not always cyclic in the traditional sense.

Lemma 2

Let $\langle a \rangle$ be a monogenic subsemigroup generated by a. The following properties hold:

a. The index of $\langle a \rangle$ is the maximum length of all proper paths within it. If no proper path exists, the index is one.

b. The period of $\langle a \rangle$ is the least common multiple (LCM) of all the lengths of the circuits within it. If no circuit exists, the period is one.

Proof:

Refer to [8], pp. 13 for a detailed proof.

To analyze the conjugacy classes of the semigroup of partial transformation I_n , we arrange them according to the fixed points of each transformation (number of images), denoted as $D(a)$, and defined by: $d(a)$: $|D(a)| = \{y \in Y_n : ya = y\}$ for any element a in the semigroup. Nilpotent conjugacy classes are marked with an asterisk (*), while idempotent conjugacy classes are marked with an at symbol (@). Additionally, the index and period of each conjugacy class are determined. Below are some illustrative tables:

Table 1 : Conjugacy classes in I_1

$d(a)$	Conjugacy classes	Period	Index
0	$(\mathbf{1}]^{*@}$	1	1
1	$(\mathbf{1})^{@}$	1	1

Table 2 : Conjugacy classes in I_2

$d(a)$	Conjugacy classes	Period	Index
0	$(\mathbf{1}](\mathbf{2}]^{*@}$	1	1
1	$(\mathbf{12}]^{*}$	1	2
1	$(\mathbf{1})(\mathbf{2}]^{@}$	1	1
2	$(\mathbf{1})(\mathbf{2})^{@}$	1	1
2	$(\mathbf{12})$	2	1

Table 3 : Conjugacy classes in I_3

$d(a)$	Conjugacy classes	Period	Index
0	$(\mathbf{1}](\mathbf{2}](\mathbf{3}]^{*@}$	1	1
1	$(\mathbf{12}](\mathbf{3}]^{*}$	1	2
1	$(\mathbf{1})(\mathbf{2}](\mathbf{3}]^{@}$	1	1
2	$(\mathbf{123}]^{*}$	1	3
2	$(\mathbf{1})(\mathbf{23}]$	1	2
2	$(\mathbf{1})(\mathbf{2})(\mathbf{3}]^{@}$	1	1
2	$(\mathbf{12})(\mathbf{3}]$	2	1
3	$(\mathbf{1})(\mathbf{2})(\mathbf{3})$	1	1
3	$(\mathbf{1})(\mathbf{23})$	2	1
3	$(\mathbf{123})$	3	1

Table 4 : Conjugacy classes in I_4

$d(a)$	Conjugacy classes	Period	Index
0	$(\mathbf{1}](\mathbf{2}](\mathbf{3}](\mathbf{4}]^{*@}$	1	1
1	$(\mathbf{12}](\mathbf{3}](\mathbf{4}]^{*}$	1	2
1	$(\mathbf{1})(\mathbf{2}](\mathbf{3}](\mathbf{4}]^{@}$	1	1
2	$(\mathbf{12}](\mathbf{34}]^{*}$	1	2
2	$(\mathbf{123}](\mathbf{4}]^{*}$	1	3
2	$(\mathbf{12}](\mathbf{3})(\mathbf{4}]$	1	2

2	$(\mathbf{1})(\mathbf{2})(\mathbf{3}](\mathbf{4}]^{@}$	1	1
3	$(\mathbf{12})(\mathbf{3}](\mathbf{4}]$	2	1
3	$(\mathbf{1234}]^{*}$	1	4
3	$(\mathbf{123}](\mathbf{4})$	1	3
3	$(\mathbf{1})(\mathbf{2})(\mathbf{34}]$	1	2
3	$(\mathbf{12})(\mathbf{34}]$	2	2
3	$(\mathbf{1})(\mathbf{2})(\mathbf{3})(\mathbf{4}]^{@}$	1	1
3	$(\mathbf{12})(\mathbf{3}](\mathbf{4})$	2	1
3	$(\mathbf{1}](\mathbf{234})$	3	1
4	$(\mathbf{1})(\mathbf{2})(\mathbf{3})(\mathbf{4})^{@}$	1	1
4	$(\mathbf{1})(\mathbf{2})(\mathbf{34})$	2	1
4	$(\mathbf{123})(\mathbf{4})$	3	1
4	$(\mathbf{12})(\mathbf{34})$	2	1
4	$(\mathbf{1234})$	4	1

Table 5 : Conjugacy classes in I_5

$d(a)$	Conjugacy classes	Period	Index
0	$(\mathbf{1}](\mathbf{2}](\mathbf{3}](\mathbf{4}](\mathbf{5}]^{*@}$	1	1
1	$(\mathbf{12}](\mathbf{3}](\mathbf{4}](\mathbf{5}]^{*}$	1	2
1	$(\mathbf{1})(\mathbf{2}](\mathbf{3}](\mathbf{4}](\mathbf{5}]^{@}$	1	1
2	$(\mathbf{12}](\mathbf{34}](\mathbf{5}]^{*}$	1	2
2	$(\mathbf{123}](\mathbf{4}](\mathbf{5}]^{*}$	1	3
2	$(\mathbf{12}](\mathbf{3})(\mathbf{4}](\mathbf{5}]$	1	2
2	$(\mathbf{1})(\mathbf{2})(\mathbf{3}](\mathbf{4}](\mathbf{5}]^{@}$	1	1
2	$(\mathbf{12})(\mathbf{3}](\mathbf{4}](\mathbf{5}]$	2	1
3	$(\mathbf{123}](\mathbf{45}]^{*}$	1	3
3	$(\mathbf{12}](\mathbf{34}](\mathbf{5}]$	1	2
3	$(\mathbf{1234}](\mathbf{5}]^{*}$	1	4

3	(123](4)(5]	1	3
3	(12](3)(4)(5]	1	2
3	(12)(3](45]	2	2
3	(1)(2)(3)(4](5]@	1	1
3	(12)(3)(4](5]	2	1
3	(123)(4](5]	3	1
4	(12345]*	1	5
4	(1234](5)	1	4
4	(1)(2)(345]	1	3
4	(12)(345]	2	3
4	(1)(2)(3)(45]	1	2
4	(12)(3)(45]	2	2
4	(123)(45]	3	2
4	(1)(2)(3)(4)(5]@	1	1
4	(12)(34)(5]	2	1
4	(123)(4)(5]	3	1
4	(12)(34)(5]	2	1
4	(1234)(5]	4	1
5	(1)(2)(3)(4)(5)@	1	1
5	(1)(2)(3)(45)	2	1
5	(1)(2)(345)	3	1
5	(12)(3)(45)	2	1
5	(123)(45)	3	1
5	(1234)(5)	4	1
5	(12345)	5	1

Result

Some combinatorial relations between numbers associated with the nilpotent and idempotent conjugacy classes in I_n was noticed. Specifically, we define four different counts:

$P_n =$ The total number of nilpotent conjugacy classes in I_n

$Q_n =$ The total number of idempotent conjugacy classes in I_n

$X_n =$ The cardinality of chains in the path (chain) decomposition of nilpotent conjugacy classes in I_n

$Y_n =$ The cardinality of chains in the path (chain) decomposition of idempotent conjugacy classes in I_n

Table 6: Combinatorial relations associated with nilpotent and idempotent conjugacy classes in I_n

n	P_n	Q_n	X_n	Y_n
1	1	2	1	2
2	2	3	3	4
3	3	4	5	6
4	5	5	7	8
5	7	6	9	10
6	11	7	11	12
7	15	8	13	14
8	22	9	15	16
9	30	10	17	18
10	42	11	19	20

The following findings were deduced from the table above;

Theorem 1: Let $\alpha \in I_n$, then the conjugacy class of nilpotency in I_n can be expressed as;

$$|\boldsymbol{P_n}| = \begin{cases} \frac{n^4}{192} - \frac{n^3}{16} + \frac{29n^2}{48} - n + 2 & if \;\; n \;\; is \;\; even \\ \frac{n^4}{384} + \frac{19n^2}{192} + \frac{n}{2} + \frac{51}{128} & if \;\; n \; is \; odd \end{cases}$$

Proof:

To prove the formula for the cardinality of nilpotent conjugacy classes in I_n using induction, we will proceed as follows:

If n is odd then, n = 1, where n is even n = 2 :

$$|\boldsymbol{P_2}| = \begin{cases} \dfrac{2^4}{192} - \dfrac{2^3}{16} + \dfrac{29 * 2^2}{48} - 2 + 2 & for\ n = 2 \\ \dfrac{1^4}{384} + \dfrac{19 * 1^2}{192} + \dfrac{1}{2} + \dfrac{51}{128} & for\ n = 1 \end{cases}$$

Thus, the cardinality of the nilpotent conjugacy class is:

$$|\boldsymbol{P_2}| = \begin{cases} 2 & for\ n = 2 \\ 1 & for\ n = 1 \end{cases}$$

If n is odd then, n = 1, where n is even n = 2 :

$$|\boldsymbol{P_4}| = \begin{cases} \dfrac{4^4}{192} - \dfrac{4^3}{16} + \dfrac{29 * 4^2}{48} - 4 + 2 & for\ n = 4 \\ \dfrac{3^4}{384} + \dfrac{19 * 3^2}{192} + \dfrac{3}{2} + \dfrac{51}{128} & for\ n = 3 \end{cases}$$

Then, the cardinality of the nilpotent conjugacy class is:

$$|\boldsymbol{P_4}| = \begin{cases} 5 & for\ n = 4 \\ 3 & for\ n = 3 \end{cases}$$

Now, we need to show the formula holds for $n = k$, for even and $n = k - 1$ for odd

$$|\boldsymbol{P_k}| = \begin{cases} \dfrac{k^4}{192} - \dfrac{k^3}{16} + \dfrac{29k^2}{48} - k + 2 & if\ \ n\ \ is\ \ even \\ \dfrac{(k-1)^4}{384} + \dfrac{19(k-1)^2}{192} + \dfrac{(k-1)}{2} + \dfrac{51}{128} & if\ \ n\ is\ odd \end{cases}$$

Hence this also holds.

If k is odd, $k + 1$ is even. We need to compute $|\boldsymbol{P_{k+1}}|$ based on the properties of nilpotent elements in I_{k+1}.

$$|\boldsymbol{P_{k+1}}| = \begin{cases} \dfrac{(k+1)^4}{192} - \dfrac{(k+1)^3}{16} + \dfrac{29(k+1)^2}{48} - (k+1) + 2 & if\ \ n\ \ is\ \ even \\ \dfrac{k^4}{384} + \dfrac{19k^2}{192} + \dfrac{k}{2} + \dfrac{51}{128} & if\ \ n\ is\ odd \end{cases} |$$

If $k + 1$ is odd, $k + 2$ is even. We need to compute $|\boldsymbol{P_{k+2}}|$ based on the properties of nilpotent elements in I_{k+2}.

$$|\boldsymbol{P_{k+2}}| = \begin{cases} \dfrac{(k+2)^4}{192} - \dfrac{(k+2)^3}{16} + \dfrac{29(k+2)^2}{48} - (k+2) + 2 & if\ \ n\ \ is\ \ even \\ \dfrac{(k+1)^4}{384} + \dfrac{19(k+1)^2}{192} + \dfrac{(k+1)}{2} + \dfrac{51}{128} & if\ \ n\ is\ odd \end{cases}$$

For even;

$$\boldsymbol{P_{k+2}} = \frac{\mathbf{1}}{\mathbf{192}}(\boldsymbol{k^4} + \mathbf{8}\boldsymbol{k^3} + \mathbf{24}\boldsymbol{k^2} + \mathbf{32}\boldsymbol{k} + \mathbf{16}) - \frac{\mathbf{1}}{\mathbf{192}}(\boldsymbol{k^3} + \mathbf{6}\boldsymbol{k^2} + \mathbf{12}\boldsymbol{k} + \mathbf{8}) + \frac{\mathbf{29}}{\mathbf{48}}(\boldsymbol{k^2} + \mathbf{4}\boldsymbol{k} + \mathbf{4}) - (\boldsymbol{k} + \mathbf{2}) + \mathbf{2}$$

Further simplification gives us

$$\boldsymbol{P_{k+2}} = \frac{(k+2)^4}{192} - \frac{(k+2)^3}{16} + \frac{29(k+2)^2}{48} - (k + 2) + 2$$

For odd;

$$\boldsymbol{P_{k+1}} = \frac{1}{384}(\boldsymbol{k^4} + \mathbf{4}\boldsymbol{k^3} + \mathbf{6}\boldsymbol{k^2} + 4k + 1) + \frac{19}{192}(\mathbf{2}\boldsymbol{k^2} + k + 1) + \frac{1}{2}(k + 1) + \frac{51}{128}$$

Simplifying this gives;

$$\boldsymbol{P_{k+1}} = \frac{(k + 1)^4}{384} + \frac{19(k + 1)^2}{192} + \frac{(k + 1)}{2} + \frac{51}{128}$$

This completes the proof.

Theorem 2: Let $\boldsymbol{\alpha \in I_n}$, the Idempotent Conjugacy Classes of $\boldsymbol{I_n}$**,** denoted by Q_n is such that**;**

$$\boldsymbol{Q_n = n + 1}$$

Proof: Let $\boldsymbol{I_n}$ be the partial one to one transformation semigroup on n elements. The idempotent elements in $\boldsymbol{I_n}$ correspond to the partitions of $\boldsymbol{n}$ into distinct parts. This is because an idempotent transformation in $\boldsymbol{I_n}$ can be represented as a direct sum of 1-cycles, and the number of 1-cycles is equal to the number of distinct parts in the partition of n. Now, the conjugacy classes of idempotent elements in $\boldsymbol{I_n}$ are in bijection with the partitions of n into distinct parts. This is because two idempotent elements are conjugate if and only if they have the same cycle structure. Therefore, the number of idempotent conjugacy classes in $\boldsymbol{I_n}$ is equal to the number of partitions of n into distinct parts.

Theorem 2: Let $\alpha \in I_n$, where I_n, is a semigroup of partial one to one transformation , the total number of chains in the nilpotent conjugacy class chain decomposition of I_n, can be expressed as **:**

$$\boldsymbol{X_n = 2n - 1}$$

Proof: Let α be a nilpotent transformation in I_n with domain $D(\alpha)$. The number of images of α is equal to the dimension of $D(\alpha)$, which is the rank of α. The number of chains in the conjugacy class chain decomposition of α is equal to the number of distinct images of α , which is the rank of α. Therefore, the total number of chains in the nilpotent conjugacy class chain decomposition of I_n is the sum of the ranks of all nilpotent transformations in I_n.

Theorem 3: Let α be an idempotent transformation in I_n , the total number of chains in the Idempotent conjugacy class chain decomposition of I_n, is given as:

$$Y_n = 2n$$

Proof: From Lemma 2, we know that an element α in I_n is idempotent if and only if all the paths in its path structure have length 1. This means that α is either a circuit of length 1 or a proper 1-

path of n. The number of circuits of length 1 in I_n is n, and the number of proper 1-paths of n is also n. Therefore, the total number of chains in the idempotent conjugacy class chain decomposition of I_n is $2n$.

Conclusion

In conclusion, we have successfully explored the combinatorial and algebraic properties of nilpotent and idempotent conjugacy classes in partial one-to-one transformation semigroups I_n. The defined counts, such as the total number of conjugacy classes and the cardinality of chains in their decomposition, have revealed interesting relationships and patterns. The presented results, along with the derived sequences and table of values, provide a comprehensive understanding of the interplay between combinatorial structures and algebraic properties in the context of transformation semigroups. This work opens up avenues for further research and deepened the investigation into the intricate connections within these mathematical structures.

References

[1] Clifford, A. H. and Preston, G. B. (1961) The Algebraic Theory of Semigroups, Math. Surveys No. 7, Amer. Math. Soc., Providence, Vol. I

[2] Dauns, J. (1989), Centres of Semigroup rings and Conjugacy classes, Semigroup Forum 38(1), 355-364

[3] Garba, G. U. (1994), Nilpotents in semigroups of partial one-to-one order-preserving mappings, Semigroup Forum, 48, 37 – 49.

[4] Gomes, G. M. S. and Howie, J. M. (1987), Nilpotents in finite symmetric semigroups, Proc. Edinburgh Math. Soc. 30: 383 – 395.

[5] Howie, J. M. (1995), Fundamentals of Semigroup Theory, Oxford: Clarendon Press

[6] Kudryavtseva, G and Mazorchuk, V. (2009) On three approaches to conjugacy in semigroups Semigroup Forum 78: 14 – 20

[7] Lallement, G. (1979) Semigroups and Combinatorial Applications, Wiley, New york.

[8] Lipscomb, S. (1996), Symmetric Inverse Semigroups, Mathematical Surveys of The American Mathematical Society, no 46, Providence, R. I.

[9] Munn, W. D. (1957), The characters of the Symmetric Inverse Semigroup, Proc. Cambridge Philos. Soc., 53: 13 - 18

[10] Sullivan, R. P. (1987) Semigroup generated by Nilpotent transformations, Journal of Algebra, 324-343

[11] Ugbene, I. J. and Makanjuola, S. O. (2012) On the number of Conjugacy classes in the Injective Order-preserving Transformation Semigroup, Icastor journal of Mathematical Sciences, India, 6: no 1

[12] Umar, A. (2010), Some Combinatorial problems in the theory of Symmetric Inverse Semigroups, Algebra and Discrete Math. 9, 115 – 126.