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ESTIMATES OF SECOND AND THIRD HANKEL DETERMINANTS FOR ANALYTIC FUNCTIONS ASSOCIATED WITH THE POISSON DISTRIBUTION

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ABSTRACT

In this work, certain new subclasses of analytic p-valent functions are defined using the generalized Poisson distribution. The sharp bounds for the second and third order Hankel determinants $H_2(2)$ *and* $\mathrm{H}_3(1)$ *respectively, including the Fekete-Szego functional belonging to the aforementioned subclasses of analytic function are also established..*

1. Introduction

Let M denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \qquad (1)$$

which are analytic and univalent in the open unit disk $U = \{z : z \in C, |z| < 1\}$, and let $S \in \mathrm{M}$ consists of univalent functions in U normalized with $f(0) = f'(0) - 1 = 0$.

Also, let $A_p(\omega)$ denote the class of normalized functions of the form

$$f_p(z) = (z-\omega)^p + \sum_{k=1}^{\infty} a_{k+p}(z-\omega)^{k+p},\ p \in N, \qquad (2)$$

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which are analytic and p-valent in the open unit disk $U=\{z: z\in C, |z|<1\}$ and normalized by $f(\omega)=0$ and $f'(\omega)=1$, where ω is arbitrarily fixed in U (see [4]and [5] among others). The analytic function $f(z)$ of the form (2) is said to be starlike and convex of order $\alpha(0\le\alpha<1)$ provided the following geometric conditions are respectively satisfied:

$$\mathrm{Re}\left\{\frac{(z-\omega)f_p'(z)}{f(z)}\right\}>\alpha$$

and

$$\mathrm{Re}\left\{1+\frac{(z-\omega)f_p''(z)}{f'(z)}\right\}>\alpha.$$

Wald in 1978, [3], introduced the class $p(\omega)\subset \mathrm{P}$, where P is the famous class of Caratheodory function, such that

$$p(z)=1+\sum_{k=1}^{\alpha}c_k(z-\omega)^k, z\in U \tag{3}$$

which are analytic in U and satisfy the conditions: $p(\omega)=1$ and $\mathrm{Re}\, p(z)>0$, where

$$|c_k|\le\frac{2}{(1+d)(1-d)^k}, k\ge 1 \text{ and } |\omega|=d. \tag{4}$$

see [3] and [5] among others.

A power series whose coefficients are probability of Poisson distribution was introduced by Porwal in 2014 [1], such that

$$X(m,z)=z+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}z^k,\quad z\in U,\quad m>0 \tag{5}$$

while the probability mass function of poisson distribution is given by,

$$P(x)=\frac{\lambda^x e^{-\lambda}}{x!}$$

$\lambda = parameter$, $x=0,1,2,3...$ see [2]. However, the radius of convergence of the series $X(m,z)$ by ratio test is infinity.

Furthermore, [1] also defined a series T(m,z) as follows:

$$T(m,z)=2z-X(m,z)=z-\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}z^k, z\in U. \tag{6}$$

Now, we recall that the Hadamard product of two analytic functions f and u denoted by $(f*u)(z)$ is given by

$$(f*u)(z)=(z-\omega)^p+\sum_{k=2}^{\infty}a_{k+p}b_{k+p}(z-\omega)^{k+p}=(u*f)(z) \tag{7}$$

where f is as defined in (2) and

$$u(z)=(z-\omega)^p+\sum_{k=2}^{\infty}b_{k+p}(z-\omega)^{k+p},\quad [7].$$

In this work, (6) is redefined such that

$$W_p(m,z)=(z-w)^p-\sum_{k=1}^{\infty}\frac{m^{k+p-1}}{(k+p-1)!}e^{-m}(z-\omega)^{k+p} \tag{8}$$

and

$$f_p(z) = (z-\omega)^p - \sum_{k=1}^{\infty} a_{k+p}(z-\omega)^{k+p} \qquad (9)$$

where $p \in \mathrm{N}, \omega \neq z, z \in C$.

Let $V(z): \mathrm{A} \to \mathrm{A}$ be a linear operator defined by

$$\mathrm{H}_p(z) = W_p(m,z) * f_p(z)$$
$$= (z-\omega)^p - \sum_{k=1}^{\infty} \frac{m^{k+p-1}}{(k+p-1)!} e^{-m} a_{k+p}(z-\omega)^{k+p} \qquad (10)$$

which is the Hadamard product of (8) and (9).

Pommerenke [6] in 1966 defined the q[th] Hankel determinant of f for $q \geq 1$ and $n \geq 1$ as

$$\mathrm{H}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+a} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+zq-z} \end{vmatrix} \qquad (11)$$

Several authors have considered this determinant and interesting useful results had been established, [7], [8], [9]. However, none of these results discussed analytic p-valent functions associated with the poisson distribution in relation to the Hankel determinants. Hence, the motivation for the present investigation.

With various choices of parameters n and q, several functional are obtained. For example:

(i) Let q=2, n=1, we obtain

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|, (a_1 = 1) \qquad (12)$$

(ii) If q=3, n=2, we have

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \qquad (13)$$

(iii) Letting q=3 and n=1, we have

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2) \qquad (14)$$

Applying the triangle inequality in (14), we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2|. \qquad (15)$$

Definition 1.1: Let the function $f_p(z)$ be of the form (9) and belong to the class $S_p(m,\beta)$ then for $0 \leq \beta < 1$,

$$\mathrm{Re}\left\{\frac{(z-\omega)H_p^{'}(z)}{H_p(z)}\right\} > \beta$$

Definition 1.2: Let the function $f_p(z)$ be of the form (9) and belong to the class $C^*{}_p(m,\beta)$ then for $0 \le \beta < 1$,

$$\mathrm{Re}\left\{1+\frac{(z-\omega)H_p^{''}(z)}{H^{'}{}_p(z)}\right\} > \beta$$

1. Preliminary Lemma

Lemma 2.1: Let $c(z)=c_1 z + c_2 z^2 + \ldots \in \Omega$ be so that $|\omega(z)|<1$ in U. If ρ is a complex number, then

$$\left|c_3 - tc_2^2\right| < \max\{1,|t|\}$$

The sharp inequality for the function c(z)=z or c(z)=z^2, (see [14])

Lemma 2.2: Let $p \in P$ and be of the form (3). Then for all $n \in N = \{1,2,3,\ldots\}$

$$|c_n| \le 2.$$

2. Main Results

In this section, the results and some of the consequences are discussed.

Theorem 3.1

Let $f_p(z) \in S_p(m,\beta)$, then for $m>0, 0 \le \beta < 1, p \in N, 0 \le d < 1$

$$|a_{p+1}a_{p+2} - a_{p+3}| \le \frac{(p-\beta)(p+1)!}{3m^{2p=2}e^{-2m}(1-d^2)^3} K$$

where

$$K = \left|6mp!(p-\beta)\left[(p-\beta)-(1+d)-m^p e^{-m}(p+2)[2(p-\beta)e^{-m}[(p-\beta)-(1+d)]-2(1+d)[2(p-\beta)-(1+d)e^{-m}]]\right]\right|.$$

Proof

Let $f_p(z) \in S_p(m,\beta)$, by definition(1.1), we have

$$\frac{\dfrac{(z-\omega)\mathrm{H}_p^{'}(z)}{\mathrm{H}_p(z)}}{(p-\beta)} = p(z)$$

this implies that

$$\frac{(p-\beta)(z-\omega)^p - \sum_{k=1}^{\infty}(k+p-\beta)\dfrac{m^{k+p-1}}{(k+p-1)!}e^{-m}a_{k+p}(z-\omega)^{k+p}}{(p-\beta)(z-\omega)^p - \sum_{k=1}^{\infty}(p-\beta)\dfrac{m^{k+p-1}}{(k+p-1)!}e^{-m}a_{k+p}(z-\omega)^{k+p}} = p(z) \qquad (16)$$

$$= 1 + c_1(z-\omega) + c_2(z-\omega)^2 + c_3(z-\omega)^3 + c_4(z-\omega)^4 + \ldots$$

Simple computation from (16) yields

$$1-\left(\frac{1+p-\beta}{p-\beta}\right)\frac{m^p}{p!}e^{-m}a_{p+1}(z-\omega)-\left(\frac{2+p-\beta}{p-\beta}\right)\frac{m^{p+1}}{(p+1)!}e^{-m}a_{p+z}(z-\omega)^2-\left(\frac{3+p-\beta}{p-\beta}\right)\frac{m^{p+2}}{(p+2)!}a_{p+3}(z-\omega)^3$$
$$-\left(\frac{4+p-\beta}{p-\beta}\right)\frac{m^{p+3}}{(p+3)!}e^{-m}a_{p+4}(z-\omega)^4-\left(\frac{5+p-\beta}{p-\beta}\right)\frac{m^{p+4}}{(p+4)!}e^{-m}a_{p+5}(z-\omega)^5+\ldots$$
$$=1-\frac{m^p}{p!}e^{-m}a_{p+1}(z-\omega)-\frac{m^{p+1}}{(p+1)!}e^{-m}a_{p+2}(z-\omega)^2-\frac{m^{p+2}}{(p+2)!}e^{-m}a_{p+3}(z-\omega)^3-\frac{m^{p+3}}{(p+3)!}e^{-m}a_{p+4}(z-\omega)^4$$
$$-\frac{m^{p+4}}{(p+4)!}e^{-m}a_{p+5}(z-\omega)^5+a(z-\omega)-c_1\frac{m^p}{p!}e^{-m} \tag{17}$$
$$a_{p+1}(z-\omega)^2-c_1\frac{m^{p+1}}{(p+1)!}e^{-m}a_{p+2}(z-\omega)^3-\frac{c_1m^{p+2}}{(p+2)!}e^{-m}a_{p+3}(z-\omega)^4-c_1\frac{m^{p+3}}{(p+3)!}e^{-m}a_{p+4}(z-\omega)^5+c_2(z-\omega)^2$$
$$-c_2\frac{m^p}{p!}e^{-m}a_{p+1}(z-\omega)^3-c_2\frac{m^{p+1}}{(p+1)!}e^{-m}a_{p+2}(z-\omega)^4-\frac{c_2m^{p+2}}{(p+2)!}e^{-m}a_{p+3}(z-\omega)^5+c_3(z-\omega)^3$$
$$-c_3\frac{m^p}{p!}e^{-m}a_{p+1}(z-\omega)^4-c_4\frac{m^p}{p!}e^{-m}a_{p+1}(z-\omega)^5+c_5(z-\omega)^5+\ldots$$

Comparing the coefficients of $(z-\omega)'s$ in (17), we obtain

$$a_{p+1}=\frac{2p!(p-\beta)}{m^pe^{-m}(1-d^2)} \tag{18}$$
$$a_{p+2}=\frac{(p-\beta)(p+1)![(p-\beta)-(1+d)]}{m^{p+1}e^{-m}(1-d^2)^2} \tag{19}$$
$$a_{p+3}=\frac{(p-\beta)(p+3)!}{3m^{p+3}e^{-m}(1-d^2)^3}[2(p-\beta)e^{-m}[(p-\beta)-(1+d)]-2(1+d)[2(p-\beta)+(1+d)e^{-m}]] \tag{20}$$
$$a_{p+4}=\frac{(p-\beta)(p+3)!}{2m^{p+3}e^{-m}(1-d^2)^4}\left[\begin{array}{l}[2(p-\beta)e^{-m}[(p-\beta)-(1+d)]-2(1+d)[2(p-\beta)-(1+d)e^{-m}]]+(p-\beta)(1+d)\\ [(p-\beta)-2(1+d)]-(1+d)^3\end{array}\right] \tag{21}$$

The desired result is obtained using equations (18),(19) and (20).

Corollary 3.1.1:

For $m>0, 0\le\beta<1, d=0, p\in N$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right|\le\frac{(p-\beta)(p+1)!}{3m^{2p+2}e^{-2m}}\left\{(p-\beta)^2[6mp!-2m^pe^{-2m}(p+2)]+(p-\beta)[2m^pe^{-2m}(p+2)-6mp!-4]-2e^{-m}\right\}$$

Corollary 3.1.2:

For $m>0, \beta=0, d\ne1, p\in N$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right|\le\frac{(p-\beta)(p+1)!}{3m^{2p+2}e^{-2m}(1-d^2)^3}\left\{\begin{array}{l}6mp!\,p(p-1+d)-m^pe^{-m}(p+2)[2pe^{-m}(p-1+d)]\\ -(1+d)[2p-(1+d)e^{-m}]\end{array}\right\}$$

Corollary 3.1.3:

For $m=1, 0\le\beta<1, d\ne1, p\in N$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right|\le\frac{(p-\beta)(p+1)!}{3e^{-2}(1-d^2)^3}\left\{\begin{array}{l}6p!(p-\beta)[(p-\beta)-(1+d)]-e^{-1}(p+2)[2(p-\beta)e^{-1}((p-\beta)-(1+d)]\\ -2(1+d)[2(p-\beta)-(1+d)e^{-1}]\end{array}\right\}$$

Corollary 3.1.4:

For $m>0, 0\le\beta<1, d\ne 1, p=1$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_2a_3-a_4|\le\frac{2(p-\beta)}{3m^4e^{-2m}(1-d^2)^3}\begin{Bmatrix}6m(1-\beta)[(1-\beta)-(1+d)]-3me^{-m}[2(1-\beta)e^{-m}[(1-\beta)-(1+d)]]\\-2(1+d)[2(1-\beta)-(1+d)e^{-m}]\end{Bmatrix}$$

Corollary 3.1.5:

For $m=1, 0\le\beta<1, d=0, p\in N$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\frac{(p-\beta)(p+1)!}{3e^{-2}}\begin{Bmatrix}6p!(p-\beta)[(p-\beta)-1]-e^{-1}(p+2)[2(p-\beta)e^{-1}((p-\beta)-1)]\\-2[2(p-\beta)-e^{-m}]\end{Bmatrix}$$

Corollary 3.1.6:

For $m=1, \beta=0, d\ne 1, p\in N$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\frac{p(p+1)!}{3e^{-2}(1-d^2)^3}\begin{Bmatrix}6p!\,p[p-(1+d)]-e^{-1}(p+2)[2pe^{-1}(p-(1+d))]\\-2(1+d)[2p-(1+d)e^{-1}]\end{Bmatrix}$$

Corollary 3.1.7:

For $m=1, 0\le\beta<1, d\ne 1, p=1$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_2a_3-a_4|\le\frac{2(1-\beta)}{3e^{-2}(1-d^2)^3}\begin{Bmatrix}6(1-\beta)[(1-\beta)-(1+d)]-3e^{-1}[2(1-\beta)e^{-1}[(1-\beta)-(1+d)]]\\-2(1+d)[2(1-\beta)-(1+d)e^{-1}]\end{Bmatrix}$$

Corollary 3.1.8:

For $m>0, \beta=0, d\ne 1, p=1$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_2a_3-a_4|\le\frac{4}{3m^4e^{-2m}(1-d^2)^3}\{3md(1-e^{-2m})-(1+d)[2-(1+d)e^{-m}]\}$$

Corollary 3.1.9:

For $m=1, \beta=0, d=0, p=1$ in theorem 3.1 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_2a_3-a_4|\le\frac{4}{3e^{-2}}(e^{-1}-2)$$

Theorem 3.2

Let $f_p(z)\in S_p(m,\beta)$, then for $m>0, 0\le\beta<1, p\in N, 0\le d<1$

$$|a_{p+2}-a_{p+1}^2|=\frac{(p-\beta)}{m^{2p+1}e^{-2m}(1-d^2)^2}\left|m^pe^{-m}(p+1)![(p-\beta)-(1+d)]-4m(p!)^2(p-\beta)\right|$$

Proof:

The method of proof is similar to that of theorem 3.1 and the desired result is obtained by simplifying (18) and (19).

Corollary 3.2.1:

For $m>0, 0\le\beta<1, d=0, p\in N$ in theorem 3.2 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_{p+2}-a_{p+1}^2|\le\frac{(p-\beta)}{m^{2p+1}e^{-2m}}\left|m^pe^{-m}(p+1)![(p-\beta)-1]-4m(p!)^2(p-\beta)\right|$$

Corollary 3.2.2:

For $m>0, \beta=0, d\ne 1, p\in N$ in theorem 3.2 and if $f_p(z)\in S_p(m,\beta)$ then

$$|a_{p+2}-a_{p+1}^2|\le\frac{p}{m^{2p+1}e^{-2m}(1-d^2)^2}\left|m^pe^{-m}(p+1)![p-(1+d)]-4m(p!)^2p\right|$$

Corollary 3.2.3:

For $m=1, 0 \le \beta < 1, d \ne 1, p \in N$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{(p-\beta)}{e^{-2}(1-d^2)^2}\left|e^{-1}(p+1)![(p-\beta)-(1+d)] - 4(p!)^2(p-\beta)\right|$$

Corollary 3.2.4:

For $m > 0, 0 \le \beta < 1, d \ne 1, p = 1$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_3 - a_2^2\right| \le \frac{2(1-\beta)m}{m^3 e^{-2m}(1-d^2)^2}\left|\beta(2-e^{-m}) + (2-e^{-m}d)\right|$$

Corollary 3.2.5:

For $m=1, 0 \le \beta < 1, d = 0, p \in N$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{(p-\beta)}{e^{-2}}\left|me^{-1}(p+1)!(p-\beta)-1) - 4(p!)^2(p-\beta)\right|$$

Corollary 3.2.6:

For $m=1, \beta = 0, d \ne 1, p \in N$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{p}{e^{-2}(1-d^2)^2}\left|e^{-1}(p+1)![p-(1+d)] - 4(p!)^2 p\right|$$

Corollary 3.2.7:

For $m=1, 0 \le \beta < 1, d \ne 1, p = 1$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_3 - a_2^2\right| \le \frac{(1-\beta)}{e^{-2}(1-d^2)^2}\left|2e^{-1}[(1-\beta)-(1+d)] - 4(1-\beta)\right|.$$

Corollary 3.2.8:

For $m=1, \beta = 0, d = 0, p = 1$ in theorem 3.2 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_3 - a_2^2\right| \le \frac{4}{e^{-2}}.$$

Theorem 3.3

Let $f_p(z) \in S_p(m,\beta)$, then for $m > 0, 0 \le \beta < 1, p \in N, 0 \le d < 1$

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le \frac{(p-\beta)^2}{3m^{2p+2}e^{-2m}(1-d^2)^4}\left|\begin{matrix} 2p!(p+2)!\left[\begin{matrix} 2(p-\beta)e^{-m}[(p-\beta)-(1+d)] - 2(1+d)[2(p-\beta)+(1+d)e^{-m}] \\ -(p+1)!^2[(p-\beta)-(1+d)^2] \end{matrix}\right] \\ -3(p+1)!^2[(p-\beta)-(1+d)^2] \end{matrix}\right|$$

Proof:

The method of proof is similar to that of theorem 3.1 and the desired result is obtained by simplifying (18),(19) and (20).

Corollary 3.3.1

For $m > 0, 0 \le \beta < 1, d = 0, p \in N$ in theorem 3.3 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le \frac{(p-\beta)^2}{3m^{2p+2}e^{-2m}}\left|\begin{matrix} 2p!(p+2)![2(p-\beta)e^{-m}[(p-\beta)-1] - 2[2(p-\beta)+e^{-m}] \\ -(p+1)!^2[(p-\beta)-1]] - 3(p+1)!^2[(p-\beta)-1] \end{matrix}\right|$$

Corollary 3.3.2:

For $m > 0, 0 \le \beta < 1, d \ne 1, p = 1$ in theorem 3.3 and if $f_p(z) \in S_p(m,\beta)$ then

$$\left|a_2a_4-a_3^2\right|\le\frac{(1-\beta)^2}{3m^4e^{-2m}(1-d^2)^4}\left|\begin{matrix}12[2(1-\beta)e^{-m}[(1-\beta)-(1+d)]-2(1+d)[2(1-\beta)+(1+d)e^{-m}]\\-4[(1-\beta)-(1+d)^2]]-12[(1-\beta)-(1+d)^2]\end{matrix}\right|$$

Corollary 3.3.3:
For $m=1, 0\le\beta<1, d=0, p\in N$ in theorem 3.3 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|\le\frac{(p-\beta)^2}{3e^{-2}}\left|\begin{matrix}2p!(p+2)![2(p-\beta)e^{-1}[(p-\beta)-1]-2[2(p-\beta)+e^{-1}]\\-(p+1)!^2[(p-\beta)-1]]-3(p+1)!^2[(p-\beta)-1]\end{matrix}\right|$$

Corollary 3.3.4:
For $m=1, 0\le\beta<1, d\ne1, p=1$ in theorem 3.3 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_2a_4-a_3^2\right|\le\frac{(1-\beta)^2}{3e^{-2}(1-d^2)^4}\left|\begin{matrix}12[2(1-\beta)e^{-1}[(1-\beta)-(1+d)]-2(1+d)[2(1-\beta)+(1+d)e^{-1}]\\-4[(1-\beta)-(1+d)^2]]-12[(1-\beta)-(1+d)^2]\end{matrix}\right|$$

Corollary 3.3.5:
For $m>0, \beta=0, d\ne1, p=1$ in theorem 3.3 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_2a_4-a_3^2\right|\le\frac{12}{3m^4e^{-2m}(1-d^2)^4}\left|2e^{-m}+2(1+d)[2+(1+d)e^{-m}]+5(d^2+d)\right|$$

Corollary 3.3.6:
For $m=1, \beta=0, d\ne1, p=1$ in theorem 3.3 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_2a_4-a_3^2\right|\le\frac{4}{e^{-2}(1-d^2)^4}\left|[2e^{-1}d-2(1+d)[2+(1+d)e^{-1}]+4(2d+d^2)]+2d+d^2\right|$$

Corollary 3.3.7:
For $m=1, \beta=0, d=0, p=1$ in theorem 3.3 and if $f_p(z)\in S_p(m,\beta)$ then

$$\left|a_2a_4-a_3^2\right|\le\frac{8}{e^{-2}}(2-e^{-1})$$

Theorem 3.4
Let $f_p(z)\in S_p(m,\beta)$, then for $m>0, 0\le\beta<1, p\in N, 0\le d<1$

$$H_3(1)\le\frac{(p-\beta)^2}{m^{3p+3}e^{-3m}(1-d^2)^6}\,|\,B(p+1)!+B^*(p+1)!(p+2)!+B^{**}(p+3)!|$$

where

$$B=\frac{1}{3}\left\{[(p-\beta)-(1+d)]\begin{pmatrix}2p!(p+2)![2(p-\beta)e^{=m}[(p-\beta)-(1+d)]]-2(1+d)[2(p-\beta)+(1+d)e^{-m}]\\-4(p+1)!^2[(p-\beta)(1+d)^2]\end{pmatrix}\right\}$$

$$B^*=\frac{1}{9}\left\{\left(2(p-\beta)e^{-m}[(p-\beta)-(1-d)]-2(1+d)[2(p-\beta)-(1+d)e^{-m}]\right)\begin{pmatrix}6(p-\beta)p![(p-\beta)-(1+d)]-e^{-m}m^{m-1}(p+2)\\ [2(p-\beta)e^{-m}[(p-\beta)-(1+d)]]\\-2m^{=1}(1+d)[2(p-\beta)-(1+d)e^{-m}]\end{pmatrix}\right\}$$

$$B^{**}=\frac{1}{2}\left\{\begin{pmatrix}2(p-\beta)e^{-m}[(p-\beta)-(1-d)]-2(1+d)[2(p-\beta)-(1+d)e^{-m}]+\\(p-\beta)(1+d)[(p-\beta)-2(1+d)]-(1+d)^3\end{pmatrix}\begin{pmatrix}m^{p-1}e^{-m}(p+1)![(p-\beta)-(1+d)]\\-4p!^2(p-\beta)\end{pmatrix}\right\}$$

Proof:
The method of proof is similar to that of theorem 3.1 and the desired result is obtained by simplifying (18),(19),(20) and (21).

Corollary 3.4.1:
For $m > 0, \beta = 0, d \neq 1, p = 1$ in theorem 3.4 and if $f_p(z) \in S_p(m, \beta)$ then

$$H_3(1) \leq \frac{2}{m^6 e^{-3m}(1-d^2)^6} | A + 6A^* + 12A^{**} |$$

where

$$A = \frac{1}{3}\left\{d[12[2e^{-m} - (1+d)] - 2(1+d)[2 + (1+d)e^{-m}] + 16(d^2 + 2d)]\right\}$$

$$A^* = \frac{1}{9}\left\{[2e^{-m}d - 2(1+d)[2 + (1+d)e^{-m}][6d - 6de^{-2m} - 2m^{-1}(1+d)[2 - (1+d)e^{-m}]]\right\}$$

$$A^{**} = \frac{1}{2}\left\{[2e^{-m}d - 2(1+d)[2 - (1+d)e^{-m}] + (1+d)^2 - (1+d)^3][2e^{-m}d - 4]\right\}$$

Corollary 3.4.2:
For $m = 1, \beta = 0, d \neq 1, p = 1$ in theorem 3.4 and if $f_p(z) \in S_p(m, \beta)$ then

$$H_3(1) \leq \frac{2}{e^{-3}(1-d^2)^6} | D + 6D^* + 12D^{**} |$$

where

$$D = \frac{1}{3}\left\{d[12[2e^{-1} - (1+d)] - 2(1+d)[2 + (1+d)e^{-1}] + 16(d^2 + 2d)]\right\}$$

$$D^* = \frac{1}{9}\left\{[2e^{-1}d - 2(1+d)[2 + (1+d)e^{-1}][6d - 6de^{-2} - 2m^{-1}(1+d)[2 - (1+d)e^{-1}]]\right\}$$

$$D^{**} = \frac{1}{2}\left\{[2e^{-1}d - 2(1+d)[2 - (1+d)e^{-1}] + (1+d)^2 - (1+d)^3][2e^{-1}d - 4]\right\}$$

Corollary 3.4.3:
For $m > 0, \beta = 0, d = 0, p = 1$ in theorem 3.4 and if $f_p(z) \in S_p(m, \beta)$ then

$$H_3(1) \leq \frac{64(2 - e^{-m})}{3m^6 e^{-3m}}.$$

Corollary 3.4.4:
For $m = 1, \beta = 0, d = 0, p = 1$ in theorem 3.4 and if $f_p(z) \in S_p(m, \beta)$ then

$$H_3(1) \leq \frac{64(2 - e^{-1})}{3e^{-3}}.$$

Conclusion

It is known that Poisson distribution is a famous subject of interest in Statistics. This present work had investigated the estimates of second and third order Hankel determinant for analytic functions associated with the Poisson distribution. Statistical distribution in geometric function theory associated with the new subclasses were also considered.
The sharp bounds of the second coefficient of normalized univalent functions readily yields the growth and distortion bounds and also helps in the investigation of univalence of analytic functions.
For recent publications on second and third Hankel determinants, interested readers can refer to [10],[11],[12],[15],[16].

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