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COMPUTATIONAL ANALYSIS OF FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATION USING EULERIAN POLYNOMIAL BASIS FUNCTION

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ABSTRACT

This study develops a novel eulerian polynomial function for fractionally-differentiated Linear Volterra-Fredholm Integro-differential equations (LVFIDEs). The procedure is developed and evaluated against the current set of Lagrange polynomials (LPs); In order to achieve the best results and implementation of these kinds, a general algorithm is recommended and examples are provided. Additionally, using MATLAB 2009 software, a special case fractional differential equation is used to assess the viability of the suggested approach. In order to assess how well the suggested strategy solves difficulties, comparisons between it and current approaches are finally provided.

1. Introduction

In order to solve Linear Volterra-Fredholm Integro-Differential Equations (LVFIDEs) with fractional derivatives of the following sort, a novel class of Eulerian polynomial functions is examined in this paper:

$$D^{a}u(x) = q(x)u(x) + f(x) + \int_{a}^{x} k_1(x,t)u(t)dt \ + \int_{a}^{x} k_2(x,t)u(t)dt \ \ ; \ 0 < a < 1 \qquad (1)$$

with initial condition

$$u(a) = u_0 \qquad (2)$$

where $D^{a}u(x)$ denote the 'Caputo fractional derivative' $u(x); q(x), k_1(x,t)$ and $k_2(x,t)$ are continuous functions, x and t are real variables in [a, b] and $u(x)$ is the indefinite function to be determined using Eulerian polynomial functions.

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Numerous applications in the sciences, engineering, and even finance include this [1]. In order to help in problem resolution, it is often necessary to find an approximate answer using numerical methods when an analytical solution cannot be found [2]. Numerous academics have examined and assessed the numerical solution of LVFIDEs, including [1], who found the numerical solution to FIDEs of the Volterra type with the Caputo fractional derivative by using the Adomian decomposition approach. In a similar vein, [3] solved a class of linear fractional integro-differential equations using Taylor expansion series. [2] also looked into the least squares method of numerically solving LFIDEs with the help of shifted Chebyshev polynomial base functions. [4] provided a numerical method for solving fractional integro-differential equations that was based on cubic B-spline wavelets. presented the Haplace Analytical Method (HATM), an analytical technique that combines the Laplace decomposition method with HAM. Fractional integro-differential equations, both linear and nonlinear, can be solved using this method. [6] studied the use of shifted Laguerre polynomials in the least squares approach for the numerical solution of linear fractional integro-differential equations. [7] presented two numerical techniques for resolving FIDEs; the least squares approach uses Bernstein polynomials as its foundation. [8] created a model using the perturbation-iteration approach to get various FIDEs' approximate solutions. Studied the fractionally-order Volterra-Fredholm integro-differential equation solution method using sinc-collocation [9]. By integrating the generalized fractional Taylor series with the residual functions, [10] adjusted the fractional power series approach to derive an approximate solution to the model. To determine the approximate solution of a nonlinear FVFIDE, [11] used the Adomian decomposition and the modified Laplace Adomian decomposition techniques. The uniqueness and existence theorems for FVFIDEs were examined by [12]. Volterra-Fredholm integral equations were solved by [13] using the Lagrange collocation method, which uses Lagrange collocation points to convert the system of linear integral equations into matrix form. [14] presented a numerical technique for resolving first-order linear Volterra-Fredholm integro-differential equations utilizing three different kinds of Lagrange polynomials: Original, Modified, and Barycentric forms. Applied Lagrange polynomials (Original Lagrange Polynomial, Modified Lagrange Polynomial, and Barycentric Lagrange Polynomial) to solve linear Volterra-Fredholm integral equations [15]. By incorporating the meshless barycentric Lagrange quadrature formula, [16] was able to solve the two-dimensional linear Fredholm integral equations of the second sort. [17] introduced a method for solving two-dimensional integral equations using modified barycentric rational interpolation. Employed the barycentric rational interpolation collocation approach [18] to address boundary value issues of a higher level. [19] used the Barycentric interpolation collocation method to numerically solve a class of nonlinear partial differential equations. [20] further provided a Lagrange polynomial-based numerical solution of the fractional Volterra-Fredholm Integro-Differential Equation.

Furthermore, the constant coefficient fractional Fredholm Integro-Differential Equations of the type

$$\sum_{i=0}^{\infty} P_i D^{\alpha} u(t) = g(t) + \lambda \int_0^x H(x,t) u(t)\, dt; \quad a \le x,\ t \le b \tag{3}$$

under the initial boundary condition

$$D^{\alpha} u(a) = u(0)$$

$$D^{\alpha} u(0) = u'(a)$$

where a is constant and $1 < \alpha \le 2$ and D^{α} is the fractional derivative in the Caputo sense.

Preliminaries

Definition 2.1: A real function f(x), x > 0 is said to be in space Cμ; μ ϵ R, if there exists a real number p > μ, such that $f(x) = x^2 f_1(x)$ where $f_1(x)$ ϵ C(0, ∞), and it is said to be in the space C_μ^n *if* $f^n \in R_\mu,\ n \in N.$

Definition 2.2[21]: The Riemann–Liouville fractional integral operator of order α ≥ 0 of a function f $f \in C_\mu,\ \mu \geq -1$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(x)}\int_0^x (x-t)^{\alpha-1} f(t)dt\ ;\ \alpha > 0, t > 0 \tag{4}$$

in particular $J^0 f(x) = f(x)$,

Some properties of the operator J^α, for β ≥ 0 and γ ≥ -1,

i. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

ii. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

iii. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 2.3[21]: The Caputo fractional derivative of $f \in C_{-1}^m,\ m \in N$ is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-n-1} f^n(t)dt;\ \ m-1 < \alpha \leq m$$

Lemma 2.4: if $m-1 < \alpha \leq m\ ;\ m \in N,\ f \in C_\mu^m,\ \mu > -1$ then the following two properties hold

i. $D^\alpha\left[J^\alpha f(x)\right] = f(x)$

ii. $J^\alpha\left[D^\alpha f(x)\right] = f(x) - \sum_{i=1}^{m-1} f^i(0)\dfrac{x^i}{i!}$

Methodology

The equation (1) is said to be convergence, if it satisfies this theorem [22]

Consider the system

$$\sum_{i=0}^{\infty} P_i D^\alpha u(t) = g(t) + \lambda\int_0^x H(x,t)u(t)\,dt;\quad a \leq x,\ t \leq b$$

$x(0) = x_0$; t ϵ I = $[t_0, t_1]$ with the same conditions on the equation parameters as in equation (1), then

i. Equation 1 is convergence on the interval I = $[t_0, t_1]$.
ii. The $W(t, t_0)$ of equation 1 is non-singular
iii. Equation 1 is stable on the interval I = $[t_0, t_1]$.

Proof:

Recall that the non-singular nature of W(t, t0) in equation (1) is equivalent to stating that W(t, t0) is positive definite. This, in turn, implies that CT times the convergence of equation (1) is almost always equal to zero on the interval [t0, t1], suggesting that C = 0, T stands for matrix transpose.

$$C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}H(t-s)d_\theta u^*(s-\theta,\theta)\right] = 0$$

which implies C = 0; C ϵ E^n given that the integral is positive, equation (1) is correct.

This demonstrates that (i) and (ii) are comparable. Then to demonstrate the equivalency of (ii) & (iii).

By the definition,

$$C^T \int_{t_0}^{t} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] = 0$$

For each s ϵ[t_0, t_1], then

$$\int_{t_0}^{t_1} C^T \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] u(s)ds$$

$$C^T \int_{t_0}^{t_1} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] u(s)ds = 0 \quad for \;\; u \in R \tag{5}$$

It follows that equation (3) is orthogonal.

$$R(t,t_0) = \left\{ \int_{t_0}^{t} \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) u(s)ds; \quad u \in C^m; \;\; |u_i| \le 1; \; i = 1,2,\ldots m \right\}$$

Assuming that equation (1) will now converge, we get R(t, t0) = Rn, which leads to C = 0. This indicates that (iii) implies (ii) or that (i) is equal to (ii) and that (ii) is identical to (iii) and vice versa.

In the event if, on the other hand, equation (1) is not convergent and R(t, t0) ≠ Rn for t > t0, then C ≠ 0, C ϵ Rn, such that CTR(t, t0) = 0 exists.

$$0 = C^T \int_{t_0}^{t_1} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] u(s)ds \;; \quad for \;\; u \in R$$

$$\int_{t_0}^{t_1} C^T \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] u(s)ds$$

Hence,

$$C^T \int_{t_0}^{t_1} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} H(t-s) d_\theta u^*(s-\theta,\theta) \right] u(s)ds = 0; \;\; s \in [t_0, \, t_1]; \;\; C \ne 0$$

By definition, it implies that the system is assume to be convergence

Eulerian description:

Euler introduces the alternating ξ- function

$$\varphi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \tag{6}$$

Whenever Re(s) > 0, this series converges. It is possible to holomorphically extend the function φ(s) to the entire s-plane, which is connected to the ν-function:

$$\varphi(s) = (1 - 2^{1-s})\xi(s) \tag{7}$$

Introduce the Eulerian polynomials $P_n(t)$ for this purpose by

$$\sum_{k=0}^{\infty} (k+1)^{n+k} = \frac{P_n(t)}{(1-t)^{n+1}} \tag{8}$$

This is how the Euclerian polynomial was produced.

$$\begin{aligned}
P_0(t) &= 1, \\
P_1(t) &= 1 \\
P_2(t) &= 1 + t \\
P_3(t) &= 1 + 4t + t^2 \\
P_4(t) &= 1 + 11t + 11t^2 + t^3 \\
P_5(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\
P_6(t) &= 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5
\end{aligned} \tag{9}$$

Naturally, the series only converges when | t | <1, but in the case of Euler

$$\varphi(-n)=P_n(-1)2^{-n-1}$$

which contrasts φ(-n) with φ(n+1); φ(-n) = 0 for n > 0 even and n – odd [23]

Theorem: Given (λ-positive, g-ample) With degree n + 1-λ, the polynomial H(t) satisfies

$$t^{n+1-\lambda}H\left(\frac{1}{t}\right)=H(t)$$

Proof**:** For the Hilbert series [23]

$$\sum_{k=0}^{\infty}\chi(X,kg)t^k=\left(\sum_{k=0}^{\infty}e^{kg}t^k\right)T=\frac{1}{1-te^{\theta}}T \tag{10}$$

T represents the entire Todd class. It is necessary to assess this statement on X's basic cycle. Likewise

$\sum_{k=0}^{\infty}\chi(X,kg)t^k$ is given by

$$\frac{t^{-1}e^{-\theta}}{1-t^{-1}e^{-\theta}}T=\frac{-1}{1-te^{\theta}}T \tag{11}$$

Thus,

$$\sum_{k=0}^{\infty}\chi(X,kg)t^k=-\sum_{k=-1}^{-\infty}\chi(X,kg)t^k \tag{12}$$

By utilizing a different summation index of –k instead of k and $\chi(X-kg)=(-1)^n\chi(X(k-\lambda)g)$ and

$$\chi(X,kg)=\dim H^0(X,\mathrm{O}(k))\quad for\ k>-\lambda,$$

$$\chi(X,kg)=0\ \ for\ -\lambda<k<0,$$

$$\chi(X,0.g)=1 \tag{13}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\chi(X,kg)t^k&=(-1)^{n+1}\sum_{k=1}^{-\infty}\chi(X,(k-\lambda)g)t^{-k}\\&=(-1)^{n+1}\sum_{k=\lambda}^{\infty}\chi(X,(k-\lambda)g)t^{-k}\\&=(-1)^{n+1}\sum_{k=0}^{\infty}\chi(X,kg)t^{-k-\lambda}\end{aligned} \tag{14}$$

this gives:

$$\begin{aligned}H(t)(1-t)^{-n-1}&=(-1)^{n+1}t^{-\lambda}H(t^{-1})(1-t^{-1})^{-n-1}\\&=t^{n+1-\lambda}H\left(\frac{1}{t}\right)(1-t)^{-n-1}\end{aligned} \tag{15}$$

Work out the equations' solutions. (1-2) first substitute equation with the Eulerian Polynomial function. (9) in the formula. (1) to possess

$$\sum_{i=0}^{n}u_iP_n(x)=f(x)+q(x)\sum_{i=0}^{n}u_iP_n(x)+\int_0^x k_1(x,t)(\sum_{i=0}^{n}u_iP_n(x))dt+\int_0^x k_2(x,t)(\sum_{i=0}^{n}u_iP_n(x))dt$$

Consequently, according to the definition of the Caputo derivative, we have

$$P_n(t) = (1-t)^{n+1}\sum_{k=0}^{\infty}(k+1)^n t^k$$

replacing the Eulerian Polynomial function's produced recurrence relation to have

$$\begin{aligned} f(x) = {} & u_0\left[P_0(x) - q(x)P_0(x) - \int_0^x k_1(x,t)P_0(t)dt - \int_0^x k_2(x,t)P_0(t)dt\right] + \\ & u_1\left[P_1(x) - q(x)P_1(x) - \int_0^x k_1(x,t)P_1(t)dt - \int_0^x k_2(x,t)P_1(t)dt\right] + \\ & u_2\left[P_2(x) - q(x)P_2(x) - \int_0^x k_1(x,t)P_2(t)dt - \int_0^x k_2(x,t)P_2(t)dt\right] + \cdots + \\ & u_n\left[P_n(x) - q(x)P_n(x) - \int_0^x k_1(x,t)P_n(t)dt - \int_0^x k_2(x,t)P_n(t)dt\right] \end{aligned} \quad (16)$$

Substitute x = x_i in eq.(16) for (i= 0, 1, 2 , . . . n)

$$\begin{aligned} f(x_i) = {} & u_0\left[P_0(x_i) - q(x_i)P_0(x_i) - \int_0^x k_1(x_i,t)P_0(t)dt - \int_0^x k_2(x_i,t)P_0(t)dt\right] + \\ & u_1\left[P_1(x_i) - q(x_i)P_1(x_i) - \int_0^x k_1(x_i,t)P_1(t)dt - \int_0^x k_2(x_i,t)P_1(t)dt\right] + \\ & u_2\left[P_2(x_i) - q(x_i)P_2(x_i) - \int_0^x k_1(x_i,t)P_2(t)dt - \int_0^x k_2(x_i,t)P_2(t)dt\right] + \cdots + \\ & u_n\left[P_n(x_i) - q(x_i)P_n(x) - \int_0^x k_1(x_i,t)P_n(t)dt - \int_0^x k_2(x_i,t)P_n(t)dt\right] \end{aligned} \quad (17)$$

Equation (17) can be solved by substituting the recurrence relation of the Eulerian polynomial basis function with different values of N. This leads to an equation system, which is then solved using MATLAB, the Gaussian elimination method, and the LU decomposition method with partial pivoting through the algorithm.

General Algorithm for Methods

The following procedures are presented in order to assess numerical solutions of LFVFIDE using the Eulerian polynomial function:

Step 1: assume $h = \frac{b-a}{n}, n\epsilon\mathbb{Z}, u(a) = u_0$ (the initial condition is given).

Step 2: put $x_i = a + ih$, with $x_0 = a$ and $x_n = b, i = 0,1,\dots,n.$

Step 3: In steps (1) and (2), the values of a linear system DU = C, where instances are handled by Eulerian polynomials, are found: Select Equation (17).

(Keep in mind that we utilize the precise value calculated in MATLAB for the Caputo fractional derivative and integral in all equations).

Step 4: Utilizing step 3 and the partial pivoting Gauss elimination approach, solve the problem (DU = C). Similarly, using Gaussian elimination, or LU, to calculate a solution has the same processing cost.

Error Estimate

An error estimate for the approximate solution to equation (17) is obtained in this section. We define as the error function of the approximate solution to, where is the approximate solution computed for different values of N, and is the precise solution.

Numerical Examples

Numerical examples that demonstrate the ease of use and relevance of the method under discussion are provided below. The mathematical program MATLAB employed the general technique to solve issues, and the outcomes were displayed.

Example 1: Consider the FLFIDE [20]:

$$D^{\alpha}u(x) = q(x)u(x) + f(x) + \int_0^x \sin t\, u(t)dt + \int_0^x (x+t)u(t)dt \qquad (18)$$

with the initial condition $u(0) = 0$, where

$$f(x) = \frac{-3x^{1/6}\,\Gamma(5/6)(-91+216x^2)}{91\pi} + (5-2e)x, \quad q(x) = 0 \;\; and \;\; 5/6$$

with the true solution $u(x) = x - x^3$.

(Assume case $k_1(x,t) = 0$)

Table 2 shows the maximum error with n=4,6,8,10, and Table 1 shows the absolute error with n=5.

Table 1 shows the Example (1) Absolute Error for n = 5.

X	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
0.2000	0	0
0.4000	0	0
0.6000	0	0
0.8000	0	0
1.0000	5.1000212345e-16	5.1000212e-16
$\|err\|_\infty$	5.1000212345e-16	5.1000212e-16
R.T.	16.9414	28.5123

Table 2. Example (1)'s Maximum Error, for n = 4, 6, 8, 10.

N	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
4	3.2477452607e-16	3.2477452607e-16
6	5.1000212345e-16	5.1000212345e-16
8	2.6172745328e-15	4.1440180104e-15
10	6.7699476050e-14	3.5980141117e-14

Table 4 shows the maximum error with n = 4, 5, 8, 10, and Table 3 shows the absolute error with n = 5.

Table 3: Example (2)'s Absolute Error with n = 5.

X	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
0.2000	0	0
0.4000	0	0
0.6000	0	0

0.8000	0	0
1.0000	0	0
$\|err\|_\infty$	0	0
R.T.	20.6366	20.1257

Table 4: Example (2)'s Maximum Error with n = 4, 6, 8, 10.

N	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
4	0	0
6	0	0
8	5.566293953540e-16	6.42376362744e-16
10	1.193253308662e-14	6.34368584141e-15

Example 3: Consider the LFDE [20]

$$D^\alpha u(x) = q(x)u(x) + f(x) \tag{20}$$

with the initial condition $u(0) = 0$

where $f(x) = \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}x^{1-\alpha} + x^2 - x, \quad q(x) = 1 \; and \; \alpha = 0.5$

with the exact solution $u(x) = x^2 - x$

(Note that, in this case, $k_1(x,t) = 0$ and $k_2(x,t) = 0$).

Table 5 uses the Eulerian polynomial with n=5 to show the absolute inaccuracy.
The maximum inaccuracy using the Eulerian polynomial with n = 4, 6, 8, 10, is shown in Table 6, while the best results utilizing the Lagrange polynomial function are obtained in [20].

Table 5. The Eulerian polynomial with n=5 Absolute Error of Example (3).

X	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
0.2000	0	0
0.4000	0	0
0.6000	0	0
0.8000	0	0
1.0000	2.1527833558 E-17	2.1527833558 E-17
$\|err\|_\infty$	2.1527833558 E-17	2.1527833558 E-17
R.T.	13.8447	30.1743

Table 6: The Eulerian Polynomial Maximum Error of Example (3) for n = 4, 6, 8, and 10.

N	Lagrange Polynomial[20]	Eulerian Polynomial (present result)
4	0	0
6	2.1527833558 E-17	2.1527833558 E-17
8	1.6518263979e-17	3.3036527958e-17
10	3.3777921562e-16	1.98693656252e-16

Discussion

The Eulerian polynomial was used in this study to solve the LVFIDEs. Based on the outcomes of the examples, we deduce that:

- The approach provides the introduced algorithms' acceptability and capacity.
- The error term decreases with increasing n (the degree of polynomials) in all techniques except in cases where the exact answer is a low degree polynomial that we can fulfill.
- Additionally, Tables (4 and 6) show that neither technique yields any results for n=4 due to the difficulties in locating the fractional derivatives, which are extremely difficult to compute either by hand or with MATLAB.
- For the same reason, we recommend utilizing numerical integration rather than exact value to circumvent the challenge of locating the integration in Equation (17).
- It is possible to extend and apply methods to nonlinear FVFIDE; in this scenario, the issue becomes a nonlinear system of equations that can be resolved through discretization or linearization.

Conclusion

When discussing the reasons why the Eulerian method has been chosen over the Lagrangian method, it is generally stated that the Eulerian method requires less computing power than the Lagrangian method and that it operates with particle concentration rather than particle positions, making it more suitable for engineering applications. But under what circumstances are the outcomes of the two approaches equal, and for which two unique geometries are the Lagrangian and Eulerian approaches different? The drawback is that, for other studies and geometries, modeling and analyzing two particular situations on two special geometries will not yield an overarching rule for the disparity between the Lagrangian and Eulerian techniques' outcomes.

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