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Finite p −Groups With Small Normal Abelian Subgroups.

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ABSTRACT

Let $p^{\alpha(X)}$ be the maximal order of a minimal non-abelian subgroup X of a finite p-group G.
We classify the p-groups G all of whose abelian subgroups of order $\leq p^{\alpha(X)-3}$ are normal..

1. Introduction

An important topic in group theory is the analysis of the structural characteristics of a group G's subgroups. If every subgroup in group G is normal, then group G is said to be Dedekindian. Dedekind asserts that all abelian groups in this situation are Dedekindian. A Hamiltonian group is a non-abelian Dedekindian group. Every Hamiltonian group G is a direct product of the form $G = Q \times A \times E$, where A is an elementary abelian of odd order, $Q \cong Q_8$, the ordinary quaternion group, and E is an elementary abelian 2-group, as shown by Dedekind and Baer in both the finite and infinite order instances [1]. A group's normal subgroups are crucial in establishing the group's structure. Studying groups with the structure of their normal subgroups is natural. For instance, Y. Berkovich classified the finite p-groups in [1] all of whose noncyclic subgroups are normal. [2] identified the finite p-groups that have few nonnormal subgroups. In [3], the small normal closure of all cyclic subgroups for finite p-groups was characterized.

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The *p*-groups with extra special normal subgroups and the finite *p*-groups, all of whose noncyclic abelian subgroups are normal, were characterized by Y. Berkovich and Z. Janko [4]. The finite *p*-groups, all of whose subgroups of order $> p$ are normal, were categorized by D.S. Passman in [5]. In [6], it was studied how to construct finite *p*-groups of nilpotency class 2 with few normal subgroups. Unlike [6], [7] determined the finite *p*-groups with numerous normal subgroups. Frontiers of Mathematics in China provided a characterisation of the finite *p*-groups whose nonnormal subgroups have small orders in [8]. Furthermore, X. Guo, Q. Lui, and L. Feng identified the finite *p*-groups in [9] that have numerous normal subgroups. P. Bai and X. Guo studied finite *p*-groups with few normal subgroups in [10]. Another method for studying finite *p*-groups through their normal subgroups was presented by Y. Berkovich in [11], where he categorized the finite *p*-groups into those whose small subgroups are all normal.

A group's normal subgroups play a critical role in defining the group's structure. We categorize the finite *p*-groups in this article, all of whose small abelian subgroups are normal. Terms are defined and some known findings are provided in section 2. The structure of finite *p*-groups, all of whose small abelian subgroups are normal, is determined in section 3.

2. Preliminaries

Some group theoretic definitions and well-known results on finite groups and finite *p*-groups that are used in the main work are provided in this preparatory part. We study just finite groups, and our notations are common to theory of finite groups.

Definition 2.1.[12]. Let p be a prime. A finite group G is called a *p-group* if the order of G is a power of p.

Definition 2.2.[12]. A subgroup H of a group G is called a maximal subgroup of G if $H < G$ and for any $K < G$, such that $H \leq K < G, H = K$.

Definition 2.3.[1]. Let G be a group. For $a, x \in G, a^x = x^{-1}ax$.

Definition 2.4.[1]. Let G be a group. An element $x \in G$ is said to invert a subgroup $H \leq G$ if $h^x = h^{-1}$ for all $h \in H$.

Definition 2.5.[12]. Let G be a group. If $H \subseteq G$, then $\langle H \rangle = \langle x | x \in H \rangle$ is the subgroup generated by H.

Definition 2.6.[12]. Let G be a group. For $a, b \in G, [a, b] = a^{-1}b^{-1}ab$ is called the commutator of the elements a, b of G.

Definition 2.7.[13]. The center, $Z(G)$, of a group G is the set of all elements in G that commute with every element of G. In symbols, $Z(G) = \{x \in G | xg = gx \ \forall g \in G\}$. The centre of a group G is a normal subgroup.

Definition 2.8.[13]. Let A and B be a groups. Then $A \times B$ is a group called the direct product of A and B.

Definition 2.9.[1]. Let G be a group and let A and B be subgroups of G. Then $G = A \cdot B$ is a semidirect product with kernel B and Complement A.

Definition 2.10.[13]. Let G be a group and let H be subgroup of G. Then H is called a normal subgroup of G if $aH = Ha$ for all $a \in G$. The fact that H is a normal subgroup of G is denoted by $H \trianglelefteq G$.

Definition 2.11.[11]. A group G is called a Dedekind group if all its subgroups are normal.

Definition 2.12.[11]. A nonabelian Dedekind group is called a Hamiltonian group.

Definition 2.13.[11]. A group G splits over $N \trianglelefteq G$ if $G = H \cdot N$ with $H \leq G$ and $H \cap N = \{1\}$. (In this case, G semidirect product of H and N with kernel N.

Definition 2.14.[11]. A group G is an extension of $N \trianglelefteq G$ by a group H if $G/N \cong H$. semidirect product with kernel B and Complement A.

Definition 2.15.[13]. Let a be a fixed element of a group G. The centralizer of a in G, $C_G(a)$, is the set of all elements in G that commute with a. In symbols, $C_G(a) = \{g \in G | ga = ag\}$.

Definition 2.16.[1]. Let G be a group. Then G^m is the direct product of m copies of G.

Definition 2.17.[1]. A *p*-group G is called an elementary abelian *p*-group if every element has order *p*. An elementary abelian group of order p^m is denoted by E_{p^m} or C_m^p.

Definition 2.18.[1]. A special *p*-group is a non-abelian *p*-group G such that $G' = \Phi(G) = Z(G)$.

Definition 2.19.[1]. A *p*-group G is called extra special if $Z(G)$ is a cyclic group of order p and $G/Z(G)$ is a nontrivial elementary abelian *p*-group.

Definition 2.20.[12]. Let G be a group. The subgroup $[G, G]$ of G generated by all commutators in G is called the derived subgroup of G or the commutator subgroup of G, and is denoted by G'.

Definition 2.21.[11]. A minimal non-abelian *p-group* *G* is said to be metacyclic if it has a cyclic normal subgroup *C* such that the quotient G/C is cyclic.

Lemma 2.1. [1]. If *G* is a nilpotent minimal non-abelian group, then *G* is a *p-group*, $|G'| = p$, $Z(G) = \Phi(G)$ is of order p^2 in *G* and one of the following holds:

(i) $p = 2$ and *G* is the ordinary quaternion group;

(ii) $G = \langle a, b | a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}}, m > 1 \rangle$ is metacyclic of order p^{m+n};

(iii) $G = \langle a, b, c | a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$ is non-metacyclic of order p^{m+n+1}.

Corollary 2.2. [11]. Let G be a *p-group*, then the following statements are equivalent:

(1) G is minimal non-abelian;

(2) $d(G) = 2$ and $|G'| = p$;

(3) $d(G) = 2$ and $Z(G) = \Phi(G)$.

Definition 2.22.[11]. Let G be a minimal non-abelian *p-group*. If G is non-metacyclic and such that ${G}/{G'}$ is abelian of the type (p^m, p^n), then G is called an -group. A metacyclic group G is said to be an $M_p(m,n)$-group if $G = AB,\ A \lhd G,\ A \cong C_{p^m}$ and $B \cong C_{p^n}$ (here, C_t is a cyclic group of order t).

One has that $\left|M_p(m,n,1)\right| = p^{m+n+1}$ and $\left|M_p(m,n)\right| = p^{m+n}$. The groups $M_p(m,n,1)$ and $M_p(m,n)$ are determined up to isomorphism by the parameters m, n. Thus, if G is a minimal non-abelian *p-group*, then $d(G) = 2, |G{:}Z(G)| = p^2, |G'| = p$ and $G \in \{M_p(m,n,1),\ M_p(m,n),\ Q_8\}$. [11].

Given a nonabelian p-group G, let $p^{\upsilon(G)}$ be the maximal order of an abelian subgroup of G.

Theorem 2.3. [11]. Let G be a non-Dedekindian *p-group* with $\upsilon = \upsilon(G) \geq 3$. If all abelian subgroups of G of order $p^{\upsilon(G)-2}$ are normal, then $G \in \{M_p(2,2), Q_{2^4}\}$.

Lemma 2.4. [11]. Suppose that all subgroups of index $\geq p^3$ of a minimal nonabelian p-group G are normal. Then either $|G| = p^3$ or $G = M_p(2,2)$.

Given a nonabelian p-group G, let $p^{a(G)}$ be the maximal order of a minimal nonabelian subgroup of G. If $H < G$ is nonabelian, then $p^{a(H)} \leq p^{a(G)}$.

Theorem 2.4. [11]. The following conditions for a nonabelian p-group G are equivalent:

(a) All abelian subgroups of G of order $p^{a(G)-2}$ are normal.
(b) $G = Q \times E$, where Q is a generalized quaternion group and exp $(E) \leq 2$.

3. Finite p −Groups With Small Normal Abelian Subgroups

In Theorem 3.0, we characterize the finite p-groups with small normal abelian subgroups. Let $p^{a(G)}$ be the maximal order of a minimal nonabelian subgroup of a nonabelian p-group, G.

Theorem 3.0. Let G be a finite non-abelian p-group. If all abelian subgroups of G of order $p^{a(G)-3}$ are normal subgroups of G, then $G = C_p \times \left(C_{p^2} \ltimes C_{p^2}\right)$

Proof:
Since the maximal order of a minimal nonabelian subgroup of G is $p^{a(G)}$, then we have $a(G) = 4 \Rightarrow a(G) - 3 = 1$. Thus, all subgroups of G of order p are G-invariant. This means $\Omega_1(G) \leq Z(G)$. Now, let $M \leq G$ be a maximal nonabelian subgroup of G. If M is of order $p^{a(G)}$, then $\upsilon(M) = a(G) - 1 \leq \upsilon(G)$. Let $A < M$ be of index p^3. Then $|A| = p^{\upsilon(M)-2} \leq p^{\upsilon(G)-2}$. Therefore, $A \lhd G$, and hence $A \lhd M$. Thus, $M \cong M_p(2,2)$ $(= C_{p^2} \ltimes C_{p^2})$.

Next, we assume that $M < G$ (otherwise, there will be nothing to prove); Then $G \geq p|M|$. Note also that $\exp\big(Z(M)\big) = p$. Now, supposing a metacyclic minimal nonabelian subgroup of G, $M_p(2,2) \cong M = A \cdot B$, a semidirect product of the cyclic subgroups $A = \langle\, a | a^{p^2} = 1 \,\rangle$ and $B = \langle\, b | b^{p^2} = 1 \,\rangle$ with kernel A. Then $G \geq p|M| = p^5$. In that case, M contains a nonnormal subgroup B of order p^2. Therefore, G has no abelian subgroup of order $> p^3$. Thus, every abelian subgroup of G is of order $.p,\ p^2$ or p^3. Since $[G\!: M] = p$, then G must be a product of one of its abelian subgroups of index p^4 (a cyclic group, C_p, of order p) and M. Now, G cannot be a semidirect product of C_p and M because M is a normal subgroup of G (index p) and C_p is also normal in G (from the hypothesis). Thus, G must be a direct product of C_p and M. That is, $G = C_p \times M$
$= C_p \times \big(C_{p^2} \ltimes C_{p^2}\big)$.

Corollary 3.1. Let G be a finite non-abelian p-group. If all abelian subgroups of G of order $p^{a(G)-3}$ are normal subgroups of G, then $exp(G) = p^2$.

Proof:

Since $G = C_p \times M = C_p \times \big(C_{p^2} \ltimes C_{p^2}\big)$, then $dl(G) = 3$. That is, G has 3 generators. Let the generators be $a, b,$ and c respectively such that $C_p = \langle a \rangle = \langle a | a^p = 1 \rangle$, $C_{p^2} = \langle b \rangle = \langle b | b^{p^2} = 1 \rangle$ And $C_{p^2} = \langle c \rangle = \langle c | c^{p^2} = 1 \rangle$. Then Now, every element of G is of the form $\big(a^i, b^j c^k\big)$ and will take either $\big|a^i\big| = p$ or $\big|b^j c^k\big| = p^2$. Thus, $exp(G) = p^2$.

Conclusion

Finally, we have shown that G is a direct product of one of its cyclic subgroups of index p and its $M_p(2,2)$ subgroup if G is a p-group whose small abelian subgroups are all normal.

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