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α-FUZZY FIXED POINTS OF HESITANT FUZZY MAPPING.

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Abstract

This paper aims to generalize some hesitant fuzzy fixed point theorems in literature. Here we obtain in the sense of Estruch and Vidal, α-fuzzy fixed point results of hesitant fuzzy maps.

Keywords: Fixed point, Hesitant fuzzy set, Hesitant fuzzy mapping.

1. Introduction

The earliest fixed points results were obtained for single-valued self-maps. See [1-4] and references therein. Thereafter, Nadler [5] defined multivalued mappings as a generalization of single-valued maps and extended the contractive maps of [2]. His result unlike other similar results was proved with less restriction on the ambient space and the map. It was only required that the metric space be complete and the map be a contractive one.

A fuzzy set [6]is a set X characterized by a membership $\mu: X \to [0,1]$ such that $\mu(x)$=0 if $x \notin X$ and $\mu(x) \in (0,1]$ if $x \in X$. It generalizes the crisp set as elements of X can have partial membership. By defining a fuzzy set of a set, one can define objects like $\alpha -$ level sets which are subsets of a fuzzy set X defined by

$F_\alpha = \{x \in X : \mu(x) \geq \alpha, \alpha \in (0,1]\}$ and $F_0 = cl(\{x \in X : \mu(x) > 0\}$.

Armed with these concepts, Heilpern [7] introduced fuzzy mapping as a generalization of the multivalued maps of Nadler. Thus, a fuzzy map is the multivalued map of Nadler where the image is a family of fuzzy sets equipped with α-level sets of fuzzy sets that are closed and compact. Following this introduction and the accompanying fixed point results are other generalizations and extensions. See [8,9-12] and refences therein. Most significant of these is the introduction of α-fuzzy fixed point by Estruch and Vidal [8]. Thus, a fuzzy fixed point is identified to a certain degree of an $\alpha \in [0,1]$, where when $\alpha = 1$, the fuzzy fixed point (respectively the fuzzy fixed point results) becomes the crisps fixed point (respectively the fixed point results) as in Nadler's.

On the other hand, research study on fuzzy sets has increased recently due its ability to represent and tackle imprecision and vagueness which exist in many complex systems. Among these is the extension of the definition to handle other situations. Some of these generalizations and or extensions of interest in this study are hesitant fuzzy set [13] and Relative fuzzy set [14]. While the former is a subset-valued membership function of a fuzzy set, the latter is a fuzzy set characterized by growth membership function. For research application of hesitant fuzzy set, see [15,16,17]. The authors of [18] recently discussed α-fuzzy fixed point of relative fuzzy maps after [14] announced relative fuzzy set as a fuzzy set characterized by a dynamic membership function.

Our interest in this study is the hesitant fuzzy set and consequently its associated maps and fixed-point results. We extend [8] some results on hesitant fuzzy mapping on metric space and prove some fixed-point results for some generalized hesitant fuzzy maps on metric space. The results generalize among other results in [8] and [18].

Next, relevant concepts and results needed in the sequel will be discussed.

A metric space (X, d) is nonempty set equipped with the function $d: X \times X \to \mathbb{R}$ such that $d(x, y) \geq 0$, $d(x, y) = 0 \leftrightarrow x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. It is called a b-metric space $(X, d)_s$ with coefficient s if $d(x, y) \leq s[d(x, z) + d(z, y)]$ instead.

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Note that $(X,d)_s$ is a metric space if $s=1$.

A Hesitant fuzzy Set [13,10] is a set of X, a family S of all subsets of the interval [0,1] whose membership map is $h: X \rightarrow S$ such that $h(x) \in S\backslash\{0\}$ if $x \in X$ and $h(x)=0$ if $x \notin X$.

Note that a hesitant fuzzy set reduces to fuzzy set when h is single-valued and a crisp set if h is single-valued and $S=\{0,1\}$.

Let $H(X)$ be a family of hesitant fuzzy sets of X. A generalized contraction for a pair of hesitant fuzzy maps H_F, H_G [1] is $H_F, H_G: \mathrm{X} \rightarrow H(X)$ such that

$$D(H_F(x), H_G(y)) \leq a_1 p(x, H_F(x)) + a_2 p(y, H_G(y)) + a_3 p(y, H_F(x)) + a_4(x, H_G(y)) + a_5 d(x,y) \quad \ldots\ldots(1)$$

for any $x, y \in X$, where $\sum_{I=1}^{5} a_I < 1$, and $a_1 = a_2$ or $a_3 = a_4$, $a_I \in \mathbb{R}_+$, (X,d) a metric space.

Theorem 1.1.
Let (X,d) be a metric space and $H_F, H_G: \mathrm{X} \rightarrow H(X)$ be a pair of hesitant fuzzy maps such that the inequality above holds. Then there exist $x^* \in$ X such that $\{x^*\} \subset H_F$ and $\{x^*\} \subset H_G$ hold.

2. Main Results

Let $H_\alpha(X)$ denote a collection of hesitant fuzzy sets of X called approximate quantities of hesitant fuzzy set if for any $F \in H(X)$ F_α is closed and compact for each $\alpha \in [0,1]$. Note that $F_\alpha = \{x \in X : h(x) \geq \alpha, \alpha \in (0,1]\}$ and $F_0 = cl(\{x \in X : h(x) > 0\})$.

Definition 2.1: (Hesitant α −fuzzy Contraction Map of b-metric space)
Let $(X,d)_s$ be a b-metric space with coefficient s. Then the hesitant fuzzy map $\mathrm{H}: \mathrm{X} \rightarrow H_\alpha(X)$ on b-metric space is said to be a hesitant α −fuzzy contraction map if

$$D_\alpha(\mathrm{H(x)}, \mathrm{H(y)}) \leq \mathrm{ad(x,y)} \quad \ldots\ldots(2)$$

for any $x, y \in \mathrm{X}$ where $a \in (0, \frac{1}{s})$.

x^* is a fixed fuzzy point of H if $x^* \in [H(x^*)]_\alpha$ or $x^*{}_\alpha \subset H(x^*)$.

Definition 2.2: (Hesitant α −fuzzy Generalized Contraction Map of b-metric space)
Let $(X,d)_s$ be a b-metric space with coefficient s and $H_\alpha(X)$ a collection of hesitant approximate quantities of hesitant fuzzy sets of . Then the pair of hesitant fuzzy maps on a b-metric space $\mathrm{H_F}, \mathrm{H_G}: \mathrm{X} \rightarrow H_\alpha(X)$ such that

$$D_\alpha(\mathrm{H_F(x)}, \mathrm{H_G(y)}) \leq \frac{1}{s}\left[a_1 p(x, H_F(x)) + a_2 p(y, H_G(y)) + a_3 p(y, H_F(x)) + a_4(x, H_G(y))\right] + a_5 d(x,y) \quad \ldots\ldots(3)$$

for any $x, y \in X$, where $\sum_{I=1}^{5} a_I < 1$, and $a_1 = a_2$ or $a_3 = a_4$, $a_I \in \mathbb{R}_+$, is called the hesitant α −fuzzy generalized contraction mapping on a b-metric space.

Theorem 2.3
Let $(X,d)_s$ be a complete b-metric space with coefficient s and $\mathrm{H_F}, \mathrm{H_G}: \mathrm{X} \rightarrow H_\alpha(X)$ be the hesitant α −fuzzy generalized maps. Then there exist $x^* \in$ X such that
$x^* \in [H_{\mathrm{F}}(x^*)]_\alpha$ and $x^* \in [H_{\mathrm{G}}(x^*)]_\alpha$ (or $x^*{}_\alpha \subset H_F(x^*)$ *and* $x^*{}_\alpha \subset H_{\mathrm{G}}(x^*)$) i.e. x^* is a α − fuzzy fixed point of H_{F} and H_{G}.

Proof:
Let $x_0 \in X$ then $H_F(x_0) \in H_\alpha(X)$. Let also $\{x_1\} \subset H_F(x_0)$, then there is $x_2 \in X$ such that
$\{x_2\} \subset H_G(x_1) \in H_\alpha(X)$. So $d(x_1, x_2) \leq D_\alpha(\mathrm{H_F}(x_0), \mathrm{H_G}(x_1))$.
Also, there is $x_3 \in X$ such that $\{x_3\} \subset H_F(x_2) \in H_\alpha(X)$.
So $d(x_1, x_3) \leq D_\alpha(\mathrm{H_G}(x_1), \mathrm{H_F}(x_2))$.
Continuing, we have that there is $x_n \in X$ such that
$\{x_{2n+1}\} \subset H_F(x_{2n})$ and $\{x_{2n+2}\} \subset H_F(x_{2n+1})$.

So that $d(\{x_{2n+1}, x_{2n+2}) \leq D_\alpha(\mathrm{H_F}(x_{2n}), \mathrm{H_G}(x_{2n+1}))$ and $d(x_{2n+2}, x_{2n+3}) \leq D_\alpha(\mathrm{H_F}(x_{2n+1}), \mathrm{H_G}(x_{2n+2}))$.

Letting $n = 0$ then
$d(\{x_{2n+1}, x_{2n+2}) = d(x_1, x_2) \leq D_\alpha(\mathrm{H_F}(x_0), \mathrm{H_G}(x_1)) \leq \frac{1}{s}\left[a_1 p(x_0, H_F(x_0)) + a_2 p(x_1, H_G(x_1)) + a_3 p(x_1, H_F(x_0)) + a_4(x_0, H_G(x_1))\right] + a_5 d(x_0, x_1) \leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_1, x_1) + a_4 d(x_0, x_2) + a_5 d(x_0, x_1) \leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_4 s[d(x_0, x_1) + d(x_1, x_2)] + a_5 d(x_0, x_1) \leq (a_1 + a_4 s + a_5) d(x_0, x_1) + (a_2 + a_4 s) d(x_1, x_2) \leq \frac{a_1 + a_4 s + a_5}{1 - a_2 - a_4 s} d(x_0, x_1)$. Now, let $= \frac{a_1 + a_4 s + a_5}{1 - a_2 - a_4 s}$.
Then $d(x_1, x_2) \leq v\, d(x_0, x_1)$.
But $a_1 + a_2 + s(a_3 + a_4) + a_5 < 1$ giving $a_1 + sa_3 + a_5 < 1 - a_2 - a_4 s$ so that if $a_3 \geq a_4$ then we have that $0 < v < 1$. Letting $n = \frac{1}{2}$ then
$d(\{x_{2n+1}, x_{2n+2}) = d(x_2, x_3) \leq \frac{a_2 + a_3 s + a_5}{1 - a_1 - a_3 s} d(x_1, x_2) \leq \frac{a_2 + a_3 s + a_5}{1 - a_1 - a_3 s} \frac{a_1 + a_4 s + a_5}{1 - a_2 - a_4 s} d(x_0, x_1)$.
Let $w = \frac{a_2 + a_3 s + a_5}{1 - a_1 - a_3 s}$. Then
$d(x_2, x_3) \leq wv\, d(x_0, x_1)$.
But $a_1 + a_2 + s(a_3 + a_{a4}) + a_5 < 1$ giving $a_1 + sa_4 + a_5 < 1 - a_2 - a_3 s$ so that if $a_4 \geq a_3$ then we have that $0 < w < 1$. Then $a_4 = a_3$ and $0 < uw < 1$ for both cases.
Letting $n = 1$ then

$$d(\{x_{2n+1}, x_{2n+2}) = d(x_3, x_4) \leq \left(\frac{a_1 + a_4 s + a_5}{1 - a_2 - a_4 s}\right)^2 \frac{a_2 + a_3 s + a_5}{1 - a_1 - a_3 s} d(x_0, x_1) \qquad = u^2 w d(x_0, x_1)$$

Then $a_4 = a_3$ and $0 < u^2 w < 1$.
Continuing in this manner we have that for any $n \in \mathbb{N}$ and n even
$d(x_1, x_n) \leq s[u + uw + u^2 w + u^2 w^2 + \cdots + (uw)^{n-1}] d(x_0, x_1)$
n is odd then we have that
$d(x_1, x_n) \leq s[u + uw + u^2 w + u^2 w^2 + \cdots + (uw)^{n-1} u] d(x_0, x_1)$
Next, we show that any sequence $\{x_n\} \in X$ is Cauchy. Let $k \in \mathbb{N}$ with $k > 1$ then if k is even we have that
$d(x_1, x_k) \leq s^2[(uw)^1 + (uw)^2 + \cdots + (uw)^{k-2}] d(x_0, x_1)$
and if k is odd we have that
$d(x_1, x_k) \leq s^2[(uw)^1 t + (uw)^2 t + \cdots + (uw)^{k-2} t] d(x_0, x_1)$
Since $\sum_{i=1}^{5} a^i < \frac{1}{s}$, then $(uw)^{\frac{n}{2}} t\left[1 + (uw)^{\frac{1}{2}} + uw\right] < \frac{1}{s^2}$ for each n. Thus as $n \to \infty$, we have that the RHS of the inequality above tends to 0. The same is true if n is even. Therefore, the sequence $\{x_n\} \in X$ is a Cauchy sequence.
Thus, there exist $x^* \in X$ such that $\{x_n\} \to x^*$ as $n \to \infty$ $ since $(X, d)_s$ is a complete space. Now

$p(x^*, H_G(x^*)) \leq d(x^*, x_{2n+1}) + D_\alpha(H_F(x_{2n+1}), H_G(x^*))] \leq d(x^*, x_{2n+1}) + D_\alpha(x_{2n}, H_G(x^*))$.
But $D_\alpha(x_{2n}, H_G(x^*)) \leq \frac{1}{s}\left[a_1 p(x_{2n}, H_F(x_{2n})) + a_2 p(x^*, H_G(x^*)) + a_3 p(x^*, H_F(x_{2n})) + a_4 p(x_{2n}, H_G(x^*))\right] + a_5 d(x_{2n}, x^*)$

Thus, the inequality above gives
$p(x^*, H_G(x^*)) \leq d(x^*, x_{2n+1}) + \frac{1}{s}\left[a_1 p(x_{2n}, H_F(x_{2n})) + a_2 p(x^*, H_G(x^*)) + a_3 p(x^*, H_F(x_{2n})) + a_4 p(x_{2n}, H_G(x^*))\right] + a_5 d(x_{2n}, x^*) \leq d(x^*, x_{2n+1}) + \frac{1}{a_1} p(x_{2n}, H_F(x_{2n})) + \frac{1}{a_2} p(x^*, H_G(x^*)) + \frac{1}{a_3} p(x^*, H_F(x_{2n})) + \frac{1}{a_4} p(x_{2n}, H_G(x^*)) + a_5 d(x_{2n}, x^*)]$
$\leq d(x^*, x_{2n+1}) + \frac{1}{a_1} d(x_{2n}, x^*) + \frac{1}{a_2} d(x^*, x^*) + \frac{1}{a_1} d(x_{2n+1}, x^*) + \frac{1}{a_3} d(x^*, x_{2n+1}) + \frac{1}{a_4} d(x_{2n}, x^*) + a_5 d(x_{2n}, x^*)]$
$\leq \left(1 + \frac{1}{a_1} + \frac{1}{a_3}\right) d(x^*, x_{2n+1}) + \left(\frac{1}{a_1} + \frac{1}{a_4} + a_5\right) d(x^*, x_{2n})) \leq \left(\frac{a_1 a_3 + a_1 + a_3}{a_1 a_3}\right) d(x^*, x_{2n+1}) + \left(\frac{a_1 a_4 a_5 + a_4 + a_1}{a_1 a_4}\right) d(x^*, x_{2n})) \to 0$ as $n \to \infty$.

Therefore $x^* \in [H_G(x^*)]_\alpha$ by Lemma 2.3.1.
Similarly, we can show that $x^* \in [H_F(x^*)]_\alpha$ and the proof complete.

Corollary 2.4 (Theorem 2.8 of [18])

Let $(X,d)_s$ be a complete b-metric space with coefficient s and $\mathrm{H_F}, \mathrm{H_G}: \mathrm{X} \to H_\alpha(X)$ be the fuzzy contraction generalized hesitant maps. Then there exist $x^* \in \mathrm{X}$ such that $x^* \in [H_\mathrm{F}(x^*)]_1$ and $x^* \in [H_\mathrm{G}(x^*)]_1$. Thus x^* is called a fuzzy fixed point.

Proof: The proof follows from the proof of Theorem 2.3 above when $\alpha = 1$.
Corollary 2.5
Let (X,d) be a complete metric space and $\mathrm{H_F}, \mathrm{H_G}: \mathrm{X} \to H_\alpha(X)$ be the $\alpha -$fuzzy contraction generalized hesitant maps. Then there exist $x^* \in$ X such that $x^* \in [H_\mathrm{F}(x^*)]_\alpha$ and $x^* \in [H_\mathrm{G}(x^*)]_\alpha$.

Proof: The proof follows from the proof of the theorem above when $s = 1$.

Corollary 2.6 (Theorem 3.2 of [8])
Let $\alpha \in (0,1]$ and let (X,d) be a complete metric space. Let F be a fuzzy mapping from X into $W(X)$ satisfying the following condition: there exists $q \in (0,1)$ such that $\mathrm{D}_\alpha(F(x),F(y)) \le qd(x,y)$ for each $x,y \in X$. Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F. In particular if α=1, then x is a fixed point of F.

Proof: The proof follows from the proof of the theorem above when $s = 1$ and $H_\mathrm{F}(x) = H_\mathrm{G}(x) = H(x)\ \forall\ x \in X$.

Also, the Theorem and corollaries considered in this work also generalizes the Banach fixed point results of fuzzy map in the sense of Heilpern [6].

Conclusion
We extended the fixed point results of hesitant fuzzy mapping to a more generalized fixed point results of fuzzy mapping in the Estruch and Vidal sense. Similar results can also be studied for relative hesitant fuzzy maps.

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