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FUNDAMENTALS OF RECENT FINDINGS ON 3 –PRIMENESS OF NEAR-RINGS WITH DERIVATIONS.

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ABSTRACT

In this paper, we demonstrate the commutativity of prime near-rings that have nonzero derivations adhering to specific differential identities. We also present examples that validate the assumptions underlying our main results.

1. Introduction

Significant progress has been made regarding the commutativity of prime and semi-prime rings with derivations that adhere to specific differential identities [6] [5] and, [1]. This naturally led us to making an inquiry into analogous results in the context of near-rings. The exploration of derivations in near-rings began with the work of Bell and Mason in 1987 [4]. There has been considerable interest and progress in the connection between the commutativity of 3-prime near-rings and various classes of derivations that satisfy particular differential identities ([3], [8] and [7]).

More recently, Ashraf [2] demonstrated the commutativity of zero symmetric right near-rings under certain identities. The concept of derivations in near-rings has been extended in various ways [9] [13]; and the references therein). As a way of probing further into these literatures, we propose in this manuscript a new perspective on the commutativity of prime near-rings by defining derivations with novel identities that differ in a way from those established previously.

A non-empty set N equipped with two binary operations $+$ and $\cdot$ is called a left near-ring provided that $(N,+)$ is a group (not necessarily abelian); $(N,.)$ Is a semigoup and $p\cdot\ (q+r\ )=p\cdot q+p\cdot r$ for all $p,q,r\in N$.

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A left near-ring N is called a zero symmetric if $0p = 0$ holds for all $p \in N$ (recall that in a left near-ring $p0 = 0$ for all $p \in N$). Throughout this paper, we will use the word near-rings denoted by N to mean zero symmetric left near-ring. A near-ring N is said to be 3-prime if $pNq = \{0\}$ implies $p = 0$ or $q = 0$. For any $p, q \in N, [p, q] = pq - qp$ and $poq = pq + qp$ will denote lie and Jordan products respectively .The set $R = \{p \in N | qp = pq \text{ for all q} \in \text{N}\}$ is called multiplicative centre of N. An addictive mapping $d: N \to N$ is a derivation if $d(pq) = pd(q) + d(p)q$ for all $p, q \in N$ or equivalently, as noted in (Wang (1994)) that $d(pq) = d(p)q + pd(q)$ for all $p, q \in N$. N is said to be 2-torsion free if $2p = 0$ implies $p = 0$ for all $p \in N$.

2. Main result

We start with the following lemmas which are necessary in developing the proofs of our theorems.

Lemma 3.1 [3]: Let N be a prime near-ring. If N admits a nonzero derivation d for which $d(N) \subset R(N)$,then N is a commutative ring.

Lemma 3.2 [5]: Let d denote an arbitrary derivation on the near-ring N. In this case, N satisfies the following partial distributive law:

(i) $\big(d(p)q + pd(q)\big)r = d(p)qr + pd(q)z$ for all $p, q, r \in N$.

(ii) $(pd(q) + d(p)q)r = pd(q)r + d(p)qr$ for all $p, q, r \in N$.

Lemma 3.3[6]: Let N be a 2 −torsion free prime near-ring. If N admits a nonzero derivation d such that $d(p, q) = 0$ for all $p, q \in N$, then N is a commutative ring.

Lemma 3.4[6]: Let N be a 2 −torsion free prime near-ring. If N admits a nonzero derivation d such that $d^2 = 0$, then $d = 0$.

We now present our main results as follows:

Theorem 3.1.1: Let N be a prime near-ring satisfying $d([p, q]) = [p, d(q)]$ for all $p, q \in N$ associated with a nonzero derivation d, Then N is a commutative ring.

Proof.

By hypothesis, we have

$$d([p, q]) = [p, d(q)]. \quad \text{for all } p, q \in N \qquad (1.1)$$

Replacing p by pq in equation (1.1), we have

$d([pq, q]) = [pq, d(q)]$ for all $p, q \in N$.

Since$[pq, q] = [p, q]q$, this becomes

$$d([p, q]q) = [pq, d(q)] \text{ for all } p, q \in N \qquad (1.2)$$

This implies that $d([p, q])q + [p, q]d(q) = [pq, d(q)]$, for all $p, q \in N$

Using the relation (1.1), we find that $[p, d(q)].q + [p, q]d(q) = [pq, d(q)]$ for all $p, q \in N$

Or $[p, d(q)]. +(pq - qp)d(q) = [pq, d(q)]$ for all $p, q \in N$

This gives

$pd(q)q = qpd(q)$ for all $p, q \in N$ (1.3)

Substituting pq for p in equation (1.3) for all $r \in N$, we get $[q, p]rd(q) = 0$ for all $p, q, r \in N$

This implies

$$[q, p]Nd(q) = \{0\} \text{ for all } p, q \in N \quad (1.4)$$

Since N is prime, equation (1.4) gives $d(q) = 0$ or $[q, p] = 0$ for all $p, q \in N$ (1.5)

From (1.5), it follows that for each fixed $q \in N$, we have $d(q) = 0$ or $q \in R(N)$. (1.6)

But $q \in R(N)$ also implies that $d(q) \in R(N)$ and (1.6) forces $d(q) \in R(N)$ for all $p, q \in N$.

Hence, $d(N) \subset R(N)$, and using Lemma 3.1, we conclude that N is a commutative ring.□

Theorem 3.1.2: Let N be a prime near-ring satisfying $[p, d(q)] = [x, y]$ for all $p, q \in N$ associated with a nonzero derivation d, Then N is a commutative ring.

Proof:

We have $[p, d(q)] = [p, q]$ for all $p, q \in N$. (2.1)

Replacing y by qp in Equation (2.1), we get

$$[p, d(qp)] = [p, qp]\,. \quad (2.2)$$

But $[p, q]p = [p, d(q)]p$ for all $p, q \in N$. Hence,

using Lemma 2.2(i), Equation (2.2) can be expressed as

$pd(q)p + pqd(p) - d(q)p^2 - qd(p)p = pd(q)p - d(q)p^2$ for all $p, q \in N$

This means $pqd(p) = qd(p)p$ for all $p, q \in N$.

Since Equation (2.2) is the same as Equation (1.3), then using the same argument as in above we conclude that N is a commutative ring.[10]

Theorem 3.1.3: Let N be a prime near-ring satisfying $[p, d(q)] \in R(N)$ for all $x, y \in N$ with a nonzero derivation d, Then N is a commutative ring.

Proof.

By hypothesis, $[p, d(q)] \in R(N)$ for all $p, q \in N$. (3.1)

Then,

$$\big[[p, d(q)], t\big] = 0 \text{ for all } p, q, t \in N\,. \quad (3.2)$$

Replacing p by $pd(q)$ in Equation (3.2), we get

$$\big[[pd(q), d(q)], t\big] = [[p, d(q)]d(q), t] = 0 \text{ for all } p, q, t \in N \quad (3.3)$$

Using Equation (3.1), Equation (3.3) becomes

$$[p, d(q)]N[p, d(q)] = \{0\} \text{ for all } p, q \in N \ . \qquad (3.4)$$

Since N is prime, Equation (3.4) yields

$[p, d(q)] = 0$ for all $p, q \in N$. Thus, $d(N) \subset R(N)$ and by Lemma 2.1, this implies that N is a commutative ring.[11]

Theorem 3.1.4: Let N be a 2-torsion free prime near-ring satisfying $pod(q) \in R(N)$ for all $p, q \in N$ associated with a nonzero derivation d, it then follows that N is a commutative ring.

Proof:

$$\text{By hypothesis, } pod(q) \in R(N) \text{ for all} p, q, t \in N. \qquad (4.1)$$

(a) If $Z(N) = \{0\}$, then Equation (4.1) gives

$$pd(q) = -d(q)p \text{ for all } p, q, r \in N. \qquad (4.2)$$

Substituting pr for p in Equation (4.2), we get

$prd(q) = -d(q)pr$ for all $p, q, r \in N$. Using equation (4.2), we have

$$x(-d(y)z) = -d(y)xz) \text{ for all } p, q, r \in N. \text{ This means}$$

$$(pd(= q) + d(q)p)r = 0 \text{ for all } p, q, r \in N. \qquad (4.3)$$

Replacing $-q$ with q in (4.3), we have

$$pd(-q) + d(q)pNr = \{0\} \text{ for all } p, q, r \in N. \qquad (4.4)$$

Since N is prime, Equation (4.4) implies that $d(N) \subset Z(N)$, and from Lemma 2.1 we can then conclude that N is a commutative ring.□

(b) Suppose that $R(N) \neq 0$ if $0 \neq r \in R(N)$ and

$$d(q) + d(q) \in (N) \text{ for all } q \in N. \qquad (4.5)$$

From Equation (4.1),

$pd(q + q) + d(q + q)p \in R(N)$ for all $p, q \in N$, and using Equation (4.5), we have

$p(d(q + q) + d(q + q)) \in R(N)$ for all $p, q \in N$

Therefore, for all $p, q, r \in N$, we get

$$rp(d(q + q) + d(q + q)) = p(d(q + q) + d(q + q))r \text{ for all } p, q, r \in N$$

$$= (d(q + q) + d(q + q))pr \text{ for all } p, q, r \in N.$$

This implies

$$d(q + q) + d(q + q)N[r, p] = \{0\} \text{ for all } p, q, r \in N. \qquad (4.6)$$

Since N is prime, (4.6) implies that either $d(y + y) + d(y + y) = 0$ and hence $d = 0$, a contradiction, or $N \subset R(N)$ in which case $d(N) \subset R(N)$. Thus N is a commutative ring, using Lemma 2.1.□

Theorem 3.1.5: Let N be a 2-torsion free prime near-ring. If there exists no nonzero derivation d of N such that $pod(q) = qop$ for all $p, q \in N$, then N is a commutative.

Proof:

Suppose $pod(q) = qop$ for all $p, q \in N$. (5.1)

Replacing q by qp in equation (5.1), we have $pod(qp) = qpop$ for all $p, q \in N$.

But $pod(qp) = pd(qp) + d(qp)p$ and $qpox = qpp + pqp = (qop)p$ for all $p, q \in N$.

This implies that $pd(qp) + d(qp)p = (qop)p$ for all $p, q \in N$.

Therefore, by definition of d and using lemma 2.2(ii), we get

$pd(q)p + pqd(p) + d(q)pp + qd(p)p = (qop)p$. for all $p, q \in N$

This implies that $(pd(q) + d(q)p)p + pqd(p) + qd(p)p = (qop)p$ for all $p, q \in N$.

But, $(pd(q) + d(q)p)p = (qop)p$ for all $p, q \in N$, hence for all $p, q \in N$.

$$pqd(p) = -qd(p)p. \quad (5.2)$$

Substituting qr for q in equation (5.2), we obtain

$$pqrd(p) = -qrd(p)p = -q(rd(p)p) = -q\big(-prd(p)\big) = -p(-p)rd(p),$$

for all $p, q, r \in N$.

This implies that

$$pqrd(p) = -q(p)rd(p) \text{ for all } p, q, r \in N\ . \quad (5.3)$$

Since $-pqrd(p) = (-p)qrd(x)$, equation (5.3) becomes

$$(-p)qrd(p) = q(-p)rd(p) \text{ for all } p, q, r \in N \quad (5.4)$$

Replacing $-p$ by p in Equation (5.4) we obtain $pqrd(-p) = qprd(-p)$ for all $p, q, r \in N$

So $(pq - qp)rd(-p)$ =0, and this means for all $p, q \in N$,

$$[p, q]Nd(-p) = \{0\} \quad (5.5)$$

Since N is prime, equation (5.5) yields $p \in R(N)$ or $d(-p) = 0$ for each $p \in N$. It follows that

$$d(p) = 0 \text{ or } p \in R(N) \text{ for all } p, q \in N. \quad (5.6)$$

Equation (5.6) is the same as equation (1.6). Therefore, arguing as in the proof of Theorem 3.1.1, we conclude that N is a commutative ring. Using equation (5.1), that is, $pod(q) = qop$ for all $p, q \in N$, we can say $pd(q) = qop$ for all $p, q \in N$. This implies $pd(qt) = qtp$ for all $p, q, t \in N$ so that $ptd(q) = 0$ and by the primeness of N and $d \neq 0$, we conclude that $d = 0$ for all $\in N$; a contradiction.[12]

3.Example

The following example shows that the hypothesis of primeness in our theorems cannot be omitted.

Let A be a non commutative left near-ring.If $N = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} | \alpha, \beta, \gamma \in A \right\}$ and we define $d: N \to N$ by $d \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then it can be easily verified that d is a non-zero derivation of N satisfying the following conditions :

(i) $d([A, B]) = [A, d(B)]$

(ii) $[A, d(B)] = [A, B]$

(iii) $[A, d(B)] \in Z(N)$

(iv) $Aod(B) \in Z(N)$

(v) $Aod(B) = BoA$

(vi) $d(AoB) + [A, B] = 0$

Meanwhile, N is not a commutative ring.

In conclusion, these findings can be further expanded to encompass generalized derivations and semi-derivations within the context of prime and semiprime near-rings. These could further offer a broader framework for understanding their implications and applications.

References

[1] Ashraf, M.and Rehman, N. (2001). On derivation and commutativity in prime rings, *East −West Journal of Mathematics*, 3 (1) , 87 − 91.

[2] Ashraf, M., Boua, A. and Raji, A. (2014). On derivations and commutivity in prime near-rings, *Journal of Taibah university for Science*,8(3) 301-306.

[3] Ashraf, M., and Sideeque, M.A. (2018). On Derivation in Near-ring and its Generalizations: A Survey; *Palestine Journal of Mathematics* vol 7 (special issue :1,2018),111-124.

[4] Bell, H.E. and Mason, G. (1987). On Derivations in Near-Rings, *Elsevier Science Publishers* (North-Holland) 31-35.

[5] Bell, H.E. and Daif, M.N. (1995). On derivations and commutativity in prime rings, *Acta Math. Hungary* 66 (4) ,337 −343.

[6] Bell, H.E. and Daif, M.N. (1992). On remark on derivations on semiprime rings, *International Journal of Mathematics and Mathematical Sciences*. 15 (1), 205−206.

[7] Boua, A. and Abderrahmane R. (2013): 3-Prime Near-rings with involutions*: International Journal of Mathematical Engineering and Science* 2 (8)

[8] Boua, A and Oukhtite, L (2011) Derivations on Prime Near-rings international journal open9 problems *Computational. Mathematics,* 4 (2)

[9] Boua, A., Raji, A., Ali., A and Farhat, A. (2015). On Generalized semi-derivations of prime near-rings, *International Journal of Mathematics and Mathematical Sciences,* article ID 867923.

[10] Cho,Y.U. (2010). Derivation properties in prime Near-rings. Journal of Algebra, *Number Theory: Advances and Applications*, 3, Number 2: 71-72.

[11] James and Clay (1992). Near-rings: Genesis and Applications, University of Arizona, Oxford University Press 1992.

[12] Kamal, A.M and Al-Shaalan, K.H. (2013): Existence of Derivations on Near rings. *Mathematical Institute, Slovakia Academy of Sciences*, 63(3)431-448.

[13] Wang, X.K (1994), Derivations in prime near-rings, *Proceedings of the American Mathematical Society*. 121:361-366.