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On the vector Lyapunov Functions and Asymptotic Eventual Stability for nonlinear Impulsive Differential Equations via Comparison Principle

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ABSTRACT

In this paper, the asymptotic eventual stability of nonlinear impulsive differential equations with fixed moments of impulse is examined using the vector Lyapunov functions, which is generalized by a class of piecewise continuous Lyapunov functions. The novelty in the use of the vector Lypunov functions lies in the fact that the "restrictions" encountered by the scalar Lyapunov function is safely handled especially for large scale dynamical systems, since the method involves splitting the Lyapunov functions into components so that each of the components can easily describe the behavior of the solution state. Together with comparison results, sufficient conditions for the asymptotic eventual stability are presented

1. Introduction

A key property of interest in the qualitative theory of differential equations is the stability of solutions, since it allows us to compare the behavior of solutions that begin at different points [1].

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Now, the stability of differential equation of solutions using the Lyapunov method has been extensively studied in the past. In many practical cases, it is essential to examine the stability of sets that are invariant under a given system of differential equations. This, however, excludes stability in the Lyapunov sense [23]. To address the issue that will arise later, [14] introduced a new concept known as eventual stability, arguing that the set in question, while not invariant in the traditional sense, remains invariant in an asymptotic sense (See also [25]).

The theory of impulsive differential equations (IDE) is more extensive and sophisticated than the theory of regular differential equations [11], as they provide crucial models for accurately representing the actual state of various real-world processes and phenomena.

Of course, many evolutionary processes are marked by sudden, abrupt changes in state at specific points in time. These processes experience short-term disturbances that last for a negligible amount of time compared to the overall duration of the process. As a result, it is reasonable to assume that these disturbances act instantaneously, in the form of impulses. For instance, impulsive effects are observed in various fields, including biological phenomena with thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency-modulated systems [11].

Now, the effective use of impulsive differential systems now requires identifying criteria to determine the stability of their solutions [21], and one of the most widely used methods for studying the stability properties of impulsive systems is the Lyapunov function method.

There are various approaches in the literature for studying the stability of solutions, one of which is Lyapunov's second method. What sets this method apart from other stability analysis techniques, such as the Razumikhin method, the use of matrix inequalities, variational methods, Banach fixed-point theory, and monotone iteration methods is that, it allows stability to be analyzed without the need to first solve the differential system. This method involves finding an appropriate continuously differentiable Lyapunov function that is positive definite, with a time derivative along the solution path that is negative semidefinite. The stability of the trivial solution for impulsive differential equations has been thoroughly studied using this approach (see [3, 6, 20]).

In this paper, the asymptotic eventual stability of the system of nonlinear impulsive differential equation is examined. By employing the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions, and together with the comparison results, sufficient conditions for the asymptotic eventual stability of the solution set is established with illustrative example.

2. Preliminary notes and Definitions

Let R^n be the n-dimensional Euclidean space with norm $\|.\|$. Let Ω be a domain in R^n containing the origin; $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, $t_0 \in R_+$, $t > 0$.

Let $J \subset R_+$. Define the following class of functions $PC[J,\Omega] = \alpha : J \to \Omega.$, $\alpha(t)$ as a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t_k^+)$ exists.

Consider the impulsive differential system

$$\begin{aligned} x' &= f(t,x), t \neq t_k,\ t \geq t_0, k = 1,2,... \\ \Delta x &= I_k(x),\ t = t_k, k \in N \\ x(t_0^+) &= x_0 \end{aligned} \tag{2.1}$$

under the following assumptions:

A_0 (i) $0 < t_1 < t_2 < \ldots < t_k < \ldots,$ and $t_k \to \infty$ $k \to \infty;$

(ii) $f : R_+ \times R^n \to R^n$ is continuous in $(t_{k-1,}\ t_k] \times R^N$ and for each $x \in R^n$, $k = 1,2,\ldots,$

$\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^{\ +}, x)$ exists;

(iii) $I_k : R^n \to R^n$

In this paper, we assume that the function f is Lipschitz continuous with respect to its second argument, and $f(t,0) \equiv 0,\ I_k(0) \equiv 0$ for all $k,$ so that we have the trivial solution for (2.1), and the points t_k, $k = 1,2,\ldots$ are fixed such that $0 < t_1 < t_2 < \ldots and \lim_{k\to\infty} t_k = \infty.$ The system (2.1) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t;t_0,x_0) \in PC([t_0,\infty),R^N).$ Note that some sufficient conditions for the existence and uniqueness of the global solutions to (2.1) are considered in [9, 15, 16, 18, 26].

Remark 2.1.

The second equation in (2.1) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 2.1.

Let $V : R_+ \times R^N \to R_+^N$ be a continuous mapping of $R_+ \times R^N$ into R_+^N. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1,}\ t_k] \times R^N$ and for each $x \in R^N$, $k = 1,2,\ldots,$ and $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^{\ +}, x)$ exists;

(ii) V is locally Lipschitz with respect to its second argument $x.$ For $(t_{k-1,}\ t_k] \times R^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^+V(t,x) = \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)] \tag{2.2}$$

Note that in (2.1), $D^+V(t,x)$ is a functional whereas V is a function.

Definition 2.2.

A function $g(t,u)$ is said to be quasimonotone non-decreasing in u, if $u \le v$ and $u_i = v_i$ for $1 \le i \le N$ implies $g_i(t,u) \le g_i(t,v)$ for all i.

Definition 2.3.

The trivial solution $x = 0$ of (2.1) is said to be

(ES1) eventually stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ *for* $t \ge t_0$;

(ES2) uniformly eventually stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that for any $x_0 \in R^n$, the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ *for* $t \ge t_0$;

(ES3) asymptotically eventually stable if it is stable and if for each $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \geq t_0 + T$ and $\|x_0\| \leq \delta$ we have $\|x(t, t_0, x_0)\| < \varepsilon.$

(ES4) uniformly asymptotically eventually stable if it is uniformly stable and $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that for $t \geq t_0 + T$, the inequality $\|x_0\| \leq \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$.

Definition 2.4.

A function $a(r)$ is said to belong to the class K if $a \in C[[0, \rho), R_+], a(0) = 0,$ *and* $a(r)$ is strictly monotone increasing in r.

In this paper, we define the following sets:

$$\overline{S}_\psi = \{x \in R^N : \|x\| \leq \psi\}$$

$$S_\psi = \{x \in R^N : \|x\| < \psi\}$$

Remark 2.2.

The inequalities between vectors are understood to be component-wise inequalities.

Definition 2.5.

A function $b(r)$ is said to belong to a class L if $b \in C[J, R_+], b(t)$ is monotone decreasing in t *and* $b(t) \to 0$ *as* $t \to \infty.$

Definition 2.6.

A function $a(t,r)$ is said to belong to the class KK if $a \in C[[0, \rho), R_+], a \in K$ for each $t \in J,$ and a is monotone increasing in t for each $r > 0$ and $a(t,r) \to \infty$ *as* $t \to \infty$ for each $r > 0.$

Definition 2.7.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be positive definite if there exists a function $a \in K$ such that the relation $V(t,x) \geq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho.$

Definition 2.8.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be negative definite if there exists a function $a \in K$ such that the relation $V(t,x) \leq -a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho.$

Definition 2.9.

A function $V(t,x) \geq 0$ is said to be decrescent if there exists a function $a \in K$ such that the relation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho.$

Alongside (2.1), we shall consider a comparison system of the form

$$\begin{aligned} u' &= g(t,u), t \neq t_k,\ t \geq t_0, k = 1,2,... \\ \Delta u &= \psi_k(u(t_k)),\ t = t_k, \\ u(t_0^+) &= u_0 \end{aligned} \tag{2.3}$$

existing for $t \geq t_0,$ where $u \in R^n,$ elation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho.$ existing for $t \geq t_0,$ $u \in R^n,$ $R_+ = [t_0, \infty),$ $g: R_+ \times R_+^n \to R^n,$ $g(t,0) \equiv 0,$ where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n,$ the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution

$u(t,t_0,u_0) \in PC([t_0,\infty), R^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [9,18, 24, 26].

Lemma 2.1. Assume that the hypotheses $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0.$ Then the interval J can be extended to the maximal interval of existence $[t_0,\infty).$

Proof.

Since the conditions $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0,$ then from the existence theorem for the equation $x' = f(t,x(t))$ [18] with impulses, it follows that the solution $x(t) = x(t,t_0,x_0)$ of the IVP (2.1) is defined on each of the intervals $(t_{k-1,}\ t_k],\ k = 1,2,...$ Again, since $0 < t_1 < t_2 < ...and \lim_{k\to\infty} t_k = \infty,$ then we conclude that the interval J can be continued to $[t_0,\infty)$ for t_0.

3. Main Results

In this section we begin by proving the comparison results, then proceed to establish the necessary conditions for the eventual stability of the set $x(t) = 0$ of the impulsive differential systems with fixed moments of impulse.

Using (2.3), Definition 2.2 can be analogously defined as follows:

Definition 3.1.

The trivial solution $u = 0$ of (2.3) is said to be

(ES_1^*) eventually stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon,t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|u_0\| < \delta$ implies $\|u(t,t_0,u_0)\| < \varepsilon\ for\ t \ge t_0$;

(ES_2^*) uniformly eventually stable if the δ in (S_1^*) above is independent of t_0 i.e. for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|u_0\| < \delta$ implies $\|u(t,t_0,u_0)\| < \varepsilon\ for\ t \ge t_0$;

(ES_3^*) asymptotically eventually stable if S_1^* is satisfied and given $\varepsilon > 0\ and\ t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0,\varepsilon) > 0$ such that for $t \ge t_0 + T$ and $\|u_0\| \le \delta$ we have $\|u(t,t_0,u_0)\| < \varepsilon,\ t \ge t_0 + T.$

(ES_4^*) uniformly asymptotically eventually stable if (ES_2^*) is satisfied (ES_3^*) is independent of t_0.

THEOREM 3.1. (Comparison results) Assume that

(i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,\ and\ g(t,u)$ is quasimonotone non-decreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

(ii) $r(t) = r(t,t_0,u_0) \in PC([t_0,T), R^n)$ is the maximal solution of (2.3) existing for $t \ge t_0$.

(iii) $V \in C[J \times S_\psi, R_+^N],\ V \in L$ such that for $(t,x) \in J \times S_\psi$

$$D^+V(t,x) \le g(t,V(t,x)),\ t \ne t_k$$

and

$V(t,x+I_k(x(t_k)) \le \rho_k(V(t,x)),\ t=t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is non-decreasing for $k=1,2,...$

(iv) $x(t)=x(t,t_0,x_0) \in PC([t_0,T],R^N)$ is a solution of (2.1) such that,

$$V(t_0,x_0) \le u_0 \tag{3.1}$$

existing for $t \ge t_0$. Then

$$V(t,x(t)) \le r(t) \tag{3.2}$$

Proof.

Let $x(t)=x(t,t_0,x_0)$ be any solution of (2.1) existing for $t \ge t_0$, such that $V(t_0,x_0) \le u_0$.

Set $m(t)=V(t,x(t))$ for $t \ne t_k$ so that for small $h>0$, using (2.2) we have

$$m(t+h)-m(t)=V(t+h,x(t+h))-V(t+h,x(t)+hf(t,x(t))+V(t+h,x(t)+hf(t,x(t))-V(t,x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h)-m(t) \le k\|x(t+h)-(x(t)+hf(t,x(t)))\|+V(t+h,x(t)+hf(t,x(t))-V(t,x)$$

Dividing through by $h>0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h)-m(t)] \le \limsup_{h\to 0^+} \tfrac{i}{h}\left[k\|x(t+h)-x(t)-hf(t,x(t)\|e\right]+\limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h,x(t)+ hf(t,x(t))-V(t,x)]$$

where k is the local Lipschitz constant and $e=(1,1,...,1)^T$

It follows that

$$D^+m(t)=D^+V(t,x(t)) \le g(t,m(t))$$

and using condition (ii) of Theorem 3.1 we arrive at

$$V(t,x(t)) \le r(t)$$

provided

$$V(t_0,x_0) \le u_0$$

Also,

$$m(t_k^+)=V(t_k^+,x(t_k)+I_k(x(t_k^+)) \le \psi_k(m(t_k^+))$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.1).

Corollary 3.2. Assume that

(i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, *and* $g(t,u)$ is quasimonotone non-decreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u)=g(t_k^+,u)$ exists;

(ii) $p(t)=p(t,t_0,u_0) \in PC([t_0,T),R^n)$ is the minimal solution of (2.3) existing for $t \ge t_0$.

(iii) $V \in C[J \times S_\psi, R_+^N]$, $V \in L$ such that for $(t,x) \in J \times S_\psi$

$$D^+V(t,x) \ge g(t,V(t,x)),\ t \ne t_k$$

and

$V(t, x + I_k(x(t_k)) \geq \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is non-decreasing for $k = 1,2,...$

(iv) $x(t) = x(t, t_0, x_0) \in PC([t_0, T], R^N)$ is a solution of (2.1) such that,

$$V(t_0, x_0) \geq u_0 \tag{3.3}$$

existing for $t \geq t_0$. Then

$$V(t, x(t)) \geq p(t) \tag{3.4}$$

Proof.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) existing for $t \geq t_0$, such that $V(t_0, x_0) \geq u_0$.

Set $m(t) = V(t, x(t))$ for $t \neq t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)) + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Since $V(t, x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \geq k\|x(t+h) - (x(t) + hf(t, x(t)))\| + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h) - m(t)] \geq \limsup_{h\to 0^+} \tfrac{1}{h}\left[k\|x(t+h) - x(t) - hf(t, x(t)\|\right]e + \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x(t) + hf(t, x(t)) - V(t, x)]$$

where k is the local Lipschitz constant and $e = (1,1,...,1)^T$

It follows from condition (ii) of Cor 3.2 we arrive at the estimate

$$D^+ m(t) = D^+ V(t, x(t)) \geq g(t, m(t)), t \neq t_k,\ m(t_0^+) \geq u_0 \tag{3.5}$$

Also,

$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k^+)) \geq \psi_k(m(t_k^+)) \tag{3.6}$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.5).

In what follows, we shall obtain sufficient conditions for the eventual stability of the system (2.3).

THEOREM 3.2. Assume that:

(i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,$ *and* $g(t,u)$ is quasimonotone non-decreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n],\ V(t,x)$ is locally Lipschitzian in x and

$\sum_{i=1}^{N} V_i(t,x) \to 0$ *as* $\|x\| \to 0$ for each t and $(t,x) \in J \times S_\rho$,

$$D^+ V(t,x) \leq g(t, V(t,x)) \tag{3.7}$$

(iii) *for* $(t,x) \in J \times S_\rho$,

$$b(\|x(t)\|) \leq \sum_{i=1}^{N} V_i(t,x) \tag{3.8}$$

where $b \in K$, whence $b \in C[J \times S_\rho, R_+]$

Then the eventual stability of the set of trivial solution $u = 0$ of the system (2.3) implies the eventual stability of the set of trivial solution $x = 0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} \leq \delta \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), t \geq t_0 \tag{3.9}$$

Since $V(t, x)$ is continuous and mildly unbounded i.e. $V(t,0) \to 0 \; as \; \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each $\in$ such that the inequalities

$$\|x_0\| < \delta_1 \;\; and \;\; V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t, t_0, x_0)\| < \varepsilon$ by the stability of $x(t)$.

Now suppose this claim is false, there would exists a point $t_1 \in [t_0, t)$ and the solution $x(t, t_0, x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \;\; and \;\; \|x(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)) \text{, implying}$$

$$b(\varepsilon) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)) \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t, x(t)) \leq r(t, t_0, u_0), \tag{3.13}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.3).

Then using equations (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \leq V_0(t_1, x(t_1) \leq \sum_{i=1}^{N} r_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the eventual stability of the set of trivial solution $u = 0$ of (2.3) implies the eventual stability of the set of trivial solution $x = 0$ of (2.1).

In what follows, we shall establish sufficient conditions for the asymptotic eventual stability of the main system (2.1).

THEOREM 3.3. Assume that:

(i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, and $g(t,u)$ is quasimonotone non-decreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n]$, $V(t,x)$ is locally Lipschitzian in x and

$\sum_{i=1}^{N} V_i(t,x) \to 0 \ as \ \|x\| \to 0$ for each t and $(t,x) \in J \times S_\rho$,

$$D^+V(t,x) \le g(t, V(t,x)) \tag{3.7}$$

(iii) $for \ (t,x) \in J \times S_\rho$,

$$b(\|x(t)\|) \le \sum_{i=1}^{N} V_i(t,x) \le a(t,\|x\|) \tag{3.8}$$

where $b \in K$, $a(t,.) \in K$ whence $a \in C[J \times S_\rho, R_+]$

Then the asymptotic eventual stability of the set of trivial solution $u = 0$ of the system (2.3) implies the asymptotic eventual stability of the set of trivial solution $x = 0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} \le \delta \text{ implies } \sum_{i=1}^{N} u_i(t,t_0,u_0) < b(\varepsilon), t \ge t_0 \tag{3.9}$$

Since $V(t,x)$ is continuous and $V(t,0) \to 0 \ as \ \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each ε such that the inequalities

$$\|x_0\| < \delta_1 \ \ and \ \ V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t,t_0,x_0)\| < \varepsilon.$

Now suppose this claim is false, there would exist a point $t_1 \in [t_0, t)$ and the solution $x(t,t_0,x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \ \ and \ \ \|x(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \le \sum_{i=1}^{N} V_i(t_1, x(t_1)) \text{, implying}$$

$$b(\varepsilon) \le \sum_{i=1}^{N} V_i(t_1, x(t_1)) \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t, x(t)) \le r(t, t_0, u_0), \tag{3.13}$$

where $r(t,t_0,u_0)$ is the maximal solution of (2.3).

Then using (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \le V_0(t_1, x(t_1) \le \sum_{i=1}^{N} r_i(t,t_0,u_0) < b(\varepsilon)$$

which leads to an absurdity.

Hence, the asymptotic eventual stability of the set of trivial solution $u=0$ of (2.3) implies the asymptotic eventual stability of the set of trivial solution $x=0$ of (2.1).

4 Application

Consider the system of fractional differential equations

$$\begin{aligned} x_1'(t) &= -15x_1 - x_2\cos x_1 + x_1\sin x_2 + x_2\sec x_1,\ t \neq t_k \\ x_2'(t) &= 5x_1 - 4x_2\sin x_1 - x_2\sec x_2 - x_1\cos x_2,\quad t \neq t_k \\ \Delta x_1 &= \varpi_k(x(t_k)),\ t = t_k \\ \Delta x_2 &= \zeta_k(x(t_k)),\ t = t_k,\ k = 1,2,... \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial conditions,

$$x_1(t_0^+) = \gamma_{10} \ \ and \ \ x_2(t_0^+) = \gamma_{20}$$

where $x_1, x_2 \in R^N$ are arbitrary functions.

Equation (4.1) is equivalent to (2.3) and $f = (f_1 f_2)$, where $f_1(t,x_1) = -15x_1 - x_2\cos x_1 + x_1\sin x_2 + x_2\sec x_1$ and $f_2(t,x_2) = 5x_1 - 4x_2\sin x_1 - x_2\sec x_2 - x_1\cos x_2$.

Consider a vector Lyapunov function of the form $V = (V_1, V_2)^T$, where $V_1(t, x_1, x_2) = |x_1|$ and $V_2(t, x_1, x_2) = |x_2|$. So that $V = (V_1, V_2)^T$ with $x = (x_1, x_2) \in R^2$, so its associated norm defined by $\|x\| = |x_1| + |x_2|$. Now,

$$\sum_{i=1}^{2} V_i(t, x_1, x_2) = |x_1| + |x_2|$$

So that, the assumption,

$$b(\|x\|) \leq \sum_{i=1}^{n} V_i(x, y) \leq a(t, \|x\|) \text{ reduces to,}$$

$$\sqrt{x_1^{\ 2} + x_2^{\ 2}} \leq x_1^{\ 2} + x_2^{\ 2} \leq 2\left(\sqrt{x_1^{\ 2} + x_2^{\ 2}}\right)^2$$

with the proviso that $b(r) = r, and\ a(r) = 2r^2$.

Furthermore, we deduce that using equation (3.4) and $V_1(t, x_1, x_2) = |x_1|$

$$\begin{aligned} D^+V(t,x) &= \limsup_{h\to 0^+} \tfrac{1}{h}\{V(t+h, x+hf(t,x)) - V(t,x)\}, t \geq t_0 \\ D^+V_1(t,x_1) &= \limsup_{h\to 0^+} \tfrac{1}{h}\{|x_1 + hf_1(t,x_1)| - |x_1|\} \\ &= \limsup_{h\to 0^+} \tfrac{1}{h}\{hf_1(t,x_1)\} \\ &\leq f_1(t,x_1) \end{aligned} \tag{4.2}$$

$$D^+V_1(t,x_1) = -15x_1 - x_2\cos x_1 + x_1\sin x_2 + x_2\sec x_1$$

$$D^+V_1(t,x_1) = x_1(-15 + \sin x_2) + x_2(\cos x_1 + \sec x_1)$$

$$= x_1(-15 + \sin x_2) + x_2(\cos x_1 + \frac{1}{\cos x_1})$$

$$\leq |x_1|(-15 + |\sin x_2|) + |x_2|(|\cos x_1|) + \frac{1}{|\cos x_1|})$$

$$\leq |x_1|(-15 + 1) + |x_2|(1 + 1)$$

$$\therefore D^+V_1(t, x_1) \leq -14V_1 + 2V_2 \tag{4.3}$$

Also for $x_0 \in S_\psi$, $for\ t = t_k, k = 1,2,...$ we have

$$V(t, x(t) + \varpi_k) = |\varpi_k + x(t)| \leq V(t, x(t))$$

Again for $V_2(t, x_1, x_2) = |x_2|$ and deducing from (4.2) we have

$$D^+V_2(t, x_1) \leq f_2(t, x_2)$$

$$D^+V_2(t, x_2) = 5x_1 - 4x_2 \sin x_1 - x_2 \sec x_2 - x_1 \cos x_2$$

$$= x_1(5 - \cos x_2) + x_2(-4\sin x_1 - \sec x_2)$$

$$\leq |x_1|(5 - |\cos x_2|) + |x_2|(-4|\sin x_1| - \frac{1}{|\cos x_1|})$$

$$\leq |x_1|(5 - 1) + |x_2|(-4 - 1)$$

$$\therefore D^+V_2(t, x_1) \leq 4V_1 - 5V_2 \tag{4.4}$$

Also for $x_0 \in S_\psi$, $for\ t = t_k, k = 1,2,...$ we have

$$V(t, x(t) + \zeta_k) = |\zeta_k + x(t)| \leq V(t, x(t))$$

Combining (4.3) and (4.4) gives

$$D^+V \leq \begin{pmatrix} -14 & 2 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V) \tag{4.5}$$

$$u' = g(t, u) = Au$$

where $A = \begin{pmatrix} -14 & 2 \\ 4 & -5 \end{pmatrix}$.

Thus, the vectorial inequality (4.5) and all other conditions of Theorem 3.2 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (4.1) is asymptotically eventually stable. Therefore, the set $x(t) = 0$ is asymptotically eventually stable.

Conclusion

This study investigated the asymptotic eventual stability of nonlinear impulsive differential equations by using auxiliary Lyapunov functions, which serve as analogues to vector Lyapunov functions. By decomposing the Lyapunov function into components, each state or solution vector can be inserted into a corresponding component of the Lyapunov function, rather than being inserted into the entire function. This approach makes it easier for the Lyapunov function to accurately predict the behavior of the solution vectors, addressing the "restrictions and challenges" typically encountered when using a scalar Lyapunov function. In conjunction with comparison results, the paper presents sufficient conditions for the asymptotic eventual stability of impulsive differential systems.

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