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Adaptive Nonparametric Regression Model via a Global Mixing Parameter for the Multi-Response Problem

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ABSTRACT

The modeling stage of Response Surface Methodology (RSM) involves using regression models to estimate the functional relationship between the response variable and explanatory variables, relying on data generated through an appropriate experimental design. Traditionally, Ordinary Least Squares (OLS) is employed to model the data using user-specified low-order polynomials. However, OLS performance deteriorates when the homoscedasticity assumption is violated. In the literature, semiparametric regression models are preferred for RSM as they combine the strengths of parametric and nonparametric approaches, unlike purely nonparametric models, which are more sensitive to the idiosyncrasies of RSM data. This paper proposes a novel integration of an adaptive nonparametric regression model with a locally adaptive bandwidth selector derived from the explanatory variables to achieve adequate data smoothing. The adaptive nonparametric regression model incorporates local linear regression (LLR) and a product of the optimal mixing parameter and the LLR residuals, providing a second chance to fit portions of the data not captured by the LLR model. Meanwhile, the locally adaptive bandwidth selector addresses challenges such as dimensionality, sparsity in RSM data, and cost-efficient design. In applying this approach to three types of RSM data, the novel integrated model demonstrated superior performance in terms of goodness-of-fit statistics, zero residual plots, optimization results, and simulations, when compared to OLS, Model Robust Regression 1 (MRR1), and Model Robust Regression 2 (MRR2).

1. Introduction

[1, 2], described Response Surface Methodology (RSM) as a statistical technique employed by engineers and industrial statisticians for experimental model building, aimed at optimizing response variables influenced by multiple explanatory variables.

RSM is appropriate for optimizing the response variable $\boldsymbol{y}$ as a function of several explanatory variables $(x_{i1}, x_{i2}, \ldots, x_{ik})$ which is given as:

$$y_i = f(x_{i1}, x_{i2}, \ldots, x_{ik}) + \varepsilon_i, \quad i = 1,2, \ldots, n \tag{1}$$

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where ε_i is the error term and assumed to be normally distributed with mean zero and variance $\boldsymbol{\sigma}^2$. The surface as given in (1) characterized by $f(x_{i1}, x_{i2}, \ldots, x_{ik})$ is termed a response surface [3].

1.1 Ordinary Least Squares (OLS)

The common method for estimating the parameter vector is usually based on the Method of Ordinary Least Squares (OLS). The parameter vector estimates $\widehat{\boldsymbol{\beta}}$ is given as:

$$\widehat{\boldsymbol{\beta}}^{(OLS)} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'y \tag{2}$$

The estimated responses for the i^{th} location can be written as :

$$\hat{y}_i^{(OLS)} = \boldsymbol{x}_i'\widehat{\boldsymbol{\beta}}^{(OLS)} = \boldsymbol{x}_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'y\,, \qquad i = 1,2,\ldots,n \tag{3}$$

where $\boldsymbol{x}_i'$ is the i^{th} row of matrix $\boldsymbol{X}$, $\boldsymbol{X}$ is a matrix with dimension $n \times (k+1)$.
$\boldsymbol{H}_i = \boldsymbol{x}_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is the i^{th} row of the OLS "HAT" matrix of dimension $n \times n$, $\boldsymbol{H}^{(OLS)}$. The estimated response in the i^{th} location is given as:

$$\hat{y}^{(OLS)} = \boldsymbol{H}y\,. \tag{4}$$

where the matrix $\boldsymbol{H}$ is given as:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \\ \vdots \\ \boldsymbol{H}_n \end{bmatrix}, \tag{5}$$

[4, 5].

1.2 Model Robust Regression 1 (MRR1)

An effective model that addresses the drawbacks inherent in both parametric and nonparametric regression models is the use of semiparametric regression model, Model Robust Regression 1 (MRR1).
The mathematical expression for the MRR1 as given in [6, 7] as:

$$\hat{y}^{(MRR1)} = \lambda\hat{y}^{(LLR)} + (1-\lambda)\hat{y}^{(OLS)} \tag{6}$$

where the parameter λ is the mixing parameter with an interval [0, 1].

1.3 Model Robust Regression 2 (MRR2)

Model Robust Regression 2 (MRR2) combines estimates of parametric regression model to the raw data, while the nonparametric regression model portion, uses the LLR Hat matrix to fit the residuals from the estimates of parametric regression model through a mixing parameter, λ.
The MRR2 was developed by [8] and is expressed as:

$$\hat{y}^{(MRR2)} = \hat{y}^{(OLS)} + \lambda\hat{r}^{(LLR)}, \hat{r}^{(LLR)} = \boldsymbol{H}_{\boldsymbol{r}}^{(LLR)}r \tag{7}$$

$\lambda \in [0,1], r = y - y^{OLS}$ is the vector of residuals that represents the structure in the data not captured by the user specified parametric regression model.

1.4 Optimization Phase in RSM

This process utilizes optimization tools, such as Genetic Algorithms, to identify the optimal settings of the explanatory variables that optimize the fitted regression model. In RSM, two types of optimization problems are commonly encountered: single-response optimization and multiple-response optimization. The final step in optimization is to compute the overall desirability, which is determined as the geometric mean of the individual desirability values [9, 10, 11, 13].

1.5 Adaptive Nonparametric Regression Model

To enhance the utilization of the flexibility of MRR2, we designate the existing adaptive nonparametric regression model as PM2 for ease of reference.

The mathematical expression of PM2 estimate, $\hat{y}_i^{(PM2)}$ is defined by

$$\hat{y}_i^{(PM2)} = \hat{y}_i^{(LLR)} + \lambda h_i^{(LLR)}[(y_i - \hat{y}_i^{(LLR)})],\ i = 1,2,\ \dots,\ n. \quad (8)$$

The PM2 is applied in the estimation of the unknown function f in Equation (1) see [12].

1.6 Locally Adaptive Bandwidths Selector

The locally adaptive bandwidth selector accounts for two key aspects of RSM data: the kkk-th number of explanatory variables in the study and the data's sparsity, as outlined in [13]. This can be expressed mathematically as:

$$b_{ij} = T_{1j}\left(\frac{1}{2} - \frac{x_{ij}}{T_{2j}}\right)^2,\ i = 1,2,\dots,n; j = 1,2,\dots,k. \quad (9)$$

where the locally adaptive optimal bandwidths from Equation (9) is obtained at an optimally selected values of T_{1j}, T_{2j}, the tuning parameters (hereafter referred to as T_{1j}^* and T_{2j}^*, respectively), $j = 1,2,\dots,k$, based on the minimization of the $PRESS^{**}$ criterion.

2.0 Methodology

Although the flexibility of nonparametric regression methods, their application in Response Surface Methodology (RSM) remains limited due to the unique challenges of RSM data, including the curse of dimensionality, data sparsity, and the need for cost-efficient designs. This paper introduces a novel integration of an existing adaptive nonparametric regression model with a locally adaptive bandwidth selector derived from the explanatory variables. The bandwidth selector is embedded within the kernel weight matrix of the adaptive regression model.

The existing nonparametric regression model combines a portion of Local Linear Regression (LLR) estimates with the product of an optimal mixing parameter and the residuals, thereby providing a second opportunity to fit data not captured by the LLR component. The locally adaptive bandwidth selector further addresses challenges related to dimensionality, data sparsity, and small sample sizes, as discussed in [13].

2.1 Integrating the adaptive nonparametric regression model and locally adaptive bandwidths

In order to address the scanty utilization of the flexibility of MRR2, we concatinate Equations (8) and (9) respectively, a novel blend or approach.

The assumptions of PM2 and locally adaptive bandwidths are given below:

1. $\boldsymbol{x_{ij}} \in [0,1]$, *is a vector of kth explanatory variables at location* $i, \forall\ i = 1,2,\dots,n; j = 1,2,\dots,k.$
2. The optimal chosen tuning parameters T_{1j}^* , $T_{2j}^* > 0$ in all k explanatory variables
3. The optimal mixing parameter $\lambda \in [0,1]$

4. The optimal chosen bandwidths $b_{ij} \in (0,]$, $\forall\, i = 1, 2, \dots, n; j = 1,2, \dots, k$; for smoothing the data at location i and k explanatory variables.

The mathematical expression of PM2 estimate, $\hat{y}_i^{(PM2)}$ is defined by

$$\hat{y}_i^{(PM2)} = \hat{y}_i^{(LLR)} + \lambda h_i^{(LLR)}[(y_i - \hat{y}_i^{(LLR)})],\ i = 1,2,\ \dots,\ n. \quad (8)$$

$$b_{ij} = T_{1j}\left(\frac{1}{2} - \frac{x_{ij}}{T_{2j}}\right)^2,\ i = 1,2, \dots, n; j = 1,2, \dots, k.$$

(9)

The PM2 is applied in the estimation of the unknown function f in Equation (1). As soon as the PM2 and b_{ij} are combined to fit the data, we have a novel blend or approach which is now referred to as ANPM2 for easy referencing.

Hence, the ANPM2 estimate $\hat{y}_i^{(ANPM2)}$ of the response is given as:

$$\hat{\boldsymbol{y}}_{\boldsymbol{i}}^{(ANPM2)} = \boldsymbol{x}_{\boldsymbol{i}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{i}}\boldsymbol{X}^{(LLR)}\right)^{-\boldsymbol{1}}\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{i}}y + \lambda\, \boldsymbol{x}_{\boldsymbol{i}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{i}}^{*}\boldsymbol{X}^{(LLR)}\right)^{-\boldsymbol{1}}\boldsymbol{X}^{\prime(\boldsymbol{LLR})}\boldsymbol{W}_{\boldsymbol{i}}^{*}\left[y - \boldsymbol{x}_{\boldsymbol{i}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{i}}\boldsymbol{X}^{(LLR)}\right)^{-\boldsymbol{1}}\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{i}}y\right]$$

(10)

where $y = (y_1, \dots y_n)'$, $\boldsymbol{x}_{\boldsymbol{i}}^{\prime(LLR)} = (1\ x_{i1} \dots x_{ik})$ is the i^{th} row of the local linear regression model matrix, $\boldsymbol{X}^{(LLR)}$ given as:

$$\boldsymbol{X}^{(LLR)} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix} \quad (11)$$

where the kernel weight matrix is given by

$$\boldsymbol{W}_{\boldsymbol{i}} = \begin{bmatrix} w_{i1} & 0 & \cdots & 0 \\ 0 & w_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{in} \end{bmatrix}, i = 1,\ \ 2\ ,\ \ .\ \ .\ \ .\ \ , n\,. \quad (12)$$

[3, 12].

The kernel function $K\left(\frac{x_{ij}-x_{1j}}{b_{ij}}\right)$ is a simplified Gaussian kernel for one explanatory variable case, given as:

$$w_{i1} = K\left(\frac{x_{ij}-x_{1j}}{b_{ij}}\right) = e^{-\left(\frac{x_{ij}-x_{1j}}{b_{ij}}\right)^2} \quad (13)$$

Otherwise, the kernel function is a product kernel given as:

$$w_{i1} = \prod_{j=1}^{k} K\left(\frac{x_{ij}-x_{1j}}{b_{ij}}\right) \Big/ \sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{1j}}{b_{ij}}\right), p = 1,2, \dots, n, j = 1,2, \dots, k, \quad (14)$$

For $i = 1$ in Equations (10) and (12), and concantinating the existing bandwidths into the regression model to obtain a novel adaptive regression model . Thus, we have:

$$\hat{\boldsymbol{y}}_{\boldsymbol{1}}^{(ANPM2)} = \boldsymbol{x}_{\boldsymbol{1}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{1}}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{1}}y + \lambda\, \boldsymbol{x}_{\boldsymbol{1}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{1}}^{*}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(\boldsymbol{LLR})}\boldsymbol{W}_{\boldsymbol{1}}^{*}\left[y - \boldsymbol{x}_{\boldsymbol{1}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{1}}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{1}}y\right] \quad (15)$$

$$\boldsymbol{W}_{\boldsymbol{1}} = \begin{bmatrix} w_{11} & 0 & \cdots & 0 \\ 0 & w_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{1n} \end{bmatrix}_{(n\times n)} \quad (16)$$

The entries from Equation (16) and the locally adaptive bandwidths of [13] are translated to estimate $\hat{y}_1^{ANPM2}$**,**

$$w_{11} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{1j}-x_{1j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{1j}}{b_{pj}}\right)}, \quad p = 1, \ 2, \ . \ . \ . \ , n; j = 1, \ 2, \dots, k. \quad (17)$$

$$w_{11} = \frac{e^{-(\frac{x_{11}-x_{11}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{12}}{b_{12}})^2} \dots e^{-(\frac{x_{1k}-x_{1k}}{b_{1k}})^2}}{\left[e^{-(\frac{x_{11}-x_{11}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{12}}{b_{12}})^2} \dots e^{-(\frac{x_{1k}-x_{1k}}{b_{1k}})^2} + e^{-\left(\frac{x_{21}-x_{11}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{12}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{1k}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{11}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{12}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{1k}}{b_{nk}})^2}\right]}$$

$$w_{12} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{2j}-x_{1j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{1j}}{b_{pj}}\right)}, \quad p = 1, \ 2, \ . \ . \ . \ , n; j = 1, \ 2, \dots, k. \quad (18)$$

$$w_{12} = \frac{e^{-(\frac{x_{21}-x_{11}}{b_{21}})^2} e^{-(\frac{x_{22}-x_{12}}{b_{22}})^2} \dots e^{-(\frac{x_{2k}-x_{1k}}{b_{2k}})^2}}{\left[e^{-(\frac{x_{11}-x_{11}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{12}}{b_{12}})^2} \dots e^{-\left(\frac{x_{1k}-x_{1k}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{11}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{12}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{1k}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{11}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{12}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{1k}}{b_{nk}})^2}\right]}$$

$$\vdots$$

$$w_{1n} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{nj}-x_{1j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{1j}}{b_{pj}}\right)}, \quad p = 1, \ 2, \ . \ . \ . \ , n; j = 1, \ 2, \dots, k. \quad (19)$$

$$w_{1n} = \frac{e^{-\left(\frac{x_{n1}-x_{11}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{12}}{b_{n2}}\right)^2} \dots e^{-\left(\frac{x_{nk}-x_{1k}}{b_{nk}}\right)^2}}{\left[e^{-(\frac{x_{11}-x_{11}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{12}}{b_{12}})^2} \dots e^{-\left(\frac{x_{1k}-x_{1k}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{11}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{12}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{1k}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{11}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{12}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{1k}}{b_{nk}})^2}\right]} \quad (20)$$

To estimate, $\hat{y}_2^{ANPM2}$ set $i = 2$ in Equation (10) and (12), we have:

$$\hat{\boldsymbol{y}}_{\boldsymbol{2}}^{(ANPM2)} = \boldsymbol{x}_{\boldsymbol{2}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{2}}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{2}}y + \lambda\, \boldsymbol{x}_{\boldsymbol{2}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\boldsymbol{W}_{\boldsymbol{2}}^{*}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(\boldsymbol{LLR})}\boldsymbol{W}_{\boldsymbol{2}}^{*}\left[y - \boldsymbol{x}_{\boldsymbol{2}}^{\prime(LLR)}\left(\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{2}}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}^{\prime(LLR)}\grave{\boldsymbol{W}}_{\boldsymbol{2}}y\right] \quad (21)$$

$$\boldsymbol{W_2} = \begin{bmatrix} w_{21} & 0 & \cdots & 0 \\ 0 & w_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{2n} \end{bmatrix}_{(n\times n)} \tag{22}$$

The entries from Equation (22) and the locally adaptive bandwidths of [13] are translated to estimate $\boldsymbol{\hat{y}_2^{ANPM2}}$,

$$w_{21} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{1j}-x_{2j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{2j}}{b_{pj}}\right)}, \quad p = 1, 2, \ldots, n; j = 1, 2, \ldots, k. \tag{23}$$

$$w_{21} = \frac{e^{-(\frac{x_{11}-x_{21}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{22}}{b_{12}})^2} \ldots e^{-(\frac{x_{1k}-x_{2k}}{b_{1k}})^2}}{\left[e^{-(\frac{x_{11}-x_{21}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{22}}{b_{12}})^2} \ldots e^{-(\frac{x_{1k}-x_{2k}}{b_{1k}})^2} + e^{-\left(\frac{x_{21}-x_{21}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{22}}{b_{22}}\right)^2} \ldots e^{-\left(\frac{x_{2k}-x_{2k}}{b_{2k}}\right)^2} + \cdots + e^{-\left(\frac{x_{n1}-x_{21}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{22}}{b_{n2}}\right)^2} \ldots e^{-(\frac{x_{nk}-x_{2k}}{b_{nk}})^2}\right]} \tag{24}$$

$$w_{22} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{2j}-x_{2j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{2j}}{b_{pj}}\right)}, \quad p = 1, 2, \ldots, n; j = 1, 2, \ldots, k. \tag{25}$$

$$w_{22} = \frac{e^{-(\frac{x_{21}-x_{21}}{b_{21}})^2} e^{-(\frac{x_{22}-x_{22}}{b_{22}})^2} \ldots e^{-(\frac{x_{2k}-x_{2k}}{b_{2k}})^2}}{\left[e^{-(\frac{x_{11}-x_{21}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{22}}{b_{12}})^2} \ldots e^{-\left(\frac{x_{1k}-x_{2k}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{21}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{22}}{b_{22}}\right)^2} \ldots e^{-\left(\frac{x_{2k}-x_{2k}}{b_{2k}}\right)^2} + \cdots + e^{-\left(\frac{x_{n1}-x_{21}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{22}}{b_{n2}}\right)^2} \ldots e^{-(\frac{x_{nk}-x_{2k}}{b_{nk}})^2}\right]} \tag{26}$$

$$w_{2n} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{nj}-x_{2j}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{2j}}{b_{pj}}\right)}, \quad p = 1, 2, \ldots, n; j = 1, 2, \ldots, k. \tag{27}$$

$$w_{2n} = \frac{e^{-\left(\frac{x_{n1}-x_{21}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{22}}{b_{n2}}\right)^2} \ldots e^{-\left(\frac{x_{nk}-x_{2k}}{b_{nk}}\right)^2}}{\left[e^{-(\frac{x_{11}-x_{21}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{22}}{b_{12}})^2} \ldots e^{-\left(\frac{x_{1k}-x_{2k}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{21}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{22}}{b_{22}}\right)^2} \ldots e^{-\left(\frac{x_{2k}-x_{2k}}{b_{2k}}\right)^2} + \cdots + e^{-\left(\frac{x_{n1}-x_{21}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{22}}{b_{n2}}\right)^2} \ldots e^{-(\frac{x_{nk}-x_{2k}}{b_{nk}})^2}\right]} \tag{28}$$

To estimate, $\hat{y}_n^{ANPM2}$ set $i = n$ in Equation (10) and (12), we have:

$$\boldsymbol{\hat{y}_n}^{(ANPM2)} = \boldsymbol{x_n}'^{(LLR)}\left(\boldsymbol{X}'^{(LLR)}\boldsymbol{W_n}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}'^{(LLR)}\boldsymbol{W_n}y + \lambda\, \boldsymbol{x_n}'^{(LLR)}\left(\boldsymbol{X}'^{(LLR)}\boldsymbol{W_n^*}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}'^{(\boldsymbol{LLR})}\boldsymbol{W_n^*}\left[y - \boldsymbol{x_n}'^{(LLR)}\left(\boldsymbol{X}'^{(LLR)}\boldsymbol{\grave{W}_n}\boldsymbol{X}^{(LLR)}\right)^{-\mathbf{1}}\boldsymbol{X}'^{(LLR)}\boldsymbol{\grave{W}_n}y\right]$$

$$\boldsymbol{W_n} = \begin{bmatrix} w_{n1} & 0 & \cdots & 0 \\ 0 & w_{n2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{nn} \end{bmatrix}_{(n\times n)} \tag{29}$$

The entries from Equation (29) and the locally adaptive bandwidths of [13] are translated to estimate $\boldsymbol{\hat{y}_n^{ANPM2}}$,

$$w_{n1} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{1j}-x_{nj}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{nj}}{b_{pj}}\right)}, \quad p = 1, 2, ... , n; j = 1, 2, \dots, k. \tag{30}$$

$$w_{n1} = \frac{e^{-(\frac{x_{11}-x_{n1}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{n2}}{b_{12}})^2} \dots e^{-(\frac{x_{1k}-x_{nk}}{b_{1k}})^2}}{\left[e^{-(\frac{x_{11}-x_{n1}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{n2}}{b_{12}})^2} \dots e^{-(\frac{x_{1k}-x_{nk}}{b_{1k}})^2} + e^{-\left(\frac{x_{21}-x_{n1}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{n2}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{nk}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{n1}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{n2}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{nk}}{b_{nk}})^2}\right]} \tag{31}$$

$$w_{n2} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{2j}-x_{nj}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{nj}}{b_{pj}}\right)}, \quad p = 1, 2, ... , n; j = 1, 2, \dots, k. \tag{32}$$

$$w_{n2} = \frac{e^{-(\frac{x_{21}-x_{n1}}{b_{21}})^2} e^{-(\frac{x_{22}-x_{n2}}{b_{22}})^2} \dots e^{-(\frac{x_{2k}-x_{nk}}{b_{2k}})^2}}{\left[e^{-(\frac{x_{11}-x_{n1}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{n2}}{b_{12}})^2} \dots e^{-\left(\frac{x_{1k}-x_{nk}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{n1}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{n2}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{nk}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{n1}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{n2}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{nk}}{b_{nk}})^2}\right]} \tag{33}$$

$$w_{nn} = \frac{\prod_{j=1}^{k} K\left(\frac{x_{nj}-x_{nj}}{b_{ij}}\right)}{\sum_{p=1}^{n}\prod_{j=1}^{k} K\left(\frac{x_{pj}-x_{nj}}{b_{pj}}\right)}, \quad p = 1, 2, ... , n; j = 1, 2, \dots, k. \tag{34}$$

$$w_{nn} = \frac{e^{-\left(\frac{x_{n1}-x_{n1}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{n2}}{b_{n2}}\right)^2} \dots e^{-\left(\frac{x_{nk}-x_{nk}}{b_{nk}}\right)^2}}{\left[e^{-(\frac{x_{11}-x_{n1}}{b_{11}})^2} e^{-(\frac{x_{12}-x_{n2}}{b_{12}})^2} \dots e^{-\left(\frac{x_{1k}-x_{nk}}{b_{1k}}\right)^2} + e^{-\left(\frac{x_{21}-x_{n1}}{b_{21}}\right)^2} e^{-\left(\frac{x_{22}-x_{n2}}{b_{22}}\right)^2} \dots e^{-\left(\frac{x_{2k}-x_{nk}}{b_{2k}}\right)^2} + \dots + e^{-\left(\frac{x_{n1}-x_{n1}}{b_{n1}}\right)^2} e^{-\left(\frac{x_{n2}-x_{n2}}{b_{n2}}\right)^2} \dots e^{-(\frac{x_{nk}-x_{nk}}{b_{nk}})^2}\right]} \tag{35}$$

with respective diagonal matrices of kernel weights, $\boldsymbol{W_2}, \boldsymbol{W_3}, \dots, \boldsymbol{W_n}$ follows pattern from Equations (30, 32 and 34).

Using matrix notation, the ANPM2 can be expressed as:

$$\hat{y}^{(ANPM2)} = \begin{bmatrix} \boldsymbol{h_1^{(LLR)}} y + \boldsymbol{\lambda h_1^{(LLR)}}(y - (\boldsymbol{h_1^{(LLR)}} y) \\ \boldsymbol{h_2^{(LLR)}} y + \boldsymbol{\lambda h_2^{(LLR)}}(y - (\boldsymbol{h_2^{(LLR)}} y) \\ \vdots \\ \boldsymbol{h_n^{(LLR)}} y + \boldsymbol{\lambda h_n^{(LLR)}}(y - (\boldsymbol{h_n^{(LLR)}} y) \end{bmatrix}, \tag{36}$$

$$\hat{y}^{(ANPM2)} = \begin{bmatrix} \boldsymbol{h}_1^{(LLR)} + \ \boldsymbol{\lambda h}_1^{(LLR)}(\boldsymbol{I} - (\boldsymbol{h}_1^{(LLR)}) \\ \boldsymbol{h}_2^{(LLR)} + \ \boldsymbol{\lambda h}_2^{(LLR)}(\boldsymbol{I} - (\boldsymbol{h}_2^{(LLR)}) \\ \vdots \\ \boldsymbol{h}_n^{(LLR)} + \ \boldsymbol{\lambda h}_n^{(LLR)}(\boldsymbol{I} - (\boldsymbol{h}_n^{(LLR)}) \end{bmatrix} y, \quad (37)$$

$$\hat{y}^{(ANPM2)} = \ \boldsymbol{H}^{(ANPM2)} y \qquad , \quad (38)$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix, the $1\times n$ vector

$\boldsymbol{h}_i^{(LLR)} + \ \boldsymbol{\lambda h}_i^{(LLR)}(\boldsymbol{I} - (\boldsymbol{h}_i^{(LLR)})$ is the i^{th} row of the $n \times n$ ANPM2 Hat matrix $\boldsymbol{H}^{(ANPM2)}$.

Using matrix notation, the ANPM2 estimate of the response is given as:

$$\hat{y}^{(ANPM2)} = \begin{bmatrix} \boldsymbol{h}_1^{(\boldsymbol{ANPM}2)} \\ \boldsymbol{h}_2^{(\boldsymbol{ANPM}2)} \\ \vdots \\ \boldsymbol{h}_n^{(\boldsymbol{ANPM}2)} \end{bmatrix} y, \quad (39)$$

$$\hat{y}^{(ANPM2)} = \boldsymbol{H}^{(ANPM2)} y, \quad (40)$$

where $\boldsymbol{h}_i^{(\boldsymbol{ANPM}2)} = \boldsymbol{h}_i^{(LLR)} + \lambda \boldsymbol{h}_i^{(LLR)}(\boldsymbol{I} - \boldsymbol{h}_i^{(LLR)})$ is the is the i^{th} row of the $n \times n$ ANPM2 Hat matrix $\boldsymbol{H}^{(ANPM2)}$.

3.0 Application (Minced Fish Quality Data)

The Minced Fish Quality Data is presented in [3]. The problem seeks the setting of three explanatory variables x_1 (washing temperature), x_2(washing time) and x_3 (washing ratio of water volume to sample weight) that would optimize four aspect of quality of minced fish, namely, springiness (y_1), thiobarbituric acid number (y_2), cooking loss (y_3), and whiteness index (y_4).

Based on the process requirements, a CCD was conducted to establish the design experiment and observed responses as presented in Table 1.

Table 1: The Minced Fish Quality Data generated through CCD

i	CODED LEVELS			y_1	y_2	y_3	y_4
	x_1	x_2	x_3				
1	-1	-1	-1	1.83	29.31	29.50	50.36
2	1	-1	-1	1.73	39.32	19.40	48.16
3	-1	1	-1	1.85	25.16	25.70	50.72
4	1	1	-1	1.67	40.18	27.10	49.69
5	-1	-1	1	1.86	29.82	21.40	50.09
6	1	-1	1	1.77	32.20	24.00	50.61
7	-1	1	1	1.88	22.01	19.60	50.36
8	1	1	1	1.66	40.02	25.10	50.42
9	-1.682	0	0	1.81	33.00	24.20	29.31
10	1.682	0	0	1.37	51.59	30.60	50.67

11	0	-1.682	0	1.85	20.35	20.90	48.75
12	0	1.682	0	1.92	20.53	18.90	52.70
13	0	0	-1.682	1.88	23.85	23.00	50.19
14	0	0	1.682	1.90	20.16	21.20	50.86
15	0	0	0	1.89	21.72	18.50	50.84
16	0	0	0	1.88	21.21	18.60	50.93
17	0	0	0	1.87	21.55	16.80	50.98

Source: [3].

3.1.2. Transformation of Data from Central Composite Design (CCD)

In nonparametric regression techniques for RSM, the explanatory variable values are scaled to range between 0 and 1. Data collected through a Central Composite Design (CCD) is transformed using the following mathematical relation:

$$x_{new} = \frac{Min(x_{old}) - x_0}{(Min(x_{old}) - Max(x_{old}))} \tag{41}$$

where x_{new} is the transformed value, x_0 is the target value that needed to be transformed in the vector containing the old coded value, represented as x_{old}, Min (x_{old}) and $Max(x_{old})$ are the minimum and maximum values in the vector x_{old} respectively, [14].

The natural or coded variables in Table 1 are transformed to explanatory variables in Table 2 using Equation (41)

Target points needed to be transformed for location 1 under the coded variables are given below:

Target points $x_0 : -1, -1, -1$; $Min(x_{old}) : -1.682, -1.682, -1.682$; $Max(x_{old})$: $1.682, 1.682, 1.682$

$$x_{new} = \frac{Min(x_{old}) - x_0}{(Min(x_{old}) - Max(x_{old}))}$$

$$Explanatory\ variable\ x_1 : \ x_{11} = \frac{-1.682 - (-1)}{((-1.682) - (1.682)} = 0.2030$$

$$Explanatory\ variable\ x_2 : \ x_{12} = \frac{-1.682 - (-1)}{((-1.682) - (1.682))} = 0.2030$$

Target points needed to be transformed for location 2 under the coded variables are given below:

Target points x_0: $1, -1, -1$; $Min(x_{old}) : -1.682, -1.682, -1.682$; $Max(x_{old})$: $1.682, 1.682, 1.682$

$$x_{new} = \frac{Min(x_{old}) - x_0}{\left(Min(x_{old}) - Max(x_{old})\right)}$$

$$Explanatory\ variable\ x_1 :\ x_{21} = \frac{-1.682 - (1)}{((-1.682) - (1.682))} = 0.7970$$

$$Explanatory\ variable\ x_2 :\ x_{22} = \frac{-1.682 - (-1)}{((-1.682) - (1.682))} = 0.2030$$

The values of the explanatory variables are transformed by the relation in Equation (41) which is coded between 0 and 1 as given in Table 2.

Table 2: The transformed Minced Fish Quality Data

i	x_1	x_2	x_3	y_1	y_2	y_3	y_4
1	0.2030	0.2030	0.2030	1.83	29.31	29.50	50.36
2	0.7970	0.2030	0.2030	1.73	39.32	19.40	48.16
3	0.2030	0.7970	0.2030	1.85	25.16	25.70	50.72
4	0.7970	0.7970	0.2030	1.67	40.18	27.10	49.69
5	0.2030	0.2030	0.7970	1.86	29.82	21.40	50.09
6	0.7970	0.2030	0.7970	1.77	32.20	24.00	50.61
7	0.2030	0.7970	0.7970	1.88	22.01	19.60	50.36
8	0.7970	0.7970	0.7970	1.66	40.02	25.10	50.42
9	0.0000	0.5000	0.5000	1.81	33.00	24.20	29.31
10	1.0000	0.5000	0.5000	1.37	51.59	30.60	50.67
11	0.5000	0.0000	0.5000	1.85	20.35	20.90	48.75
12	0.5000	1.0000	0.5000	1.92	20.53	18.90	52.70
13	0.5000	0.5000	0.0000	1.88	23.85	23.00	50.19
14	0.5000	0.5000	1.0000	1.90	20.16	21.20	50.86
15	0.5000	0.5000	0.5000	1.89	21.72	18.50	50.84
16	0.5000	0.5000	0.5000	1.88	21.21	18.60	50.93
17	0.5000	0.5000	0.5000	1.87	21.55	16.80	50.98

The process requirements for each response given in [3] are as follows:

Maximize y_1 with lower bound L=1.70, and target value $\emptyset$= 1.92;
Minimize $y_2$with target value $\emptyset$ =20.16 and upper bound U=21.00;
Minimize $y_3$with target value $\emptyset$ =16.80, and upper bound U =20.00;
Maximize y_4 with lower bound L =45.00, and target value $\emptyset$ =50.98.

In the minced fish quality data as given in section 3.0 4, we seek to show the performance of

$ANPM2_{PAB}$ over OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$ based on the goodness-of-fit statistics and the process requirements.

Table 3 represents the mixing parameters for the multi-response minced fish quality data and the values of the mixing parameters are obtained via genetic algorithm tool in MATLAB 7.10.0.499 (R2010a).

Table 3: Mixing Parameters of different models for Minced Fish Quality Data

Response	Model	$\boldsymbol{\lambda}$
y_1	OLS	NOT APPLICABLE
	$MRR1_{PAB}$	1.0000
	$MRR2_{PAB}$	1.0000
	$ANPM2_{PAB}$	1.0000
y_2	OLS	NOT APPLICABLE
	$MRR1_{PAB}$	0.7085
	$MRR2_{PAB}$	1.0000
	$ANPM2_{PAB}$	0.9631
y_3	OLS	NOT APPLICABLE
	$MRR1_{PAB}$	0.9320
	$MRR2_{PAB}$	1.0000
	$ANPM2_{PAB}$	0.1318
y_4	OLS	NOT APPLICABLE
	$MRR1_{PAB}$	0.9999
	$MRR2_{PAB}$	1.0000
	$ANPM2_{PAB}$	0.0900

Table 4: Model *goodness –of- fit* of the Minced Fish Quality Data

Response	Model	$\boldsymbol{DF}$	$\boldsymbol{PRESS^{**}}$	$\boldsymbol{PRESS}$	$\boldsymbol{SSE}$	$\boldsymbol{MSE}$	$\boldsymbol{R^2}$(%)	$\boldsymbol{R^2_{Adj}}$(%)
y_1	OLS	14.0000	-	-	0.0231	0.0017	92.1250	91.0000
	$MRR1_{PAB}$	12.0072	0.0019	0.0497	**0.0123**	**0.0010**	94.3907	**95.7905**
	$MRR2_{PAB}$	12.0393	0.0015	0.0294	0.0124	**0.0010**	95.7900	94.4000
	$ANPM2_{PAB}$	12.0053	**0.0010**	**0.0272**	**0.0123**	**0.0010**	**95.7910**	94.3905
y_2	OLS	12.0000	-	-	90.9033	7.5753	93.3850	91.1800
	$MRR1_{PAB}$	8.2177	7.4867	162.1356	37.8103	4.6011	97.2486	94.6430
	$MRR2_{PAB}$	8.0060	10.7141	173.4754	**37.7558**	4.7159	**97.2500**	94.5100
	$ANPM2_{PAB}$	8.2934	**6.9898**	**151.8825**	38.0158	**4.5838**	97.2336	**94.6630**
y_3	OLS	9.0000	-	-	41.1338	4.5704	84.0250	71.6600
	$MRR1_{PAB}$	2.0443	8.0901	120.7925	2.0489	1.0023	99.2060	93.7860
	$MRR2_{PAB}$	2.2192	10.1660	147.5776	2.2228	**1.0016**	99.1400	**93.7900**
	$ANPM2_{PAB}$	2.0641	**4.7756**	**71.3928**	**2.0676**	1.0017	**99.1988**	93.7895
y_4	OLS	14.0000	-	-	198.8048	14.2003	54.1238	47.5700
	$MRR1_{PAB}$	12.0072	17.4184	461.3785	12.1886	1.0151	97.1875	96.2522
	$MRR2_{PAB}$	12.0214	9.5980	250.5095	12.2085	1.0156	97.1800	96.2500
	$ANPM2_{PAB}$	12.0001	**6.6035**	**174.8811**	**12.1387**	**1.0116**	**97.1990**	**96.2654**

In Table 4, $ANPM2_{PAB}$, outperformed OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$ in terms of $PRESS$ and $PRESS^{**}$ with respect to springiness (y_1), $ANPM2_{PAB}$ *and* $MRR1_{PAB}$ are jointly better in terms of SSE and MSE. Whereas, $ANPM2_{PAB}$ is better in terms of R^2, and $MRR1_{PAB}$ is better in terms of R^2_{Adj}. In terms of thiobarbituric acid number (y_2), $ANPM1_{PAB}$ is better with respect to PRESS, PRESS**, MSE and R^2_{Adj} than OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$. while $MRR2_{PAB}$ performed better than OLS, $MRR1_{PAB}$, and $ANPM2_{PAB}$ in terms SSE and R^2. For cooking loss (y_3), $ANPM2_{PAB}$ outperformed other models such as OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$ in terms of $PRESS$, $PRESS^{**}$, SSE and R^2 statistics while $MRR2_{PAB}$ performed better than existing models in terms of MSE and R^2_{Adj}.

In terms of whiteness index (y_4), $ANPM2_{PAB}$ performed better than OLS, $MRR1_{PAB}$, and $MRR2_{PAB}$ with respect to $PRESS$ and $PRESS^{**}$, SSE, MSE, R^2 and R^2_{Adj} statistics.

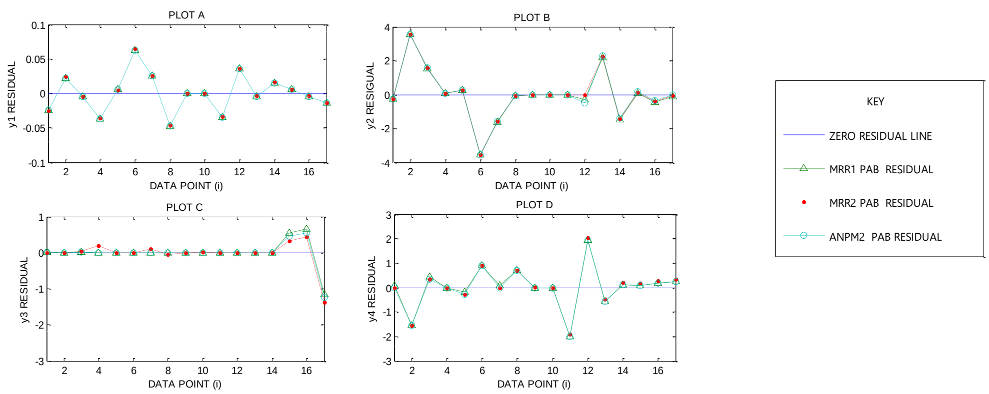

Figure 1. Plot A: maximize Springiness; Plot B: minimize thiobarbituric acid number; Plot C: minimize Cooking Loss; Plot D: maximize whiteness Index.

The multi-response residual plots in Figure 1, is a clear indication that the models $ANPM2_{PAB}$ both having the smallest residual response for the respective residual data points.

Table 5: Model optimal solution via the Desirability function in the minced fish data

Model	x_1	x_2	x_3	$\hat{y}_1$	$\hat{y}_2$	$\hat{y}_3$	$\hat{y}_4$	*d1*	*d2*	*d3*	*d4*	D (%)
OLS	0.38	1.00	0.72	1.91	19.50	17.22	50.30	0.94	1.00	0.87	0.89	92.29
$MRR1_{PAB}$	0.57	0.00	0.55	1.99	18.68	19.18	48.63	1.00	1.00	0.26	0.61	62.82
$MRR2_{PAB}$	0.13	1.00	0.00	1.85	26.40	30.46	0.00	0.68	0.00	0.00	0.00	0.00
$ANPM2_{PAB}$	**0.26**	**1.00**	**0.63**	1.94	20.16	8.33	53.89	1.00	1.00	1.00	1.00	**100.00**

In the overall desirability, the model $ANPM2_{PAB}$ outperforms other models since in their respective settings of the explanatory variables that simultaneously optimized the responses more than any other model. Obviously, $ANPM2_{PAB}$ further justify the performance over OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$.

4.0 SIMULATION STUDY

In this section, we compare the performances of the respective regression models, $MRR1_{PAB}$, $MRR2_{PAB}$, and $ANPM2_{PAB}$ using simulated data. Each simulation comprises of 500 data sets based on the following underlying polynomial models:

4.1 Simulation Study 1: Multi-Response Chemical Process Problem

This problem is analyzed by [10] with the aim to get the setting of the explanatory variables x_1 and x_2 (representing reaction time and temperature, respectively).

Table 6: The CCD for the Simulating Data for Models 1-5

i	x_1	x_2
1	0.8536	0.8536
2	0.1464	0.8536
3	0.8536	0.1464
4	0.1464	0.1464
5	1.0000	0.5000
6	0.0000	0.5000
7	0.5000	1.0000
8	0.5000	0.0000
9	0.5000	0.5000
10	0.5000	0.5000
11	0.5000	0.5000
12	0.5000	0.5000
13	0.5000	0.5000

4.2. Simulation Study 2: Multi-Response Chemical Process Problem

In this section, we compare the performances of the respective regression models using simulated data $MRR1_{PAB}$, $MRR2_{PAB}$ and $ANPM2_{PAB}$. Each simulation comprises 500 data sets based on the following underlying polynomial models:

Model 1: $20 - 10x_1 - 25x_2 - 15x_1x_2 + 20x_1^2 + 50x_2^2 + \gamma\big(2\sin(4\pi x_1) + 2\cos(4\pi x_2) - 2sin(4\pi x_1 x_2)\big)$;

Model 2: $66 + 22x_1 + 10x_2 + 13x_1x_2 - 23x_1^2 - 25x_2^2 + \gamma(2\sin(3\pi x_1) - 2\cos(3\pi x_2) + 2\sin(2\pi x_1 x_2))$;

Model 3: $38 - 17x_1 + 19x_2 + 21x_1x_2 - 23x_1^2 + 29x_2^2 + \gamma(2\sin(8\pi x_1) + 2\cos(8\pi x_2) - 2\sin(8\pi x_1 x_2))$;

Model 4: $45 - 27x_1 + 9x_2 - 22x_1x_2 + 10x_1^2 + 13x_2^2 + \gamma(2\sin(3\pi x_1) - 2\cos(3\pi x_2) + 2\sin(3\pi x_1 x_2))$;

Model 5: $90 - 59x_1 - 41x_2 - 36x_1x_2 + 15x_1^2 + 25x_2^2 - \gamma(2\sin(4\pi x_1) - 2\cos(33\pi x_2) + 2\sin(3\pi x_1 x_2))$

Table 7: Comparison of the AVESSE of each method for each model in the simulation studies

Model	γ	$MRR1_{PAB}$	$MRR2_{PAB}$	$ANPM2_{PAB}$
(1,7)	0.00	6.7900	7.1812	**4.0705**
	0.50	16.3440	16.3791	**4.1749**
	1.00	36.3603	43.0333	**4.1869**
(2, 8)	0.00	7.0896	6.2485	**3.9548**
	0.50	12.3478	14.4442	**3.9690**
	1.00	33.1176	33.8967	**4.1850**

(3, 9)	0.00	6.3701	7.0739	**3.7469**
	0.50	13.2077	15.4709	**3.8466**
	1.00	28.9234	40.5788	**4.0296**
(4, 10)	0.00	5.7516	5.9733	**4.0996**
	0.50	9.9041	21.1713	**4.2393**
	1.00	12.0239	63.0711	**4.2839**
(5, 11)	0.00	6.7111	6.7820	**3.4760**
	0.50	12.6459	13.0680	**4.0113**
	1.00	31.7956	30.1685	**4.5220**

In Table 7, $ANPM2_{PAB}$ is stable in terms of AVESSE for model 1 through model 5 as the misspecification parameters increase from zero to one, while, $MRR1_{PAB}$ and $MRR2_{PAB}$ increases in their respective AVESSE as the misspecification parameter increases from zero to one. This suggest that $ANPM2_{PAB}$ is the most suitable model over $MRR1_{PAB}$ and $MRR2_{PAB}$.

Discussion of Results

In this paper, we have shown a new blend between locally adaptive bandwidth that is driven by local variability in the data and the adaptive nonparametric regression model for RSM data.

We have compared results of the adaptive nonparametric regression model ($ANPM2_{PAB}$) with OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$ using the same data sets in section 3.0 to 4.2. The $ANPM1_{PAB}$ for single response chemical process data performed better in terms of goodness-of-fit statistics and optimization result over OLS, $MRR1_{PAB}$ and $MRR2_{PAB}$. While, for the multi-response problem, $ANPM2_{PAB}$ performed better than OLS, $MRR1_{PAB}\ and\ MRR2_{PAB}$ with respect to goodness-of-fit statistics and the process requirements for the data considered. Furthermore, the locally adaptive bandwidth enhanced the performance of $MRR1_{PAB}$, $MRR2_{PAB}$ and $ANPM2_{PAB}$ in terms of goodness-of-fit statistics for the data type examined.

Simulation studies were also carried out to further investigate the performance of the regression models by varying the misspecification parameters from zero to one. It was observed that $ANPM2_{PAB}$ was stable in terms AVESSE as the misspecification parameter increases from zero to one. However, $MRR1_{PAB}\ and\ MRR2_{PAB}$ tend to increase with respect to AVESSE as misspecification parameter increases from zero to one.

Conclusions

In addition, the model $ANPM2_{PAB}$ compared with OLS, $MRR1_{PAB}\ and\ MRR2_{PAB}$ performed satisfactorily in terms of goodness-of-fit tests. The model $ANPM2_{PAB}$, again on the average performance did better than existing models $MRR1_{PAB}\ and\ MRR2_{PAB}$ in terms of goodness-of-fits statistics and process requirements.
Lastly, simulation study was carried out on nonparametric regression models to further investigate the effect of the misspecification parameter as it increases from zero to one. It was observed that the $ANPM2_{PAB}$ was considerably stable over $MRR1_{PAB}$ and $MRR2_{PAB}$ for models 1 to 5. Evidently, the model $ANPM2_{PAB}$ appear to have better performance over the models, $MRR1_{PAB}$ and $MRR2_{PAB}$ in all the RSM data considered in this paper. Conclusively, the adaptive nonparametric regression model incorporate local linear regression (LLR) portion and product of

the optimal mixing parameter and, theresiduals of the LLR to provide a second opportunity of fitting part of the data that were not captured by the LLR model and while the locally adaptive bandwidths perform adequate smoothing of the dataset by location for kth number of explanatory variables and provides a better estimates for the dataset utilized in this paper.

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References

[1] Nair, A. T., Makwana, A. R. and Ahammed, M. M., (2014). The use of Response Surface Methodology for modelling and analysis of water and waste – water treatment processes: *A Review. Water Science and Technology, 69(3): 464 – 478.*

[2] Yeniay, O., (2014). Comparative study of algorithm for response surface optimization. *Journal of Mathematical and Computational Applications, 19(1): 93 – 104.*

[3] Wan, W. and Birch, J.B., (2011). A semi-parametric technique for multi-response optimization. *Journal of Quality and Reliability Engineering. International*, *27(1): 47-59.*

[4] Carley, M. K., Kamneva, Y. N. and Reminga, J., (2004). Response surface Methodology. Technical report written to Center for Computational Analysis of Social and Organizational Systems (CASOS), *Institute for Software Research International - Carnegie Mellon University, School of Computer Science, Pittsburgh, PA, 15213*.

[5] Rivers, D. L., (2009). A Graphical Analysis of Simultaneously Choosing the Bandwidth and Mixing Parameter for Semiparametric Regression Techniques, M. Sc. Thesis and Dissertations, *Virginia Commonwealth University.*

[6] Einsporn, R., (1987). HATLINK: A Link Between Least Squares Regression and Nonparametric Curve Estimation., Ph.D. dissertation, *Department of Statistics, Virginia Polytechnic Institute and State University, Blacksburg, VA.*

[7] Einsporn, R., (1993). Model robust regression: Using Nonparametric Regression to improve Parametric Regression Analysis. Technical Report 93-5, *Department of Statistics, Virginia Polytechnic Institute and State University, Blacksburg*.

[8] Mays, J., Birch, J. B. and Starnes, B. A., (2001). "Model robust regression: combining parametric, nonparametric and semiparametric methods." *Journals of Nonparametric Statistics, 13: 245 – 277.*

[9] Pickle, S. M., Robinson, T. J., Birch, J. B. and Anderson-Cook, C.M., (2008). A semi-parametric method to robust parameter design. *Journal of Statistical Planning and Inference, 138: 114-131*.

[10] He, Z., Zhu, P. F., and Park, S. H., (2012). A robust desirability function for multi-response surface optimization. *European Journal of Operational Research*, *221: 241-247*.

[11] Adalarasan, R. and Santhanakumar, M., (2015). Response surface methodology and desirability analysis for optimizing μWEDM parameters for A16351/20% Al_2O_3 composite. *International Journal of ChemTech Research, 7(6): 2625 – 2631*.

[12] Eguasa, O., Mbegbu, J. I. and Edionwe, E., (2019). 'On the Use of Nonparametric Regression Model for Response Surface Methodology (RSM)', Benin Journal of Statistics, 2(5): 61-75.

[13] Eguasa, O., Edionwe, E. and Mbegbu, J. I. (2022). "Local Linear Regression and the problem of dimensionality: a remedial strategy via a new locally adaptive bandwidths selector", Journal of Applied Statistics, Vol.50, No. 6, pg. 1283 – 1309.

[14] Myers, R., Montgomery, D. C., and Anderson-Cook, C. M., (2009). Response Surface Methodology Process and Product Optimization Using Designed Experiments, *Third Edition, Wiley, ISBN: 978-0-470-17446-3. New York.*