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BINOMIAL OPTION VALUATION MODEL WITH ADAPTIVE SWING FACTOR

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ABSTRACT

This paper presents a new Binomial option valuation model which is with an adaptive swing factor. The existing versions of the Binomial model are developed based on fixed swing factor and results from fixed swing factor models are commonly associated with Snon-linear error propagations which translates to non-monotonic convergence and reduced accuracy in application to option pricing. In order to overcome this challenge, we adopt swing factors which are functions of the step number(n). The accuracy, convergence and stability behavior of the Binomial option pricing model with adaptive swing factor (up and down move size) are all investigated. The Adaptive Factor Model when compared with two popular versions of the traditional Binomial models - the Cox, Ross and Rubinstein (CRR) model [3], the Jarrow and Rudd (JR) model [5], a more recent Leisen and Reimer (LR) [3] model registered more accurate performances, especially with respect to option pricing

1. Introduction

In 1979, the Binomial model was developed and published by Cox, Ross and Rubinstein [3] and since then various versions of the model have been widely used in option valuation due to its simplicity and flexibility. Numerous authors have since been analyzing and extending the ideas with the aim to improve on it with respect to the various features of the model. Of particular interest to researchers have been the pricing accuracy, the rate of convergence and stability behavious of any version of the model. All versions of the binomial model differ basically in terms of the parametization approach adopted by the author.

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Based on the two conditions established from risk-neutral process, and properties of lognormal distribution as well as having three parameters $p, u \wedge d$to determine, Cox, Ross and Rubinstein (1979) [3] leveraged on the one degree of freedom available and chose $u=1/d$ which leads to:

$$p=\frac{e^{rdt}-d}{u-d}, \quad u=e^{\sigma\sqrt{dt}}, \qquad d=e^{-\sigma\sqrt{dt}}$$

Jarrow and Rudd (1983) [5] considered choice of parameters such that, for large values of the step number n, the binomial stock price process approximates the lognormal stock price process used in the Black-Scholes model. They showed that to obtain that assumed approximation, the up and down move sizes u and d respectively should be

$$u=\left(r-\frac{1}{2}\sigma^2\right)\frac{T}{n}+\sigma\sqrt{\frac{T}{n}}$$

$$d=\left(r-\frac{1}{2}\sigma^2\right)\frac{T}{n}-\sigma\sqrt{\frac{T}{n}}$$

And as n tends to infinity, $p=\frac{1}{2}$

William F. Sharpe in1978 [6], proposed the idea of the binomial model. Cox, Ross, and Rubinstein (1979) formalized and pioneered the basic approach to the binomial model. They developed a discrete-time binomial approach, the CRR model, to option valuing, and published it with the title: “Option Pricing: A Simplified Approach”. The CRR model is based on the assumption that stock return is proportional to the risk-free interest rate. The essence of their approach is the construction of a binomial lattice of stock prices where the risk-neutral valuation rule is maintained. The fundamental principle of option valuation by arbitrage methods is particularly clear in this setting. Its development requires only elementary Mathematics, yet it contains, as a special limiting case, the Black-Scholes model, which had previously been developed only by much more difficult methods. The model readily lends itself to generalization in many ways, and its construction is based on the assumption that stock prices have two possible movement directions at each time point: Up or Down.

Rendleman and Bartter in 1979 [7], applied the CRR methodology to the pricing of put and call options on debt securities.

Robert Jarrow and Andrew Rudd in1983 [8] modified the binomial model by simply assuming that the up and down moves have equal probabilities of 0.5 each, having all the inputs which describe price movements reflected in up and down move sizes; and that logarithmic stock return matches the actual mean return of the stock.

Yuen and Yang in 2010 [9], continued the work on binomial models, proposing enhancements to Boyle's trinomial tree by incorporating measures to handle different regime states. Their modifications ensured that the tree model could accommodate data from various regimes simultaneously while preserving its structural integrity. This approach aimed to provide a more robust and flexible modeling framework.

The CRR and JR models do value options to some degree of accuracy, especially with a reasonably high step number n, but there remains associated with the models the challenges of non-monotonic convergence.

In 1996, Leisen and Reimer [1] introduced into the parametization, the idea of using the Peizer-Pratt Inversion function which is a discrete equivalent to the cumulative normal probability distribution function that models continuous random variable. With Leisen-Reimer model, the challenge of convergence rate was overcome and higher accuracy obtained in option pricing process.

The Peizer-Pratt [4] inversion function, $h^{-1}(z)$, used by Leisen and Reimer was prsented as follows:

$$h^{-1}(z)=\frac{1}{2}+\frac{sign(z)}{2}\sqrt{1-e^{¿¿}¿}$$

With

$$m=n+\frac{1}{3}+\frac{0.1}{n+1}$$

2. Theoretical Analysis

An option provides the holder with the right, but not the obligation, to buy an underlying asset (if it's a call option) or to sell the underlying asset (if it's a put option) at a specific price (the Exercise price) on or before a specified date in the future (expiration date)

The holder of an option will exercise his right at the expiration date T, only if the option has value. A call option has value at expiration time T, if the stock price s_T is greater than the exercise price x, in which case, the option is worth $s_T - x$

Hence if $C_T$is the value of the Call Option at expiration, then:

$$C_T = max(0, s_T - x)(1)$$

Taking the Expectation, we get:

$$E(C_T) = E\{max(0, s_T - x)(2)$$

Based on risk-neutral argument, the value of the option at time t is equal to the value of the option at maturity T discounted at the risk-free rate of interest r. Therefore:

$$C_t = e^{-r(T-t)} E(C_T)(3)$$

Which gives:

$$C_t = e^{-r(T-t)} E[max(0, s_T - x)](4)$$

Define τ=(T-t) (5)

Hence,

$$C_t = e^{-r\tau} E[max(0, s_T - x)](6)$$

Suppose P_t is the Put Option Value at T, then similar arguments leads to:

$$P_t = e^{-r\tau} E[max(0, x - s_T)] \quad (7)$$

Essentially, these existing versions of binomial model, (CRR, JR, and LR models) only differ in the unique definitions of the parameters; that is the swing factors u and d, as well as the probabilities $p_u \wedge p_d$.

Taking Expectation, in equation 3.6, we obtain:

$$E(s_T) = \sum_{j=0}^{n} c_j^n p_u^{n-j} p_d^j s_t u^{n-j} d^j ; j = 0,1,2,\ldots,n \quad (8)$$

From equation (3.6)

$$C_t = e^{-r\tau} E\{max(0, s_T - x)\}$$

Based on the properties of Expectation, we can rewrite this as:

$$C_t = e^{-r\tau}\{max[0, E(s_T) - x]\}$$

Substituting $E(s_T)$ from equation (3.8), we obtain:

$$C_t = e^{-r\tau} \sum_{j=0}^{n} c_j^n p_u^{n-j} p_d^j \{0, s_t u^{n-j} d^j - x\} \quad (9)$$

Similarly,

$$P_t = e^{-r\tau} \sum_{j=0}^{n} C_j^n p_u^{n-j} p_d^j \{0, x - s_t u^{n-j} d^j\} \quad (10)$$

j=0,1,2, ,n

Equations (9) and (10) are the basic binomial models for call and put options respectively assuming that there is no payable dividend on the underlying stock prior to maturity.

2.1 The Adaptive Factor Model (AFM):

The Adaptive Factor Model (AFM) herein introduced, modifies the Peizer-Pratt [4] inversion function as used in Leisen-Reimer [1] by converting the constant decimal 0.1 to a variable w, dependent on the strike price x.

2.2 Parametization of Cox, Ross and Rubinstein (CRR) Model

Estimation Of p_u

From risk-neutral economy, the expected yield from all assets equals the risk-free rate of interest. Therefore;

$$E(s_{t+1}) = s_t e^{rdt} \quad (11)$$

Where dt is an incremental change in time

Also from the stock price and associated probability lattice

$$E(s_{t+1}) = p_u s_t u + p_d s_t d$$

$$¿\, p_u s_t u + s_t d(1 - p_u),$$

Implying that;

$$E(s_{t+1}) = p_u s_t u - p_u s_t d + s_t d \quad (12)$$

Combining equations (3.11) and (3.12), we have:

$$p_u s_t u - p_u s_t d = s_t e^{rdt} - s_t d$$

We strike out the commom term s_t and obtain

$$p_u u - p_u d = e^{rdt} - d$$

Or

$$p_u(u - d) = e^{rdt} - d$$

Which gives us

$$p_u = \frac{e^{rdt} - d}{u - d} \quad (13)$$

Estimation of u

We go by matching variance on the lattices as follows

$$Var(s_{t+dt}) = E(s_{t+dt}^2) - E^2(s_{t+dt}) = s_t^2(p_u u^2 + p_d d^2) - s_t^2 e^{2rdt} \quad (14)$$

From properties of lognormal distribution, it can be shown that

$$Var\left(\frac{s_{t+dt}}{s_t}\right) = e^{2rdt}(e^{\sigma^{2dt}} - 1) \quad (15)$$

Putting (14) and (15) together, we have

$$s_t^2 e^{2rdt}(e^{\sigma^{2dt}} - 1) = s_t^2(p_u u^2 + p_d d^2) - s_t e^{2rdt}$$

giving

$$e^{2rdt + \sigma^{2dt}} = p_u u^2 + p_d d^2 \quad (16)$$

From (13) we substitute p_u with $\frac{e^{rdt} - d}{u - d}$ in the rhs of (16) as follows;

$$\frac{e^{rdt}-d}{u-d}u^2+\frac{u-e^{rdt}}{u-d}d^2$$

Simplifying, we obtain

$$\frac{u^2e^{rdt}-u^2d+ud^2-d^2e^{rdt}}{u-d}$$

$$¿\frac{\left(x^2-x^2\right)e^{rdt}-(u-d)}{u-d}$$

$$¿(u+d)e^{rdt}-1$$

Putting this back into (16), we have:

$$e^{rdt+\sigma^2dt}=(u+d)e^{rdt}-1 \qquad (17)$$

We have ine degree of freedom. Hence we assume u=1/d which transforms equation (17) as:

$$u^2e^{rdt}-u\left(1-e^{2rdt+\sigma^2dt}\right)+e^{rdt}=0$$

Solving, we obtain a positive root:

$u=¿¿$

We simplify the term inside the square root by first-order Taylor expansions, limiting the result to powers of dt. This gives;

$$\left(1+e^{2rdt+\sigma^{2dt}}\right)-4e^{2rdt}$$

$$\approx\left[2+\left(2r+\sigma^2\right)dt\right]^2-4(1+2rdt)$$

$$\approx 4\sigma^2dt$$

Hence

$$u=\frac{2+\left(2r+\sigma^2\right)dt+2\sigma\sqrt{dt}}{2\sigma^2dt}$$

$$\approx 1+rdt+\frac{1}{2}\sigma^2dt+\sigma\sqrt{dt}-rdt$$

$$\approx 1+\sigma\sqrt{dt}+\frac{1}{2}\sigma^2dt$$

Recall Taylor Series expansion of the exponential function e^x:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

Therefore

$1+\sigma\sqrt{dt}+\frac{1}{2}\sigma^2dt=e^{\sigma\sqrt{dt}}$, to order 2

Hence

$u=e^{\sigma\sqrt{dt}}$ and $d=e^{-\sigma\sqrt{dt}}$

This completes the parametization of the CRR model. The parameters for the three versions of the binomial model that we are comparing this work with are shown in the following table:

CRR (79)	**JR (83)**	**LR (96)**
$u=e^{\sigma\sqrt{dt}}$ $d=e^{-\sigma\sqrt{dt}}$	$u=e^{\left(r-\frac{1}{2}\sigma^2\right)dt+\sigma\sqrt{dt}}$ $d=e^{\left(r-\frac{1}{2}\sigma^2\right)dt-\sigma\sqrt{dt}}$	$u=\frac{p'}{p}e^{rdt}$ $d=\frac{1-p'}{1-p}e^{rdt}$
$p_u=\frac{e^{rdt}-d}{u-d}$ $p_d=1-p_u$	$p_u=p_d=\frac{1}{2}$	$p_u=\frac{p'}{p}$

		$p_d=\frac{1-p'}{1-p}$ $p^{'}=h^{-1}(d_1)$ $p=h^{-1}(d_2)$

$$d_1=\frac{\ln\left(\frac{s}{x}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2=d_1-\sigma\sqrt{T}$$

$$h^{-1}(z)=\frac{1}{2}+\frac{sign(z)}{2}\sqrt{1-e^{-¿¿}¿}$$

$$m=n+\frac{1}{2}+\frac{0.1}{n+1}$$

2.3 Parametization for the Adaptive Factor Model

The Adaptive Factor Model introduced here adopts t0he same parameters as Leisen-Reimer with a modification to the Peizer-Pratt Inversion function $h^{-1}(z)$ by replacing the constant $0.1 \in the\, term\, \frac{0.1}{n+1}$ with a variable $w=f(x)$. The relationship between w anddx is unique for every strike price (x) determined by a process of quadratic regression as follows:

Consider for example, a European-style call option with parameters:

s = 24.78, x =27.00, r = 0.0526, σ= 0.25, T = 0.5

We set w = [0.05, 1.5¿ with a step size of $\sigma = 10^{-5}$ and x = [24, 30] with step size of dx = 1

For each x in the closed interval x-3:1: x+3, we seek the best w (Optimum w) value in the interval 0.05:0.00001:1.5 that optimizes $C_t=e^{-r\tau}\sum_{j=0}^{n} c_j^n p_u^{n-j} p_d^j \{0, s_t u^{n-j} d^j - x\}$ when adopting LR parameter, but with $m=n+\frac{1}{3}+\frac{w}{n+1}$ instead of $m=n+\frac{1}{3}+\frac{0.1}{n+1.}$.

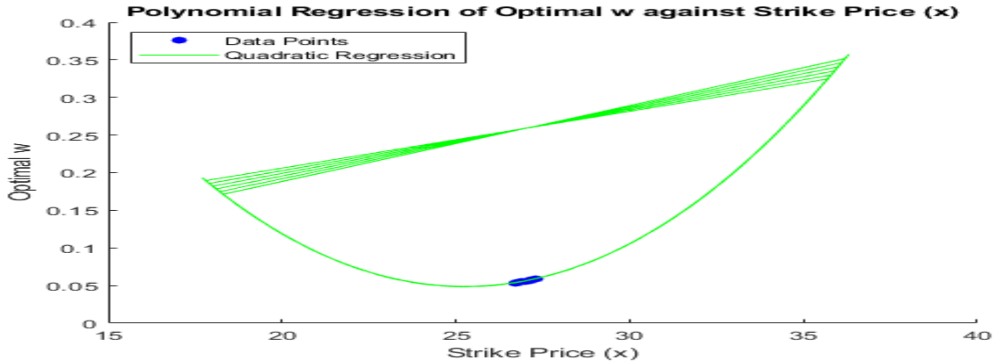
Table 1 below shows the set of optimal w values and corresponding strike price x; forming a set of ordered pairs (x, w)

Table 1: X AND CORRESPONDING OPTIMAL W

x	24	25	26	27	28	29	30
w	0.05565	0.05	0.05	0.05594	0.06726	0.08321	0.1033

We then simulate a 2-degree polynomial regression with these ordered pairs to determine an appropriate algebraic relationship between w and x

. Figure 1**: Regression Curve for Optimal w Against Strike Price**



w = 0.000118333*(x – 27.000000000)^2 + 0.001904875*(x - 27.000000000) + 0.055935714
This result suggests that our optimal Ct value is obtained when
$w = 0.000118333(x-27)^2 + 0.001904875(x-27) + 0.055935714$ replaces 0.1 in

$$m = n + \frac{1}{3} + \frac{0.1}{n+1}.$$

Similarly, with a step number n=25, we now price the European option in consideration using a Matlab Code:
The table below shows the result correct to 7 decimal places.

n	C_t (Our Formula)	C_t (Black-Scholes)	Error
25	1.1398663	1.1398653	0.0000010

3.0. Research Analysis and Discussion

We are comparing our Adaptive Factor Model (AFM) with the three previous works by Cox-Ross-Rubinstein (CRR), Jarrow-Rudd (JR) and Leisen-Reimer (LR) to determine the relative pricing accuracy. We base our findings, analysis and discussion on the results from valuing the option with the following parameters:
s =100, r = 0.07, $\sigma = 0.3$, T = 0.5 years, n = 25;
The strike x is ranged from 80 to 120; the same option tasks considered in Leisen-Reimer (1996).
The values obtained from numerical and statistical simulations are presented in Tabless 2 and 3 as follows.

Table 2: Call Option Parameters:
s =100, r = 0.07, σ=0.3, T = 0.5 years, n = 25; the strike x, ranged from 80 to 120.

OBTAINED VALUES					
CALL					
STRIKE (X)	TRUE VALUE	CRR	JR	LR	AFM
80	23.7579868	23.7408240	23.7623839	23.7582162	23.7579868
90	16.0996347	16.1337624	16.0843294	16.0994138	16.0996347
100	0.1337700	10.2131669	10.2010125	10.1331627	10.1337701
110	5.9494636	6.0121794	6.0245208	5.9488863	5.9494637
120	3.2828995	3.3188995	3.3341033	3.2825816	3.2828006

Table 3:

ERRORS			
CALL			
CRR	JR	LR	AFM
0.0171627	0.0043971	0.0002294	0.0000000
0.0341277	0.0153053	0.0002209	0.0000000
0.0793969	0.0672424	0.0006073	0.0000001
0.0627158	0.0750572	0.0005773	0.0000001
0.0360989	0.0513027	0.0002190	0.0000000

Put Option Parameters:

s =100, r = 0.07, σ = 0.3, T = 0.5 years, n = 25; the strike x, ranged from 80 to 120.

PUT (OBTAINED VALUES)					
STRIKE (X)	TRUE VALUE	CRR	JR	LR	AFM
80	1.0064201	0.9892573	1.0114918	1.0066495	1.0064201
90	3.0382499	3.0041221	2.9894915	3.0039013	3.0041221
100	6.7737085	6.7737085	6.7622288	6.6937043	6.6943117
110	12.2287752	12.2287752	12.2417913	12.1654821	12.1660595
120	19.1915495	19.1915495	19.2074279	19.1552315	19.1554506

PUT(ERRORS)			
CRR	JR	LR	AFM
0.0171627	0.0050718	0.0002294	0.0000000
0.0341277	.0146307	0.0002209	0.0000000
0.0793969	.0679171	0.0006073	0.0000001
0.0627158	0.0757319	0.0005773	0.0000001
0.0360989	0.0519774	0.0002190	0.0000000

3.1. Comparative Convergence Plots

Figure 2. Convergence graph of CRR (with n=200)

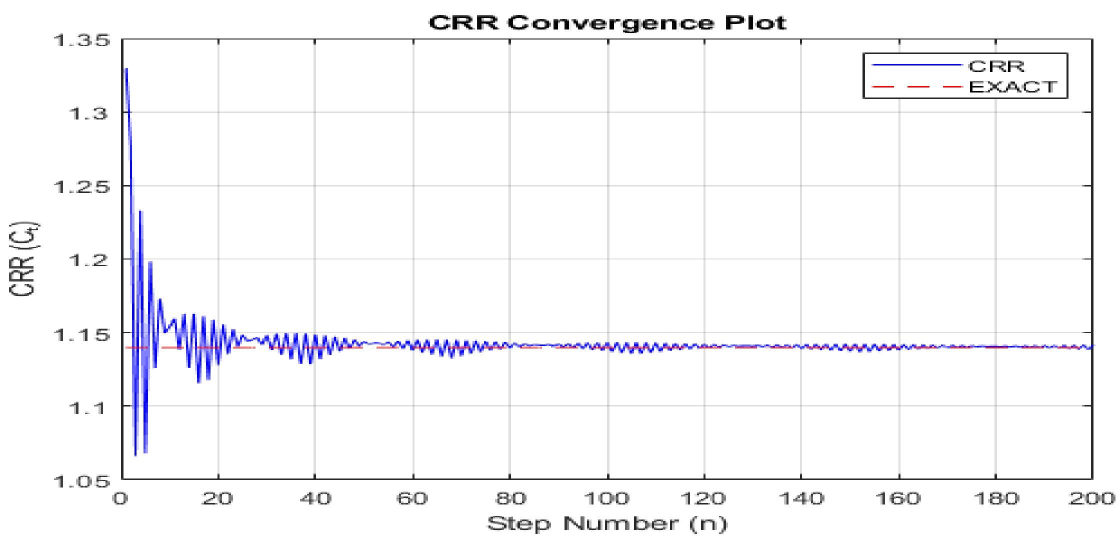


Figure 3. Convergence Graph of JR (n=200)

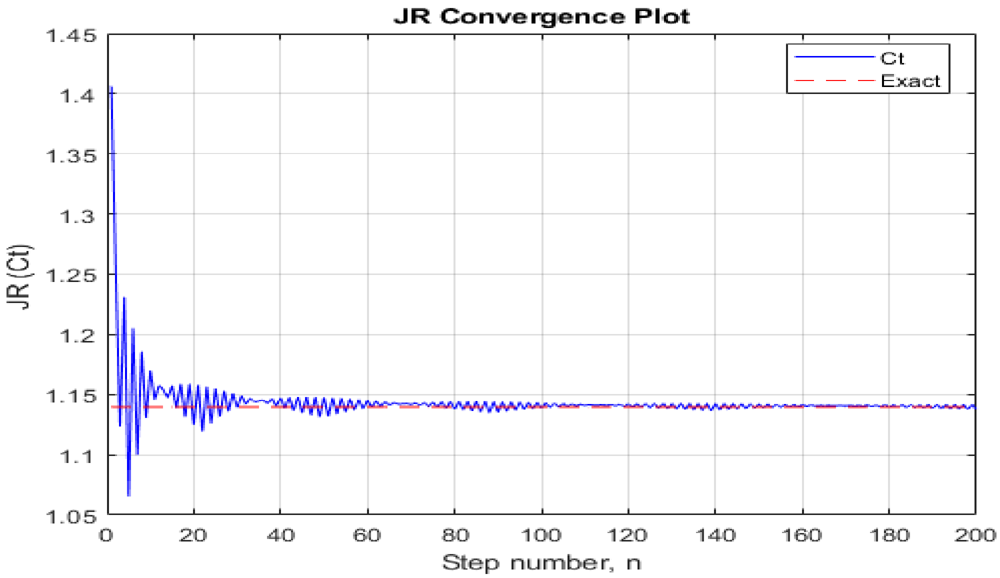


Figure 4. Convergence Graph of LR (n=200)

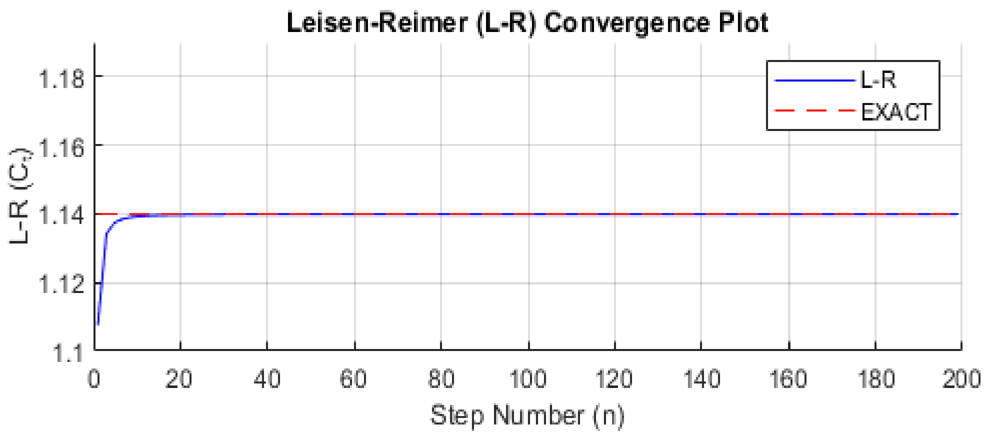
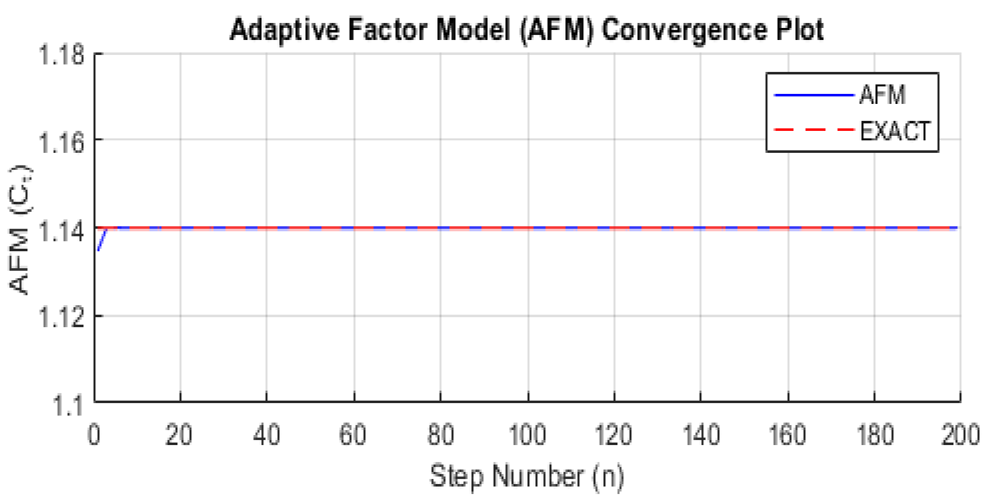


Figure 5: Convergence Graph of AFM (n=200)



3.2. Comparative Error Decay Behaviours

Figure 6: CRR Error Decay Plot

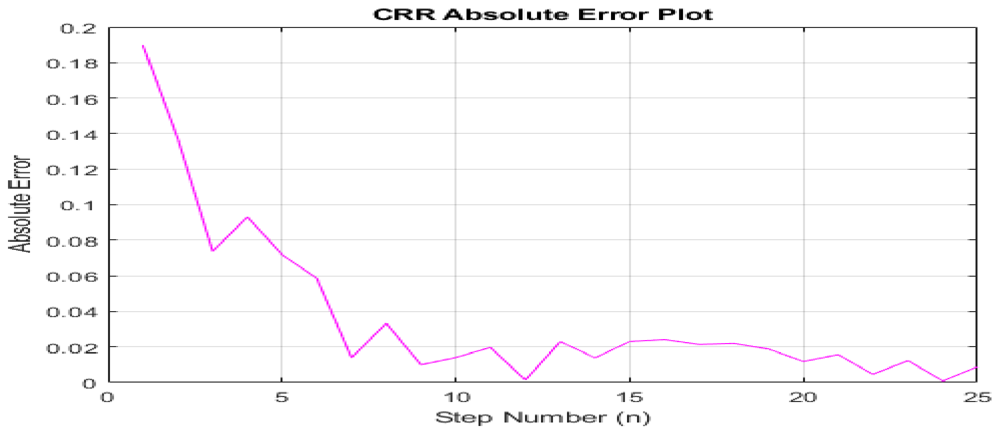


Figure 7: J-R Error Decay Plot

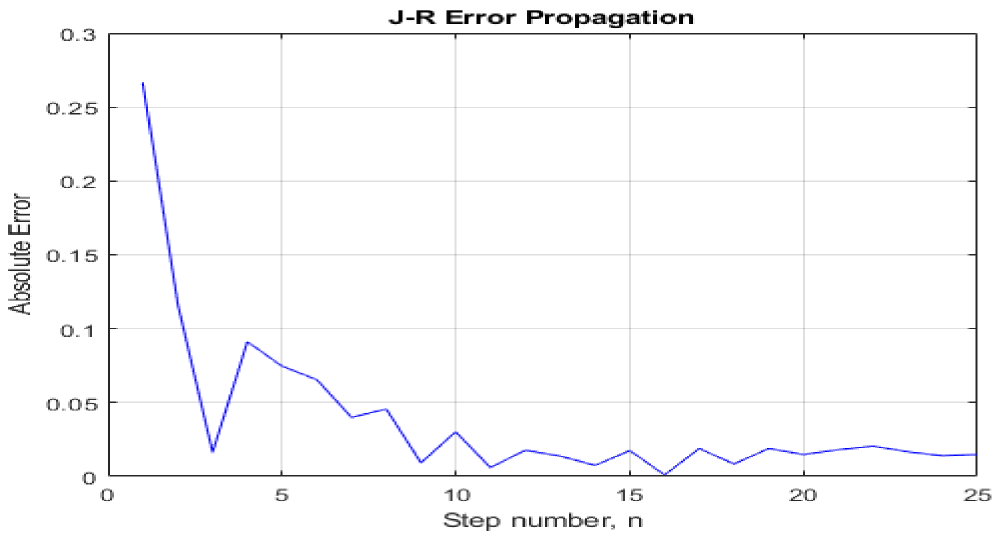


Figure 8. LR Error Decay Plot

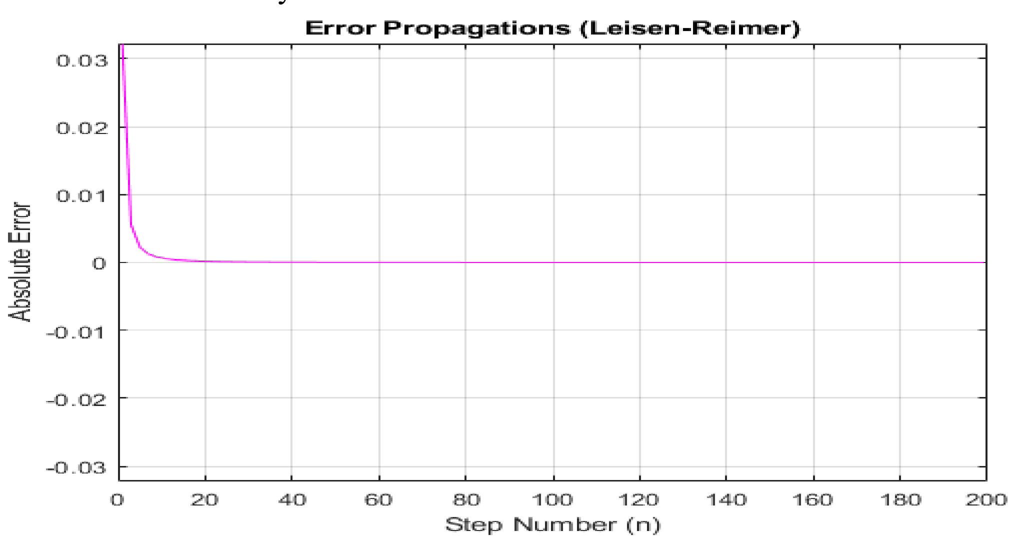
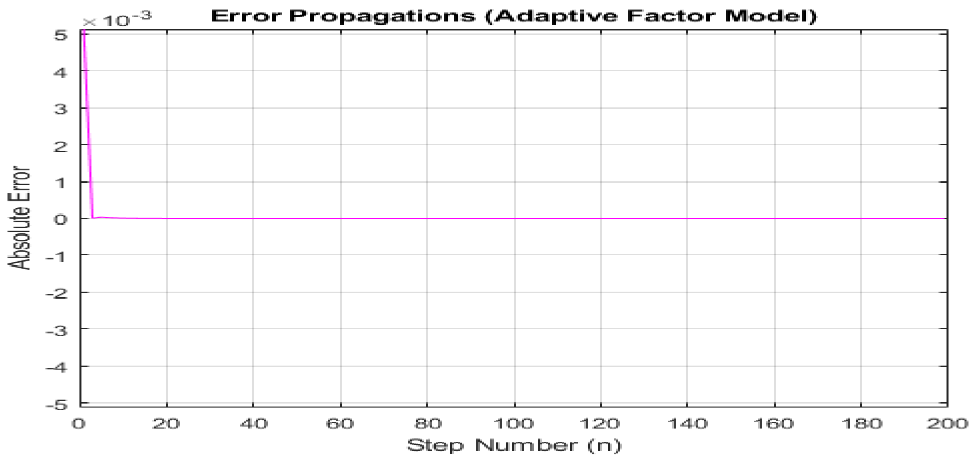


Figure 9: AFM Error Decay Plot



3.3. Comparative Stability Behaviours

Figure 10: CRR Stability Plot

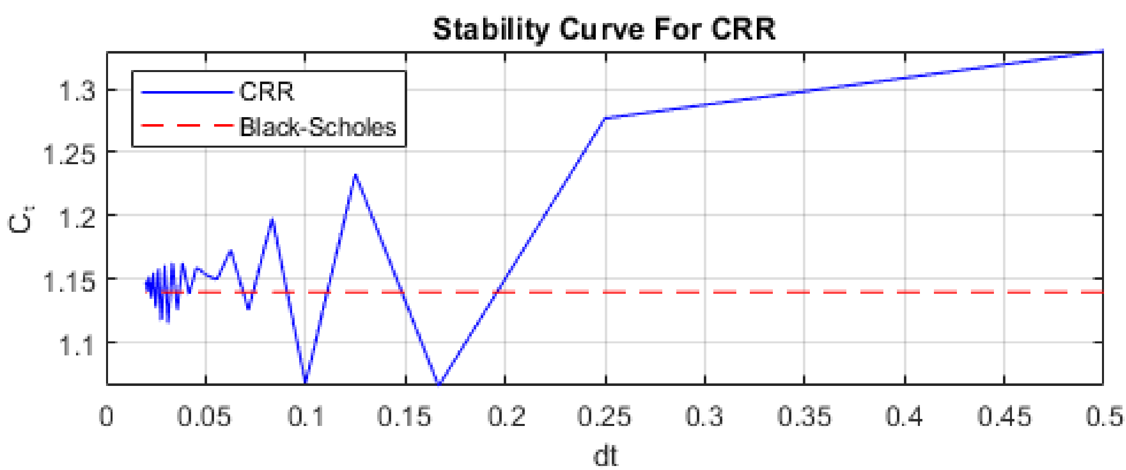


Figure 11: J-R Stability Plot

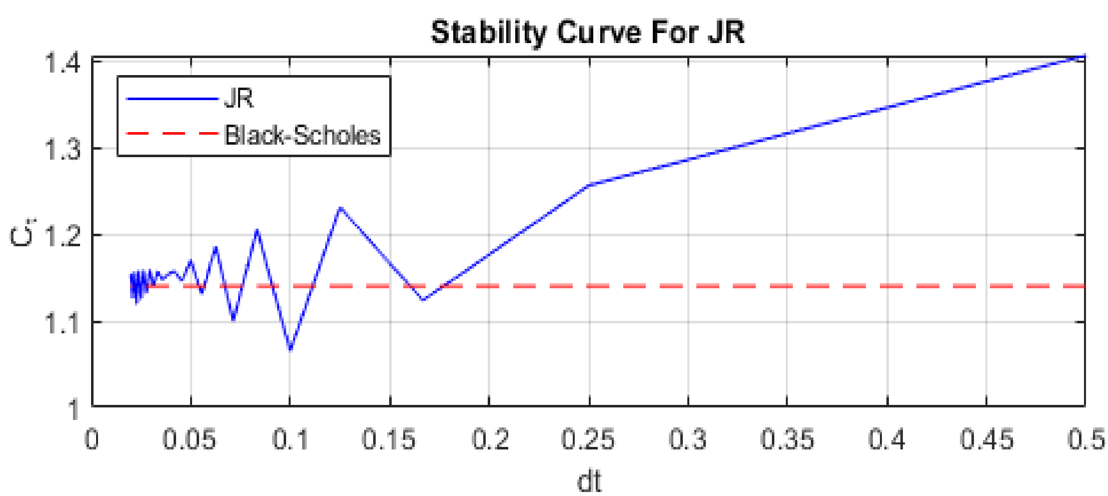


Figure 12: LR Stability Plot

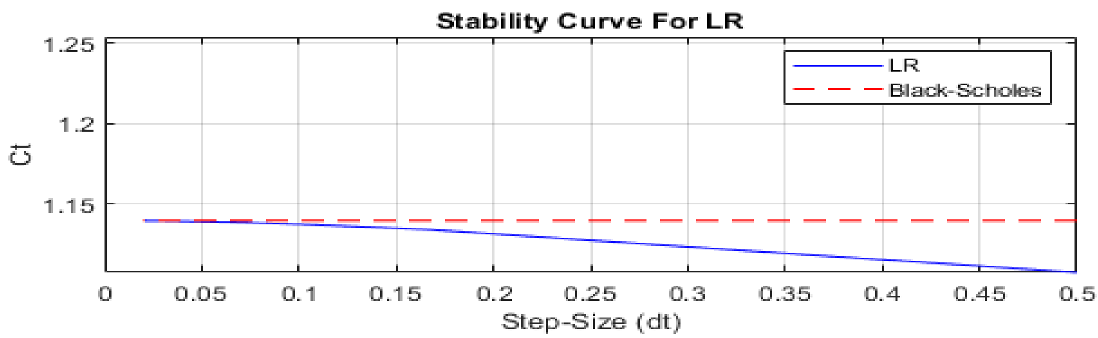
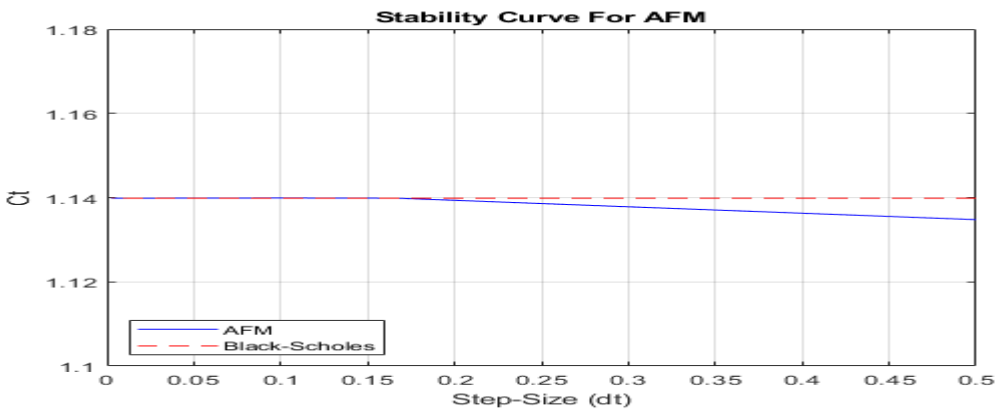


Figure 13: AFM Stability Plot



Findings and Discussions

1. **Comparative Pricing Accuracy:** It is clear from tables 2 and 3, that AFM is more accurate compared to CRR, J-R and LR in terms of option valuation.
2. **Comparative Convergence Ra**tes: The Convergence plots in Figures 2, 3, 4 and 5, we see that CRR and JR exhibit similar patterns, a non-monotonic and slow convergence pattern. However, in the case of LR and AFM, the convergence is smoother and faster. As early as at when n=15, LR converge to a value, with a good degree of accuracy, and AFM converges even faster. We infer that models based on adaptive-swing factor have higher rate of convergence than the traditional versions built on fixed swing factors.
3. **Comparative Error Decay Patterns:** From the error decay plots in Figures 6 to 9, we observe that the errors in both cases of CRR and JR reduce in a non-definite manner, along growing step number n, but in LR and AFM, the errors decay smoothly and vanish within a short time.
4. **Comparative Stability Behaviours:** Figures 10 to 13 are plots of the option values Ct, against the step size dt for graphical illustrations of the relative stability behaviours. Based on observed number of divergent points within the interval of step size plotted, we observe that AFM and LR have better stability properties than CRR and JR.

CONCLUSION

Results from this research suggests preference for models based on adaptive swing factor (AFM and LR) when valuing options with considerations for good accuracy, rate of convergence and stability. Among the models compared, the Adaptive Factor Model (AFM) is most accurate for option valuation. AFM is recommendable to the financial market practitioners for option valuation

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