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POSITIVE DEFINITE TEMPERATURE FUNCTIONS ON THE EUCLIDEAN MOTION GROUP

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ABSTRACT

Let SE(2) be a two dimensional Euclidean motion group realized as the semi-direct product of R2 and SO(2). The Fourier transform of spherical function on SE(2) and its boundedness are presented. Furthermore, a description of temperature function and positive definite temperature functions on SE(2) is presented, among other things. This temperature function is realized as the positive definite solution of the Laplace-Beltrami operator on SE(2).

1. Introduction

Special functions introduced in analysis have been shown to have tight ties to the idea of linear representations of Lie groups. The spherical functions are prominent among these functions. Both the continuous features of Lie groups and the classical Laplace spherical harmonics are generalized by the theory of spherical functions. In the current theory of infinite dimensional linear representation of Lie groups, spherical functions are crucial. Spherical functions for $SE(2)$ are the solutions of the radial part of Laplace-Beltrami operator $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2}$ on $SE(2)$. They are the Bessel functions for n in general. For $n = 2$, they are the Bessel functions of order zero and are also positive definite. This work is centered on the study of positive definite temperature functions on $SE(2)$ which are positive definite solutions of the Laplace-Beltrami operator on $SE(2)$.

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The work is organized as follows. Section two deals mainly on preliminaries concerning the Euclidean motion group. Section three is concerned with the representation, Fourier transform of *SE*(2), the explicit determination of spherical function for *SE*(2) using the method of separation of variables and the computation of its Fourier transform. Results concerning temperature functions on *SE*(2) are presented in section four.

2. Preliminaries

2.1 The Euclidean Motion Group. The group *SE*(*n*) is realized as the semi-direct product of R^n with *SO*(*n*). Any member of *SE*(*n*) may be denoted as $g = (\bar{x}\xi),$, where $\xi \in SO(n)$ and $\bar{x} \in R^n$. For any $g_1 = (\bar{x}_1, \xi_1)$ and $g_2 = (\bar{x}_2, \xi_2) \in SE(n)$, multiplication on SE(n) may be defined as

$$g_1 g_2 = (\bar{x}_1 + \xi_1 \bar{x}_2, \xi_1 \xi_2)$$

and the inverse is defined as

$$g^{-1}\left(-\xi\bar{x}, \xi^1\right)$$

Here $\left(\xi^t\right)$ denotes a transpose. Alternatively, *SE*(*n*) may also be identified as a matrix group whose arbitrary element may be identified as (*n* + 1) × (*n* + 1) matrix, given in the form

$$H(g) = \begin{pmatrix} \xi & \bar{x} \\ 0^t & 1 \end{pmatrix}$$

where $\xi \in SO(n)$ and $0^t = (0,0...,0)$. It is observed that $H(g_1)H(g_2) = H(g_1 g_2)$, $H(g^{-1}) = H^{-1}(g)$ and $g \mapsto H(g)$ an isomorphism between *SE*(*n*) and H(g).

We can now give the matrix representation of the element of $SE(2) \subset GL(3,R)$ by

$$g(x_1, x_2), \phi = \begin{pmatrix} \cos\phi - \sin\phi & x_1 \\ \sin\phi & \cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\phi \in [0,2\pi]$, $(x_1,x_2) \in R^2$ ([13], p.3), ([2], p.207, [6]). A measure on SE(2) is realized as the product of the measures on R^2 and SO(2)([4],[7]).

The universal covering group of *SE*(2) is the semi direct product group $R^2 \rtimes R$ whose multiplication and covering map are defined respectively as

$$(x, \alpha)\,(x^1, \alpha) = (x + e^{it}x^1, \alpha + \alpha^1)$$

and

$$(x, \alpha) \mapsto (x, e^{it}).$$

A sub-Laplacian on *SE*(2) has the form $\Sigma = -\sum_j X_j$, where $X_j = \alpha_j X_3 + U_j$ for some $\alpha_j \in R$ and $U_j \in Span\{X_1, X_2\}$. It is an hypo elliptic operator if $\alpha_j = 0$ [11]

2.2 Spaces of Test Functions on *SE*(2), Their Topologies and Distributions.

Here we give a brief description of spaces of distributions and their respective topologies.

2.2.1 The space $C^\infty(G)$. Given a solvable Lie Group G endowed with invariant measure $d\mu(g)$, and g its Lie algebra. Lets denote by m the dimension of g. Fix $\{X_1,...,X_m\}$ a basis of g. To each $\alpha = (\alpha_1,...,\alpha_m) \in N^m$, we put $|\alpha| = \alpha_1 + ... + \alpha_n$ and associate a differential operator X^α, which is left invariant, on G acting on $f \in C^\infty(G)$, the space $C^\infty(G)$ of infinitely differentiable functions on G, by

$$X^\alpha f(g) = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} ... \frac{\partial^{\alpha_m}}{\partial t_m^{\alpha_m}} f(g\ exp(t_1 X_1)...exp(t_m X_m))|_{t_1=...=t_m=0}.$$

The space $C^\infty(G)$ may be given a topology defined by a system of seminorms specified as

$$|f|_{a,m} = Sup_{|\alpha|\le m} |X^\alpha f(g)|$$

With this topology, C^∞(G) is metrizable, locally convex and complete, hence, it is a Frechet space. This Frechet space may be denoted as $\xi(G)$

2.2.2 The space $C_c^\infty(G)$. This space $C_c^\infty(G)$ is the space of complex-valued C^∞ function on G with compact support. For any $\epsilon > 0$, put

$$B_\epsilon = \{(\xi,\theta) \in G : ||\xi|| \le \epsilon\}$$

and

$$D_\epsilon = \mathfrak{D}(B_\epsilon) = \{f \in C_c^\infty(G) : f(\xi,\theta) = 0, if\ ||\xi|| > \epsilon\}.$$

Then $D(B_\epsilon)$ is a Frechet space with respect to the family semi norms defined as

$$\left\{P_\alpha(f) = \|D^\alpha f\|_\infty : \alpha \in \mathbb{N}^3\right\}.$$

$D(G) = \bigcup_{n=1}^\infty \mathfrak{D}(B_n)$ is topologised as the strict inductive limit of $D(B_n)$. A linear functional on the topological vector space D(G) that is continuous is known as a distribution on G. Then $D'(G)$ is the space of distribution on G.

Given a manifold M and a distribution T, T is said to vanish on a subset $V \subset M$, which is open, if $T = 0$. Let $\{U_\alpha\}_{\alpha\in\omega}$ represents the collection of all open sets on which T vanishes and let U stand for the union of $\{U_\alpha\}_{\alpha\in\omega}$. $M - U$, regarded as the complement of M, is the support of T. We denote $\xi'(G)$ a distributions space with compact support.

2.2.3 The Schwartz space S(G). Consider the Euclidean motion group $SE(2)$ realized as $R \times T$ where $R \cong R/2\pi Z$. If we choose a system of coordinates (x,y,θ) on G with $x,y \in R$ and $\theta \in T$, then a complex - valued C^∞ function f on $G = SE(2)$ is called rapidly decreasing if for any $N \in N$ and $\alpha \in N^3$ we have

$$p_{N,\alpha}(f) = Sup_{\theta\in T, \xi\in R2} |(1 + ||\xi||^2)^N (D^\alpha f)(\xi,\theta)| < +\infty,$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}\left(\frac{\partial}{\partial \theta}\right)^{\alpha_3},$$

($\alpha = (\alpha_1, \alpha_2, \alpha_3); \xi = (x, y)$). The space of all rapidly decreasing functions on G is denoted by S = S(G). Then S is a Frechet space in the topology given by the family of semi-norms $\{P_{N,\alpha} : N \in \mathbb{N}, \alpha \in \mathbb{N}^3\}$.

The space $S'(G)$ of (continuous) linear functionals on $S(G)$ is referred to as the space of tempered distributions on $G = SE(2)$. This space can be topologised by strong dual topology, which is defined as the topology of uniform convergence on the bounded subsets of S(G) generated by the seminorms $p_\varphi(u) = |u(\varphi)|$, where $u : S(G) \to \mathbb{R}$ and $\varphi \in S(G)$. We close this section by defining the concept of convolution on the space S(G).

Let $f_1, f_2 \in S(G)$ or $L^2(G)$. The convolution of f_1 and f_2 is defined as

$$(f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g) d\mu_G(h)$$
$$= \int_G f_1(gh) f_2(h^{-1}) d\mu_G(h)$$

The convolution operation obeys the associativity property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3),$$

whenever all the integrals are absolutely convergent (cf: [1], [10]).

3. Fourier transform of Spherical function on $SE(2)$

The Fourier transform of the group $SE(2)$ is needed in what follows. The following preparations concerning the representation of $SE(2)$ is presented first.

Let $L^2([0, 2\pi], \frac{d\alpha}{2\pi})$ be the space of square integrable functions on $T \cong [0, 2\pi] \cong SO(2)$. A representation of $SE(2)$ on $L^2([0, 2\pi], \frac{d\alpha}{2\pi})$ is given as

$$U^{(p)}(g)\overline{\psi}(X) = e^{-ip(\alpha, X)}\overline{\psi}(A^T X)$$

for each $g \in SE(2)$, where $p \in \mathbb{R}^+$ and $X.Y = x_1 y_1 + x_2 y_2$. $A = \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}$, $A^T = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}$, so that

$$A^T X = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 cos\phi + x_2 sin\phi \\ -x_s in\phi + x_2 cos\phi \end{pmatrix}.$$

The representation $U^{(p)}(g)$ given above is unitary and irreducible (see [2],[10]).

The following definition of group Fourier transform may be found in ([1] [8]).

3.1 Definition. The Fourier transform of $f \in S(G)$ (or $f \in L^1(G)$) is a map

$$F(f) : \mathbb{R}^+_* \to B(L^2(G))$$

defined as

$$(Ff)(\sigma) = \int_G f(g) U^\sigma(g^{-1}) d\mu$$

and the inverse is defined as

$$f(g) = \int_0^\infty Tr(U_g^\sigma f(\sigma))\sigma d\sigma,$$

where $g = \bar{x},\xi)$ and $d\mu(g)$ stands for a measure on G. F(f)(σ) may also be denoted by $\widehat{f}(\sigma)$ in what follows.

Given a locally compact group G and a subgroup K that is compact, the pair (G,K) is known as a Gelfand pair if $L^1(G//K)$ is abelian under convolution. Also, let $C_c(K\backslash G/K)$ stand for the space of continuous functions with support on G that are compact. Functions on $C_c(K\backslash G/K)$ that satisfy $f(k_1gk_2) = f(g)\ \forall k_1,k_2 \in K$ are referred to as spherical functions and $C_c(K\backslash G/K)$ equipped with convolution as a binary operation forms a Banach algebra that is commutative. The pair ($SE(2)$, $SO(2)$) is a Gelfand pair (see [3], [12].).

A function ϕ defined on a K-bi-invariant continuous function that satisfies $\phi(e) = 1$ in such a way that

$f \to f * \phi(e)$ defines an algebra homomorphism of $C_c(K\backslash G/K)$ is termed spherical function.

The Laplace-Beltrami operator on $SE(2)$(see [3],[5]) is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2}. \qquad (1)$$

A radial solution of the above operator is given as follows. Let the operator act on $\varphi = \varphi(r,\theta,\psi)$, then

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial \theta^2} + \frac{\partial^2\varphi}{\partial \psi^2}. \qquad (2)$$

We have the following elleptic partial differential equation

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial \theta^2} + \frac{\partial^2\varphi}{\partial \psi^2} = 0. \qquad (3)$$

Let us assume that the above equation has a solution of the form

$$\varphi(r,\theta,\psi) = R(r)F(\theta)\Psi(\psi). \qquad (4)$$

Now,

$$\begin{cases} \dfrac{\partial\varphi}{\partial r} = F(\theta)\Psi(\psi)\dfrac{\partial R}{\partial r} \; and \; \dfrac{\partial^2\varphi}{\partial r^2} & = F(\theta)\Psi(\psi)\dfrac{\partial^2 R}{\partial r^2} \\ \dfrac{\partial^2\varphi}{\partial \theta^2} & = R(r)\Psi(\psi)\dfrac{\partial^2 F}{\partial \theta^2} \\ \dfrac{\partial^2\varphi}{\partial \psi^2} & = R(r)F(\theta)\dfrac{\partial^2 \Psi}{\partial \psi^2} \end{cases} \qquad (5)$$

.

Substituting (5) into (3), we have

$$F(\theta)\Psi(\psi)\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}F(\theta)\Psi(\psi)\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}R(r)\Psi(\psi)\frac{\partial^2 F}{\partial \theta^2} + R(r)F(\theta)\frac{\partial^2\Psi}{\partial \psi^2} = 0. \qquad (6)$$

Divide (6) by $R(r)F(\theta)\Psi(\psi)$, then we get

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial \theta^2} + \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial \psi^2} = 0.$$

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{R(r)r}\frac{\partial R(r)}{\partial r}+\frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial\theta^2}=-\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial\psi^2}. \quad (7)$$

The Left Hand Side (LHS) of (7) depends only on (r,θ) while the Right Hand Side (RHS) depends only on ψ. We can equate each side to a constant, say $-m^2$. Thus we get

$$-\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2}=-m^2 \Rightarrow \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2}=m^2 \Rightarrow \frac{\partial^2\Psi}{\partial\Psi^2}=m^2\Psi(\psi), \quad (8)$$

and

$$\frac{1}{R(r)}\frac{\partial^2 R}{\partial r^2}+\frac{1}{R(r)r}\frac{\partial R}{\partial r}+\frac{1}{F(\theta)r^2}\frac{\partial^2 F}{\partial\theta^2}=-m^2. \quad (9)$$

Multiplying (9) by r^2, to get:

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R}{\partial r}+\frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2}=-m^2r^2$$

so that

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R}{\partial r}+m^2r^2=-\frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2}. \quad (10)$$

Again equate both sides of (10) to n^2, where n is a constant:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{r}{R(r)}\frac{dR(r)}{dr}+m^2r^2=n^2 \quad (11)$$

$$-\frac{1}{F(\theta)}\frac{d^2F(\theta)}{d\theta}=n^2 \quad (12)$$

We can now solve the ordinary differential equation:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{r}{R(r)}\frac{dR(r)}{dr}+m^2r^2-n^2=0. \quad (13)$$

Next, we transform this equation into Bessel equation. To do this, we let mr = x so that $\frac{\partial x}{\partial r}=m$.

Then

$$\begin{aligned}\frac{dR(r)}{dr}&=\frac{dR(r)}{dx}\frac{dx}{dr}\\&=m\frac{dR(r)}{dx}\end{aligned}$$

$$\begin{aligned}\frac{d^2R(r)}{dr^2}&=\frac{d}{dr}\left(\frac{dR}{dr}\right)\\&=\frac{d}{dr}\left(m\frac{dR(r)}{dx}\right)\\&=m\frac{d^2R(r)}{dx^2}\frac{dx}{dr}\\&=m\frac{d^2R(x)}{dx^2}m\\&=m^2\frac{d^2R(x)}{dx^2}.\end{aligned}$$

Equation (13) becomes

$$m^2 \frac{r^2}{R(r)} \frac{d^2R(r)}{dx^2} + \frac{mr}{R(r)} \frac{R(r)}{dx} + (m^2r^2 - n^2) = 0. \qquad (14)$$

Multiply (14) by $R(r)$ to get

$$m^2r^2 \frac{^2R(r)}{dx^2} + mr\frac{R(r)}{dx} + (m^2r^2 - n^2)R(r) = 0.$$

$mr = x$; $m^2r^2 = x^2$, therefore,

$$x^2 \frac{d^2R(r)}{dx^2} + x\frac{R(r)}{dx} + (x^2 - n^2)R(r) = 0.$$

This may be re-written as

$$\frac{d^2R(r)}{dx^2} + \frac{1}{x}\frac{dR(r)}{dx} + \left(1 - \frac{n^2}{x^2}\right)R(r) = 0. \qquad (15)$$

The differential equation (15) is a Bessel differential equation and it has a solution of the form

$$\begin{aligned} J_\lambda(mr) &= \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!\Gamma(k+\frac{n}{2})}\left(\frac{mr}{2}\right)^{2k} \qquad (16) \\ &= \Gamma\left(\frac{n}{2}\right)\left(\frac{\sqrt{\lambda}r}{2}\right)^{\frac{2-n}{2}} I_{\frac{n-2}{2}}\left(\sqrt{\lambda}r\right) \qquad 17 \end{aligned}$$

where I_v is the Bessel function of index v. Different values of λ will give different solutions. In our own case, we are considering $SE(2)$, that is $n = 2$. Therefore, (18) can be further simplified to be

$$\begin{aligned} J_\lambda(mr) &= \Gamma(1)\left(\frac{\sqrt{\lambda}r}{2}\right)^{0} I_{\frac{2-2}{2}}\left(\sqrt{\lambda}r\right) \qquad (18) \\ &= I_0\left(\sqrt{\lambda}r\right). \qquad \square \end{aligned}$$

Expression (19) is the desired spherical function for $SE(2)$, generally rreferred to as the Bessel function of order zero. Here after, it is denoted as $J_0(\lambda)$

Next, the Fourier transform of expression (19) is considered. The integral representation of $J_0(\lambda)$ is defined as

$$J_0(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda cos\phi} d\phi$$

Now

$$\begin{aligned} \mathcal{F}(J_0(\lambda)) &= \int_{-\infty}^{\infty} J_0(\lambda)e^{it\lambda}d\lambda \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda cos\phi}e^{it\lambda}d\phi d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\lambda cos\phi}e^{i\lambda t}d\phi d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda(-t-cos\phi)}d\phi d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi}\delta(-t-cos\phi)d\phi. \end{aligned}$$

We evaluate $\delta(-t - cos\phi)$ and substitute the result as follows. It is known that $\delta(x)$=$\delta(-x)$, therefore, $\delta(-t-cos\phi) = \delta(t+cos\phi)$. $t+cos\phi$ is a function,and we use one of the properties of Dirac function that says

$$\delta(g(\phi)) = \sum_{i=1}^{n} \frac{\delta(\phi - \phi_i)}{|g'(\phi_i)|}.$$

ϕ_i are the roots of $g(\phi) = 0$, $g^1(\phi)$ is the first order derivative of $g(\phi)$. Now, $g(\phi) = t + cos\phi$, therefore, $g'(\phi) = sin\phi$. The two possible roots of $t + cos\phi$ are $\phi_1 = cos^{-1}(-t)$ and $\phi_2 = 2\pi - cos^{-1}(-t)$. So therefore,

$$\delta(t + cos\phi) = \frac{\delta(\phi - cos^{-1}(-t))}{|sin(cos^{-1}(-t))|} + \frac{\delta(\phi - (2\pi - cos^{-1}(-t)))}{|sin(2\pi - cos^{-1}(-t))|}.$$

We know that $sin^2\phi + cos^2\phi = 1$ and $sin\phi = \sqrt{1 - cos^2\phi}$. Let $\phi = cos^{-1}(-t)$,

$$\begin{aligned} sin(cos^{-1}(-t)) &= \sqrt{1 - cos(cos^{-1}(-t))cos(cos^{-1}(-t))} \\ &= \sqrt{1 - cos(cos^{-1}(-t))^2} \\ &= \sqrt{1 - t^2}. \end{aligned}$$

Therefore, $\sin(\cos^{-1}(-t) = \sqrt{1-t^2}.$ Now,

$$\begin{aligned} \delta(t - cos\phi) &= \frac{\delta(\phi - cos^{-1}(-t))}{|sin(cos^{-1}(-t))|} + \frac{\delta(\phi - (2\pi - cos^{-1}(-t)))}{|sin(2\pi - cos^{-1}(-t))|} \\ &= \frac{\delta(\phi - cos^{-1}(-t))}{\sqrt{1 - t^2}} + \frac{\delta(\phi - (2\pi - cos^{-1}(-t)))}{\sqrt{1 - t^2}} \end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \delta(t - \cos\phi)d\phi = \frac{2}{2\pi}\frac{\delta(\phi - \cos^{-1}(-t)}{\sqrt{1-t^2}} + \frac{\delta(\phi - (2\pi - \cos^{-1}(-t)))}{\sqrt{1-t^2}}d\phi$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1 = t^2}}\int_0^{2\pi} \left[\delta(\phi = \cos^{-1}(-t) + \delta(\phi - (2\pi - \cos^{-1}(-t)))\right]d\phi$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1 = t^2}}\int_0^{2\pi} \left[\delta(\phi = \cos^{-1}(-t)d\phi + \int_0^{2\pi} \delta(\phi - (2\pi - \cos^{-1}(-t)))\right]d\phi$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1 = t^2}}(1+1)$$

$$= \frac{2}{2\pi}\frac{1}{\sqrt{1-t^2}}$$

$$= (\pi)^{-1}\sqrt{1-t^2}$$

This is the Fourier transform of the spherical function of $SE(2)$. It is our interest to use it in studying signal analysis and a Paley Wiener type theorem in our future research.

4. Temperature function on the motion group $SE(2)$

4.1 Definition. Let $\triangle_{SE(2)}$ be the Laplace-Beltrami operator on $SE(n)$. A temperature function on $SE(2)$ is a C^∞ function $F \in S(G)$ that satisfies

$$\left(\left(\frac{\partial}{\partial \xi} - \triangle_G\right)F\right)(\xi,\theta) = 0, (\xi,\theta) \in G.$$

Let $f \in L^\infty(SE(2))$, then f is positive definite on G if

$$\int_G f(\xi,\theta)(\varphi * \varphi^*)(\xi,\theta)d\xi d\theta \geq 0, \varphi \in \mathcal{S}(SE(2), \varphi^*(\xi,\theta) = \varphi((\xi,\theta)^{-1}).$$

A function f that is continuous on $SE(n)$ is called $U(n)$- invariant if

$$f(\alpha\xi,\theta) = f(\xi,\theta), (\xi,\theta) \in SE(2), \forall\alpha \in U(n),$$

where $U(n)$ stands for a collection of square unitary matrices of order n. Let $f \in S'(SE(2))$, then for all $\alpha \in U(n)$, f_α is defined as

$$f_\alpha(\varphi) = f(\varphi_{\alpha-1})$$

where $\varphi \in S(SE(2))$ and $\varphi_{\alpha}-1(\xi,\theta) = \varphi(\alpha^{-1}\xi,\theta)$. A function $\varphi \in S(SE(2)$ is $U(n)$-invariant if

$$\varphi_\alpha = \varphi,$$

a distribution $f \in S'(SE(2))$ is $U(n)$-invariant if and only if $f = f^\natural$ where

$$f^\natural(\varphi) = f(\varphi^\natural),$$

and

$$\varphi^\#(\xi,\theta) = \int_{U(n)} \varphi(\alpha\xi,\theta)d\alpha$$

$d\alpha$ is the normalized measure on $U(n)$ such that

$$\int_{U(n)} d\alpha = 1.$$

It therefore means that $\varphi \in S(SE(2))$ is $U(n)$-invariant if and only if $\varphi^\natural = \varphi$. Let $f \in S'(SE(2))$ and $\varphi \in S(SE(2)$, then the convolution $f * \varphi$ is defined as

$$(f * \psi)(\xi,\theta) = \int df(\xi',\theta')\psi(\xi',\theta')^{-1}(\xi,\theta), (\xi,\theta) and (\xi',\theta') \in SE(2),$$

and

$$(\psi * f)(\xi,\theta) = \int df(\xi',\theta')\psi(\xi,\theta)(\xi',\theta')^{-1}(\xi,\theta), (\xi',\theta') \in SE(2).$$

The linear map

$$S'(SE(2)) \times S(SE(2)) \ni (f,\varphi) \rightarrow f * \varphi \in S'(SE(2))$$

is separately continuous and the convolution $f * \varphi$ is a smooth function on $SE(2)$. The following lemma is needed in the proof of theorem 4.3, which is the main result of this section.

4.2 Lemma. The Bessel function $\{J_0(\lambda) : \lambda > 0\}$ associated with the Laplace-Beltrami Operator of $SE(2)$ satisfies the following conditions.

(i) $J_0(\lambda) \in S(G), \lambda > 0$.

(ii) For every $\psi \in S(G)$, $J_0(\lambda) * \psi \to \psi$ in $S(G)$ as $\lambda \to 0$.

The following result is the main result of this section.

4.3 Theorem. Let $f \in S'(SE(2))$. The function F defined on G by

$$F(\xi,\theta) = (f * J_0(\lambda))(\xi,\theta)$$

satisfies the following conditions.

(i) $((\frac{\partial}{\partial\lambda} - \Delta_G)F)(\xi,\theta) = 0, (\xi,\theta) \in SE(2)$.

(ii) There exist positive constants C and N such that

$$| F(\xi,\theta) | \leq C \underset{\theta\in R, \xi\in R^2}{Sup} | (1+ \| \xi \|^2)^{-N} \| f \| L^1(SE(2)) \tag{20}$$

where $\|f\|_{L^1(SE(2))}$ is the L^1- norm of $SE(2)$.

Proof. (i) Let $f \in S'(SE(2))$. Then by (i) of lemma 4.2, the function defined by $F(\xi,\theta) = (f * J_0(\lambda))(\xi,\theta)$ is a smooth function on $SE(2)$. Also, since the linear map $S'(SE(2))\times S(SE(2)) \ni (f,\varphi) \to f * \varphi \in S'(SE(2))$ is separately continuous, it therefore means that $f * \varphi$ is smooth. Let $U(g)$ be the universal enveloping algebra of $SE(2)$. Elements of this algebra are left invariant differential operators on $SE(2)$. The Laplace-Beltrami operator on $SE(2)$ is a member of this algebra, therefore $\triangle_G$ is a left vector field on $SE(2)$. Therefore, it stands to reason that

$$\left(\left(\frac{\partial}{\partial r} - \triangle_G\right)F\right)(\xi,\theta) = 0.$$

(ii) For $f \in S'(SE(2))$, there exists C and N such that

$$| f(\psi) | \leq C \sup_{\theta\in T, \xi\in R^2} | (1+ \| \xi \|^2)^N (D^\alpha f)(\xi,\theta) \|$$

By (20),

$$|F(\xi,\theta)| = |(f * J_0(\lambda))(\xi,\theta)| \leq \sup_{\theta\in T, \xi\in R^2} |(1 + \|\xi\|^2)^N (D^\alpha f * J_0(\lambda))(\xi,\theta)|.$$

But

$$| D^\alpha (f * J_0(\lambda))(\xi,\theta) | = | f * D^\alpha J_0(\lambda)(\xi,\theta)$$

$$\leq \int_{SE(2)} | f(\eta) \| D^\alpha (\eta^{-1}\xi) d\eta$$

$$\leq \int_{SE(2)} | Ff(\eta) \| D^\alpha \xi | d\eta \quad \text{since } SE(2) \textit{ is unimodular}$$

$$\leq \int_{SE(2)} | f(\eta) d\eta \frac{C}{|(1+\|\xi\|^2)^N|}$$

$$= C\,|(1+\|\xi\|^2)^{-N}| \int_{SE(2)} |f(\eta)|\,d\eta$$

$$=|(1+\|\xi\|^2)^{-N}\|\,f\,\|L^1(SE(2))$$

$\eta = (\xi\theta) and \quad \xi = (\xi, \theta)$

4.4 Positive definite temperature functions

Let $f \in D'(SE(2))$, and let D^i and E^i be vector fields. f is called a positive definite distribution on $SE(2)$ if it can be expressed as

$$f = \sum_{i=1}^{N} D^i E^i f_i,\ f_i \in L^\infty(SE(2))$$

D^i and E^i are respectively left and right invariant on $SE(2)$ [9]

A description of positive definite temperature function on $SE(2)$ is presented in the next theorem.

4.4. Theorem. Let F be a temperature function that is smooth and is defined as presented in (ii) of theorem 4.3 such that

$$f_{SE(2)} F(\xi,\theta)(\psi * \psi^*(\xi,\theta) d\xi d\theta \geq 0$$

for all $\varphi \in S(SE(2))$. Then there exists $f \in S'(SE(2))$ that is positive definite and unique such that

$$f(\varphi) = \lim_{s\to 0^+} \int_{SE(2)} F(\xi,\theta)\varphi(\xi,\theta) d\xi d\theta.$$

On the other hand, every positive definite distribution $f \in D'(SE(2))$ that is positive definite defines a positive definite temperature function on $SE(2)$ that satisfy the second condition of theorem 4.3 .

Proof. $\exists$ a unique $f \in\in S'(SE(2))$ such that (see Theorem 4.3)

$$f(\varphi) = \lim_{r\to 0} \int_{SE(2)} F(\xi,\theta)(\varphi * \varphi^*)(\xi,\theta) d\xi d\theta.$$

Let us assume that $F(\xi,\theta)$ is positive definite on $SE(2)$, it follows that

$$f(\varphi * \varphi^*) = \lim_{r\to 0} \int_{SE(2)} F(\xi,\theta)(\varphi * \varphi^*)(\xi,\theta) d\xi d\theta \geq 0$$

Conversely, let f be a distribution on $SE(2)$ that is positive definite, then $f \in S'(SE(2))$. Let F be a the function on $SE(2)$ defined by

$$F(\xi,\theta) = (f * J_0(\lambda))(\xi,\theta),$$

then

$$(f * J_0(\lambda))(\varphi * \varphi^*) = J_0(\lambda) * (\check{f}(\varphi * \varphi^*)),$$

where $\check{\psi}(\xi,\theta) = \psi((\xi,\theta)^{-1}$, $\psi \in S(SE(2))$, $\check{f}(\psi) = f(\check{\psi})$. Since $J_0(\lambda)$ is a positive function on $SE(2)$ and f is a positive definite distribution on $S'(SE(2))$, the following follows

$$(f * J_0(\lambda))(\varphi * \varphi^*) = J_0(\lambda) * \left(\check{f} * (\varphi * \varphi^*)\right) \geq 0, \ \lambda > 0.$$

Conclusion.

In this work, the spherical function of SE(2) has been explicitly shown to be the Bessel function of order 2. It is also observed that this function is positive definite and bounded in section three. A temperature function, obtained as the product of a Schwartz function on SE(2) and its spherical function is also observed in section four to be positive definite. It is our interest to study this temperature function in our next research to see if it is tempered and radial.

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