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COMPUTATIONAL FINITE DIFFERENCE SCHEME FOR SOLVING SECOND-ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

M. T. Raji[1a], O. Fagbemiro[2a], J. O. Hamzat[3b], A. L. Ijaola[4a] and B. I. Olajuwon[5a]

[a]*Department of Mathematics, Federal University of Agriculture Abeokuta*
[b]*Department of Mathematics, University of Lagos.*

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ABSTRACT

In this study, the computational finite difference scheme of Non-Standard Finite Difference Methods (NSFDM) for solving second-order Fredholm Integro-differential equations will be built. The numerical technique and the exact answer coincide at any point inside the interval of integration, according to the method's structure. A methodological study was also performed to demonstrate that the solutions to the second-order Fredholm integro-differential problem have a matching finite difference scheme. The resulting approach was then used to solve various current second-order Fredholm integro-differential equation problems, and the numerical comparison of results demonstrate the closeness and dependability of the derived finite difference scheme.

1. Introduction

One of the equations used in science and engineering the most is the integral-differential equation. It deals with a function's derivatives as well as integrals. After doing extensive research and basing his study on hereditary effects, Volterra examined population expansion. As a result, integro-differential equations were developed [1]. The integro-differential equations, which fall under the categories of Fredholm and Volterra equations, assert that the unknown function (x) and one or more of its derivatives, such as y'(x), y''(x),..., appear both out and under the integral sign [2]. When it comes to the integral component of Volterra type, the upper bound of the region is variable, while for the Fredholm type, it is a fixed value. The Fredholm integro-differential equation has gained the interest and the attention of several researchers and these are observed in [7], [8], [9], [11] and [12] just to mention few among many over the the years.

*Corresponding author: Olalekan Fagbemiro
E-mail address: fagbemiroo@funaab.edu.ng
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This researchers have used several numerical schemes in great deal to large advantage over the analytic approach and despite these great contribution from the use of numerical schemes there is still exist short-comings in terms of closeness, dependability and stability. In this investigation, we seek to address these short-comings by employing the derived finite scheme from an interesting perspective.

Let $y(x)$ be the exact solution of the integro-differential equation,

$$Dy(x)-\lambda\int_a^b m(x,t)y(t)dt = f(x), \qquad x\in[a,b] \tag{1}$$

with

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}y^{(m-1)}(a)+c_{jm}^{(2)}y^{(m-1)}(b)\right]=d_j, \quad j=1,\ldots,v, \tag{2}$$

where $f(x)$ and $m(x,t)$ are given continuous functions $\lambda, a, b, c_{jm}^1, c_{jm}^2$ and d_j some given constants.

The form of the second order Fredholm integro-differential equations is to be examined here is

$$y''(x) = f(x,y)+\int_a^b k(x,t)y(t)dx, \qquad a\le x\le b \tag{3}$$

subject to the initial conditions

$$y(a)=\alpha,\ y'(a)=\beta \tag{4}$$

where α *and* β are real constants. The function $f(x\ \ y)$ and the kernel $k(x, t)$ are known.

It is necessary to ascertain the solution y(x), which demonstrates the holding of the following properties;

- a solution must exists,
- the solution is unique, and
- the solution's behavior changes continuously with the initial conditions.

Integro-Differential Equations (IDEs) have garnered increasing attention in recent times. Applications of IDEs in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, and electrostatics are numerous and span a wide range of fields. Since it is typically challenging to solve the integro-differential equations analytically, a numerical approach is needed. Circuit analysis is one of the many scenarios in science and engineering that are modeled by Fredholm integro-differential equations. A system of integro-differential equations can be used to represent the activity of interacting excitatory and inhibitory neurons. Additionally, they play a major role in the modeling of several physical processes, including neural networks and signal processing [3]. The Fredholm integro-differential equations have several applications in the fields of electromagnetic theory, dispersive waves, and ocean circulation [2]. One particular use of the Non-Standard Finite Difference Method (NSFDM) is the finite difference schemes. In the beginning, [4] described the precise linearization methodology. His thought was on the subject of whether it was possible to find a linear ordinary differential equation that shares the same general solution as the supplied linear ordinary differential equation. According to [5], if there is a solution to an ordinary differential equation, then it has the exact discretization. Several writers have created several approaches to solving equation (1). Lagrange interpolation method [6], Tau operational method [7], Legendre polynomial method [8], generalized minimal residual method [9], differential transform method with Adomian polynomials [10], canonical basis function method [11], power series and chebyshev series approximation methods [12], Bessel function method [13], cubic spline collocation method [14],

nonstandard finite difference method [15], homotopy analysis transform method [2], on exact finite difference scheme, [16] among others.
However, the computation of Fredholm integro-differential equations will be done in this study using a finite difference approach.

Defnition [17] 1:
A Finite Difference Scheme is one for which the solution to the difference equation has the same general solution as the associated differential equation. Below, we give the standard finite discrete representations for some derivatives;

$$\frac{dy}{dt} \to \frac{y_{n+1} - y_n}{h} \qquad 5$$

$$\frac{d^2 y}{dx^2} = \frac{y_{n+1} - 2y_n - y_{n-1}}{h^2} \qquad 6$$

Defnition[18] 2:
A finite difference scheme is called non-standard finite difference method, if at least one of the following conditions is met;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function such that

$$\phi(h) = h + o(h^2),\ as\ h \to 0 \qquad 7$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable function of several points of the mesh. For example,

$$\left. \begin{aligned} y^2 &\approx y_n y_{n+1}, y_{n-1} y_n \\ y^3 &\approx y_{n-1} y_n y_{n+1}, y_n^2 y_{n+1} \end{aligned} \right\} \qquad 8$$

MATERIALS AND METHODS

It is crucial to note from the analysis of the Finite Difference Scheme that the solution to equation (1) can be expressed as,

$$y(x) = \phi(\lambda, y_0, y_0^{'}, x_0, x) \qquad 9$$

with

$$\left. \begin{aligned} \phi(\lambda, y_0, x_0, x_0) &= y_0 \\ \phi(\lambda, y_0', x_0, x_0) &= y_0' \end{aligned} \right\} \qquad 10$$

Consider a discrete model of equation (1) given by,

$$y_{n+1} = g(\lambda, h, y_n, y_{n+1}, x_n, x_n) = h \qquad 11$$

Its solution can be expressed in the form,

$$y_n = \psi(\lambda, h, y_0, y_0', x_0, x_n) \qquad 12$$

with

$$\left.\begin{array}{l}\psi(\lambda, y_0, x_0, x_0) = y_0 \\ \psi(\lambda, y_0', x_0, x_0) = y_0'\end{array}\right\} \qquad 13$$

Defnition 3

Equations (1) and (11) are said to have the same general solution if and only if

$$y_n = y(x_n) \qquad 14$$

for any values of h .

Theorem 1

The differential equation (1) has a Finite Difference Scheme by the expression,

$$y_{n+1} = \phi\left[\lambda, y_n, y_{n+1}, x_{n-1}, x_n, x_{n+1}\right] \qquad 15$$

where ϕ is that of equation (9).

Proof

The group characteristic of equation (1)'s solutions provides,

$$y(x+h) = \phi[\lambda, y(x), x-h, x, x+h] \qquad 16$$

If we now apply the changes,

$$x \to xn, \quad y(x) \to y_n \qquad 17$$

then, equation (16) becomes,

$$y_{n+1} = \phi\left[\lambda, y_n, y_{n+1}, x_{n-1}, x_n, x_{n+1}\right] \qquad 18$$

equation here is the necessary ordinary difference equation, and its universal solution is the same as equation (1), the following ramifications of the above theorem are significant.

(i) Equation (16) holds for all x and h if all of the solutions to (1) exist for all x = ∞. If not, the relationship is taken to hold whenever (16)'s right side is clearly defined.
(ii) The existence theorem is all that the theorem is. That is, a Finite Difference Scheme exists if an ordinary differential equation can be solved. No instructions are provided on how to actually build such a scheme, according to [17].

(iii) The theorem's main result is that, for a fixed but variable step-size, the solution to the difference equation is precisely equal to the solution to the ordinary differential equation on the computing grid.

Formulation of the New Exact Finite Difference Scheme

The previously mentioned Theorem 1 will be used to build a new finite difference method for second-order Fredholm integro-differential equations of the form (1), for which there are explicit, exact general solutions. One of the characteristics of this scheme is that there are no numerical insolvabilities un its solutions. But it's crucial to remember that, given a collection of functions that are linearly independent,

$$\left\{y'(x)\right\};\ i=1,2,\ldots,N \qquad 19$$

The equivalent discrete functions can always be found as solutions to a Nth order linear difference equation [19]. Let's

$$y_n^i \equiv y^i(x_n),\ x_n = (\Delta x)n = hn \qquad 20$$

Afterward, the necessary difference equation is given by the subsequent determinant:

$$\begin{vmatrix} y_n & y_n^1 & y_n^2 & \cdots & y_n^k \\ y_{n+1} & y_{n+1}^1 & y_{n+1}^2 & \cdots & y_{n+1}^k \\ . & . & & & . \\ . & . & & & . \\ y_{n+k} & y_{n+k}^1 & y_{n+k}^2 & \cdots & y_{n+k}^k \end{vmatrix} = 0 \qquad 21$$

Let us consider the equation of form (1) and suppose that the numerical solution of the difference equation at the same position x = xn marked by yn has the same general solution as the exact solution of the problem (1) at x = xn represented by y(xn). Consequently, the matching difference equation from equation (21), is provided by,

$$\begin{vmatrix} y_n & y_n^1 & y_n^2 \\ y_{n+1} & y_{n+1}^1 & y_{n+1}^2 \\ y_{n+2} & y_{n+2}^1 & y_{n+2}^2 \end{vmatrix} = \begin{vmatrix} y_n & y(x_n) & y'(x_n) \\ y_{n+1} & y(x_{n+1}) & y'(x_{n+1}) \\ y_{n+2} & y(x_{n+2}) & y'(x_{n+2}) \end{vmatrix} = 0 \qquad 22$$

Evaluating the determinant of (22) to have

$$\begin{aligned} & y_{n+2}\left(y(x_n)y'(x_{n+1}) - y'(x_n)y(x_{n+1})\right) + y_{n+1}\left(y'(x_n)y(x_{n+2}) - y'(x_{n+2})y(x_n)\right) \\ & + y_n\left(y'(x_{n+2})y(x_{n+1}) - y'(x_{n+1})y(x_{n+2})\right) = 0 \end{aligned} \qquad 23$$

On solving (23) to have

$$y_{n+2} = \frac{-\left[y_{n+1}\left(y'(x_n)y(x_{n+2}) - y'(x_{n+2})y(x_n)\right) + y_n\left(y'(x_{n+2})y(x_{n+1}) - y'(x_{n+1})y(x_{n+2})\right)\right]}{\left(y(x_n)y'(x_{n+1}) - y'(x_n)y(x_{n+1})\right)} \qquad 24$$

Shifting downward the index by one unit, we obtain

$$y_{n+1} = \frac{-\left[y_n\left(y'(x_{n-1})y(x_{n+1}) - y'(x_{n+1})y(x_{n-1})\right) + y_{n-1}\left(y'(x_{n+1})y(x_n) - y'(x_n)y(x_{n+1})\right)\right]}{\left(y(x_{n-1})y'(x_n) - y'(x_{n-1})y(x_n)\right)} \qquad 25$$

The Finite Difference Scheme in Equation (25) can solve any problem in Equation (1). It is noteworthy that equation (25) has the form (15).

Numerical Stability

As numerical computation advances, stability guarantees that the error between the numerical solution and the exact answer stays restricted. In other words, the computed solution, f*(x), is close to f(x*), the solution of a slightly perturbed issue. Stability issues with the way that $\left| f_{i,j} - (fi\Delta t, j\Delta S) \right|$ for fixed discretization steps Δt and ΔS behave as numerical computing advances.

Conditional Issues: The concept "near" can be measured by further information about the particular problem: $\left\| \frac{f(x) - f(x^{*})}{f(x)} \right\| \leq C \frac{\|\delta x\|}{\|x\|} \qquad f(x) \neq 0$ where C is called *condition number* of this problem. If C is large, the problem is ill-conditioned.

Error Estimate

In this section, an error estimator for the approximate solution of (25) is obtained. We defined $e_N(x) = y(x) - y_N(x)$ as the error function of the approximate solution $y_N(x)$ to $y(x)$, where, $y(x)$ is the exact solution and $y_N(x)$ is the approximate solution computed for various values.

RESULTS AND DISCUSSION

Numerical Experiments

The Finite Difference Scheme developed in this research shall be adopted in solving some modeled real-life Fredholm integrodifferential equations of the form (1). The following notations shall be used in the tables below.

EMGAA - Absolute error in Mohammed *et. al*., (2016)

EGEA - Absolute error in Gegele *et. al.*, (2014)

ETG - Absolute error in Taiwo and Gegele (2014)

Exec t / sec . - Execution time per seconds for computation at each stage

Problem 1:

Consider the model Fredholm integro-differential equation,

$$y''(x) = 32x + \int_{-1}^{1} (1 + xt) y(t) dt, \qquad -1 \leq x \leq 1 \qquad 26$$

subject to the initial conditions,

$$y(0) = 1, \; y'(0) = 1 \qquad 27$$

The exact solution to the problem is given by $y(x) = 1 + \frac{3}{2}x^2 + 5x^3$. See: Gegele *et. al.*, (2014)

On the application of the newly derived Finite Difference Scheme (23) on Problem 1 we obtain the result presented in Table 1 below.

Table 1: Showing the result for Problem 1

X	Exact	Approx.	Error	EMGAA	Exec
1	1.0199999999999998	1.0199999999999998	0.0000	1.250E-005	0.0136
2	1.1000000000000001	1.1000000000000001	0.0000	5.000E-004	0.0215
3	1.2700000000000000	1.2700000000000000	0.0000	4.375E-004	0.0218
4	1.5600000000000001	1.5600000000000001	0.0000	3.700E-004	0.0220
5	2.0000000000000000	2.0000000000000000	0.0000	2.625E-004	0.0223
6	2.6200000000000006	2.6200000000000006	0.0000	2.000E-004	0.0225
7	3.4500000000000002	3.4500000000000002	0.0000	1.875E-004	0.0226
8	4.5200000000000005	4.5200000000000005	0.0000	1.300E-003	0.0317
9	5.8600000000000003	5.8600000000000003	0.0000	1.212E-003	0.0319
10	7.5000000000000000	7.5000000000000000	0.0000	2.500E-003	0.0321

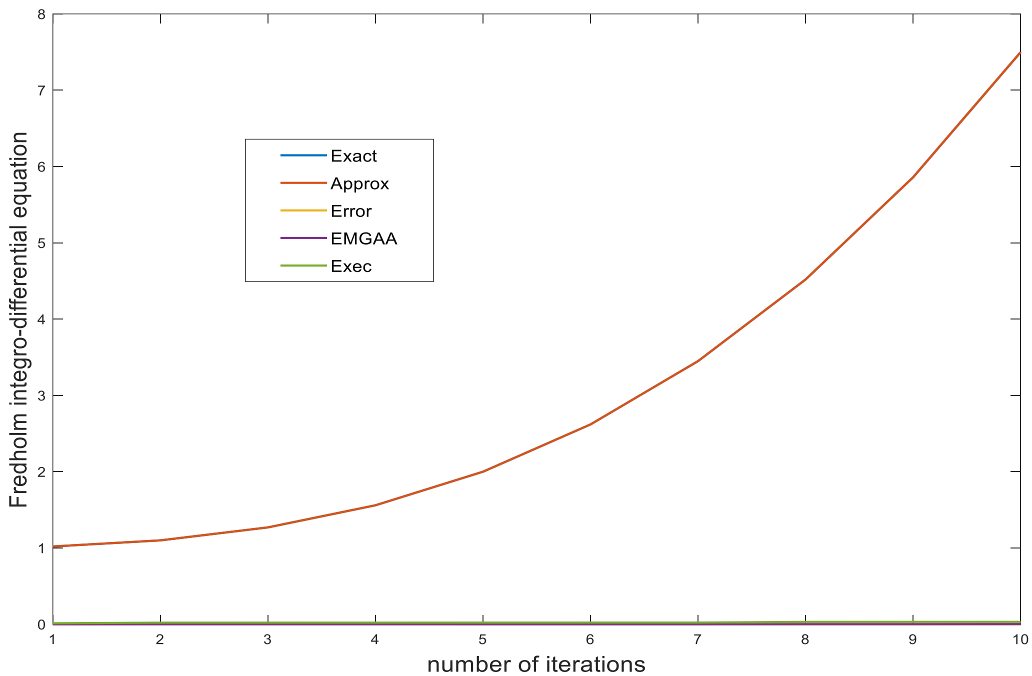

figure 1: graphical representation of the Fredholm integro-differential equation (problem 1)

Problem 2:

Consider the model Fredholm integro-differential equation,

$$y''(x) = e^x - x + \int_0^t xty(t)dt, \qquad 0 \le x \le 1 \tag{28}$$

subject to the initial conditions,

$$y(0) = 1, \;\; y'(0) = 1 \tag{29}$$

The exact solution to the problem is given by $y(x) = e^x$. See: Mohammed *et. al.*, (2016)

On the application of the newly derived FDS (23) on Problem 2 we obtain the result presented in Table 2 below.

Table 2: Showing the result for Problem 2

X	Exact	Approx.	Error	EMGAA	Exec
1	1.1051709180756477	1.1051709180756477	0.0000	2.01e-008	0.0294
2	1.2214027581601699	1.2214027581601699	0.0000	1.27e-008	0.0396
3	1.3498588075760032	1.3498588075760032	0.0000	1.36e-007	0.0463
4	1.4918246976412703	1.4918246976412703	0.0000	5.25e-007	0.0568
5	1.6487212707001282	1.6487212707001282	0.0000	2.29e-006	0.0664
6	1.8221188003905091	1.8221188003905091	0.0000	3.98e-006	0.0667
7	2.0137527074704766	2.0137527074704766	0.0000	1.59e-005	0.0668
8	2.2255409284924679	2.2255409284924679	0.0000	7.76E-004	0.0670
9	2.4596031111569499	2.4596031111569499	0.0000	3.67E-004	0.0671
10	2.7182818284590455	2.7182818284590455	0.0000	5.65E-004	0.0672

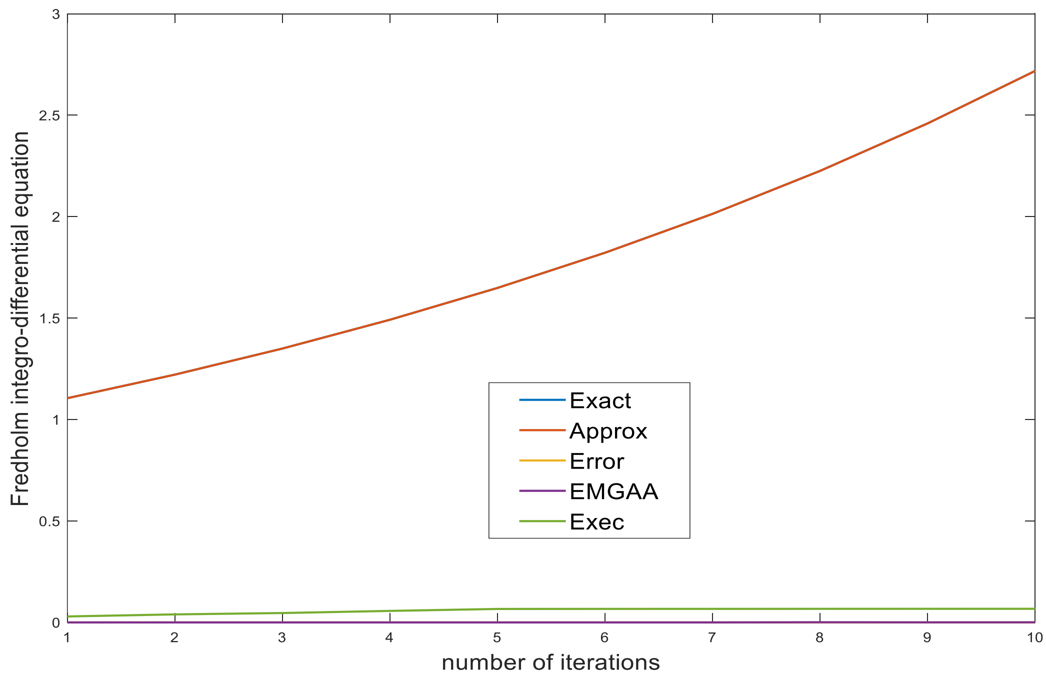

Figure 2: graphical representation of the Fredholm integro-differential equation (problem 2)

Problem 3:

Consider the model Fredholm integro-differential equation,

$$y''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} xty^2(t)dt, \qquad -1 \le x \le 1 \qquad 30$$

subject to the initial conditions,

$$y(0) = 1, \; y'(0) = 0 \qquad 31$$

The exact solution to the problem is given by $y(x) = 1 + 5x^2 - x^3$. See: Taiwo and Gegele (2014)
On the application of the newly derived FDS (23) on Problem 3 we obtain the result presented in Table 3 below.

Table 3: Showing the result for Problem 3

X	Exact	Approx.	Error	ETG	Exec
0.1	1.0490000000000002	1.0490000000000002	0.0000	6.008E-008	0.0124
0.2	1.1919999999999999	1.1919999999999999	0.0000	7.918E-008	0.0218
0.3	1.4230000000000003	1.4230000000000003	0.0000	8.432E-007	0.0312
0.4	1.7360000000000002	1.7360000000000002	0.0000	6.884E-007	0.0316
0.5	2.1250000000000000	2.1250000000000000	0.0000	5.718E-007	0.0320
0.6	2.5840000000000005	2.5840000000000005	0.0000	5.623E-007	0.0321
0.7	3.1070000000000007	3.1070000000000007	0.0000	4.009E-007	0.0322
0.8	3.6880000000000011	3.6880000000000011	0.0000	2.929E-007	0.0323
0.9	4.3210000000000006	4.3210000000000006	0.0000	2.887E-007	0.0325
1.0	5.0000000000000000	5.0000000000000000	0.0000	1.999E-007	0.0326

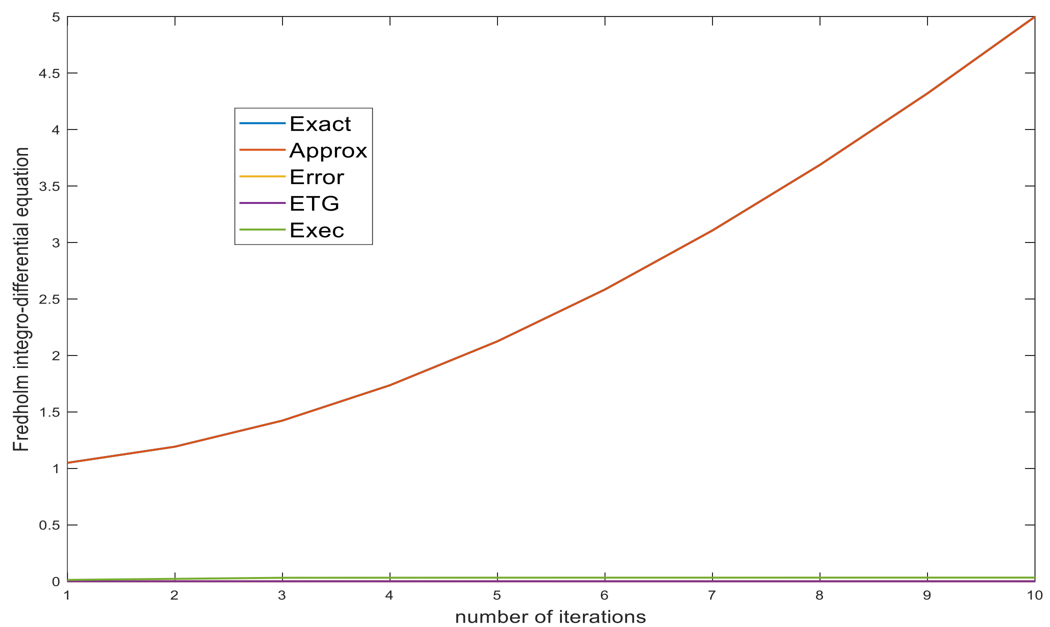

figure 3: graphical representation of the Fredholm integro-differential equation (problem 3)

Problem 4:

Consider the model Fredholm integro-differential equation,

$$y''(x) = \frac{5}{3} - 11x + \int_0^1 y(t)dt, \qquad 0 \le x \le 1 \qquad 32$$

subject to the initial conditions,

$$y(0) = y'(0) = 1 \qquad 33$$

The exact solution to the problem is given by $y(x) = 1 + x + \frac{5}{6}x^2 - \frac{5}{3}x^3$. See: Taiwo and Gegele (2014)

On the application of the newly derived EFDS (23) on Problem 4 we obtain the result presented in Table 4 below.

Table 4: Showing the result for Problem 4

X	Exact	Approx.	Error	ETG	Exec
0.1	1.1066666666666667	1.1066666666666667	0.0000	3.489E-006	0.0116
0.2	1.2200000000000000	1.2200000000000000	0.0000	3.410E-006	0.0231
0.3	1.3300000000000001	1.3300000000000001	0.0000	2.983E-006	0.0236
0.4	1.4266666666666665	1.4266666666666665	0.0000	2.837E-006	0.0237
0.5	1.5000000000000000	1.5000000000000000	0.0000	2.602E-006	0.0241
0.6	1.5400000000000000	1.5400000000000000	0.0000	2.591E-006	0.0242
0.7	1.5366666666666666	1.5366666666666666	0.0000	2.429E-006	0.0243
0.8	1.4800000000000000	1.4800000000000000	0.0000	1.994E-006	0.0244
0.9	1.3599999999999999	1.3599999999999999	0.0000	1.405E-006	0.0245
1.0	1.1666666666666667	1.1666666666666667	0.0000	1.067E-008	0.0247

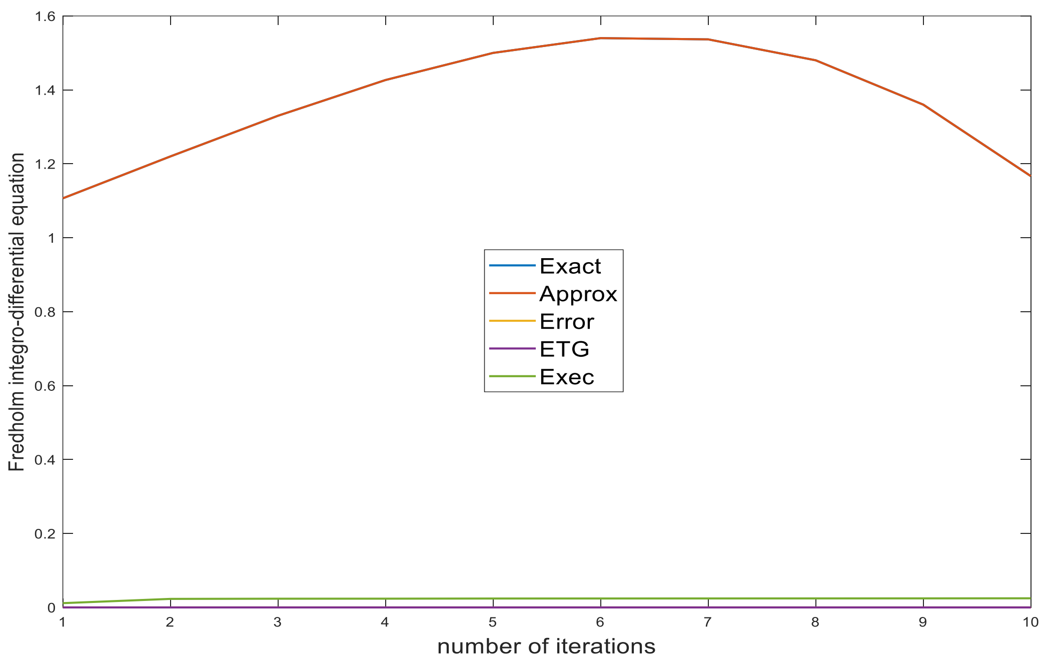

figure 4: graphical representation of the Fredholm integro-differential equation (problem 4)

From the results generated in Tables 1-4, it is clear that the Finite Difference Scheme in equation (23) is computationally reliable and efficient. This is because the computed solution matches exactly with the exact solution for each of the problems. It is also obvious from the results that the method performed better than the ones with which we compared our results with. The method is also efficient because from the tables, the execution times per seconds are very small. This shows that the method generates results very fast.

CONCLUSION

In this study, a new Finite Difference Scheme for solving second order Fredholm Integro-differential equations of the form (1) has been devised. The methods EGAA and ETG demonstrated weak numerical instability while the method FDS showed numerical stability during computation procedures.

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