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LINEAR PROGRAMMING PROBLEMS AS SKEW-SYMMETRIC AND NONSYMMETRIC GAMES: FORMULATION AND SIGNIFICANCE

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ABSTRACT

Game theory models competitive interactions, like military, political, or business conflicts, mathematically. Two-person zero-sum games, where one's gain equals another's loss, are central. Optimal strategies are often found by converting these games into linear programming (LP) problems solvable via methods like the simplex algorithm. However, converting LP problems into game formulations is less explored. This study bridges this gap by reformulating LPPs as skew-symmetric and nonsymmetric games. This approach benefits economic applications like comparative advantage and diet problems with positive constraints and prices. Skew-symmetric games' Super LPP offers computational efficiency due to sparse structures. By enabling bidirectional translation between games and LP, this work expands game theory's scope and enhances understanding of two-person zero-sum game properties, strategic frameworks, and their mathematical underpinnings.

1. INTRODUCTION

Game theory is the examination of strategic interactions employed to assess scenarios where two or more individuals act based on their individual self-interest; and the final results of such interactions are contingent on the choices made by each participant [1]. Most conventionally, these decision makers are called Players, and they engage in interactions within a framework known as the game. The players in the game are rational, meaning they actively strive to achieve particular objectives, taking into account their understanding or predictions of other players' behavior, and employing strategic thinking.

A game is defined by its players/decision makers, the rules, the resulting payoffs, the values assigned to these payoffs, and the variables controlled by each player. In a game, a player makes decisions independently. A player is not necessarily one person; it may be a group of individuals acting in an organization, a firm, or an army. The key characteristic of a player is their specific objective within the game, and that they act autonomously to pursue that objective [4].

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Games can be classified based on several criteria in game theory. Some common categories include cooperative versus non-cooperative games, zero-sum versus non-zero-sum games, simultaneous versus sequential games, perfect information versus imperfect information games, and so many others. These categorizations enable researchers and analysts to better study and comprehend the dynamics and outcomes of strategic interactions in different types of games. Notable among these classes is the zero-sum versus non-zero-sum Games: Zero-sum games have a constant total payoff, meaning any gain by one player is balanced by an equal loss by another player. Non-zero-sum games allow for situations where the total payoff can increase or decrease.
The two-person zero-sum game is characterized by the principle that one player's loss equals the other player's gain. The key features of this game can be illustrated using a payoff matrix. When both players have more than two operational strategies with no inferior strategies (i.e., the players must use each of their strategies in a given proportion) then we can determine the optimal mixed strategies of the game problem (i.e., the optimal proportion each strategy should be used) by converting it to a linear programming problem and solving the LP problem by the appropriate method [2].

There are abounding research reports in the literature on converting game problems into linear programming problems. Notably among these are the reports of Hillier and Lieberman [4], Taha [7], and Ekoko [2].

Hameed, Iman and Sahar [3] explored the effectiveness of Linear Programming (LP) and Genetic Algorithms (GAs) in solving game theory problems, particularly in the context of basketball strategy.

Olofinlade and Joshua [5] aimed to determine the optimal advertising strategies for telecommunications firms in Nigeria's telecommunications industry, like MTN, Airtel, Globacom, and 9mobile by applying game theory principles.

However, there appears to be limited research in the existing literature on the reverse process. In other words, few studies have focused on transforming linear programming problems (LPP) into game problems. This study, therefore, focuses on formulating an LPP as a game problem, with the primary objective of converting an LPP into both a skew-symmetric game and a nonsymmetric game. For the sake of clarity, and to focus on the research’s objective, we’ll limit the scope of this paper to the two-person zero-sum game.

METHOD

We seek to obtain the Skew-symmetric game from a linear programming problem.

First, we examine the following basic linear programming problem of the maximization form:

$$\left.\begin{array}{l} \text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ x_1, x_2, \dots, x_n \geq 0 \end{array}\right\} \quad (1)$$

Next, we derive the dual of the given LPP in system (1) by interchanging the rows and columns, reversing the inequality signs, and converting the objective from maximization to minimization, as demonstrated below;

$$\left.\begin{array}{l} \text{Minimize } z^* = b_1y_1 + by_2 + \cdots + b_my_m \\ \text{subject to} \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\ \qquad\qquad\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\ y_1, y_2, \ldots, y_m \geq 0 \end{array}\right\} \quad (2)$$

The next step is to combine the LPP in system (1) and its dual LPP in system (2) together to form a super LPP. To merge the two LPPs, they must be of the same type—either both should have a maximization objective function or both should follow a minimization objective function.

We shall first of all make sure both of them have a maximization objective function by We convert the minimization dual LPP in system (2) to maximization type by multiplying the objective function and constraints by -1, then, all the "≥" changes to "≤" . The dual LPP (2) in maximization objective function type is:

$$\left.\begin{array}{l} \text{Maximize } -z^* = -b_1y_1 - by_2 - \cdots - b_my_m \\ \text{subject to} \\ -a_{11}y_1 - a_{21}y_2 - \cdots - a_{m1}y_m \leq -c_1 \\ -a_{12}y_1 - a_{22}y_2 - \cdots - a_{m2}y_m \leq -c_2 \\ \qquad\qquad\vdots \\ -a_{1n}y_1 - a_{2n}y_2 - \cdots - a_{mn}y_m \leq -c_n \\ y_1, y_2, \ldots, y_m \geq 0 \end{array}\right\} \quad (3)$$

The combination of the LPP in system (1) and its dual LPP in system (3) produces a **super LPP** given in system (4) below:

$$\left.\begin{array}{l} \text{Maximize } z - z^* = -b_1y_1 - by_2 - \cdots - b_my_m + c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \\ 0y_1 + \cdots + 0y_m + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ 0y_1 + \cdots + 0y_m + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \qquad\qquad\vdots \\ 0y_1 + \cdots + 0y_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ -a_{11}y_1 - a_{21}y_2 - \cdots - a_{m1}y_m + 0x_1 + \cdots + 0x_n \leq -c_1 \\ -a_{12}y_1 - a_{22}y_2 - \cdots - a_{m2}y_m + 0x_1 + \cdots + 0x_n \leq -c_2 \\ \qquad\qquad\vdots \\ -a_{1n}y_1 - a_{2n}y_2 - \cdots - a_{mn}y_m + 0x_1 + \cdots + 0x_n \leq -c_n \\ y_1, y_2, \ldots, y_m \geq 0;\; x_1, x_2, \ldots, x_n \geq 0 \end{array}\right\} \quad (4)$$

Taking a look at (4), it is easy to note its skew symmetric (anti symmetric) form. The coefficients a_{ij}'s are present in two instances: once with a plus sign and once in transposed form a_{ji} with a minus sign. Similarly, both the b's and c's appear twice, once in the vertical and once in the horizontal, with opposite algebraic signs.

In a game, all columns share a similar structure, prompting the question: How can we manipulate the right-hand b_i's and coefficients c_j's of the programming problem to resemble ordinary a_{ij}'s? To address this, we multiply each coefficient and right-hand side values in (4) by a positive constant p. Since, the new optimal solution and objective will only differ by the multiple, p. Then, we move the right-hand coefficients of the constraints to the left-hand side, since games are usually written without any right-hand coefficients. We now have $n+m+1$ new variables, namely, $(py_1, \dots, py_m, px_1, \dots, px_n, p)$, the super problem, as given by (4) can be written as:

$$\left.\begin{array}{l}\text{Maximize } z - z^* = -b_1py_1 - \cdots - b_mpy_m + c_1px_1 + \cdots + c_npx_n + 0 \\ \text{subject to} \\ 0py_1 + \cdots + 0py_m + a_{11}px_1 + \cdots + a_{1n}px_n - pb_1 \le 0 \\ \vdots \\ 0py_1 + \cdots + 0py_m + a_{m1}px_1 + \cdots + a_{mn}px_n - pb_m \le 0 \\ -a_{11}py_1 - \cdots - a_{m1}py_m + 0px_1 + \cdots + 0px_n + pc_1 \le 0 \\ \vdots \\ -a_{1n}py_1 - \cdots - a_{mn}py_m + 0px_1 + \cdots + 0px_n + pc_n \le 0 \\ py_1, \dots, py_m \ge 0;\ px_1, \dots, px_n \ge 0, p > 0\end{array}\right\} \quad (5)$$

It is worthy of note that the duality theorem guarantees that the **optimal** objective value, z of the original LP problem and that of its dual, z^* must be exactly equal (i.e., $z - z^* = 0$). In other words, the optimal objective value of the super LPP, $z - z^*$ is zero. Since we know that $z - z^* = 0$, we replace the objective function by the equivalent constraint;

$$-b_1y_1 - by_2 - \cdots - b_my_m + c_1x_1 + c_2x_2 + \cdots + c_nx_n \ge \qquad (6)$$

The ">" sign has been intentionally included, even though it is redundant, with the acknowledgment that no feasible solution will necessitate its use. The system (5) becomes

$$\left.\begin{array}{l}0py_1 + \cdots + 0py_m + a_{11}px_1 + \cdots + a_{1n}px_n - pb_1 \le 0 \\ \vdots \\ 0py_1 + \cdots + 0py_m + a_{m1}px_1 + \cdots + a_{mn}px_n - pb_m \le 0 \\ -a_{11}py_1 - \cdots - a_{m1}py_m + 0px_1 + \cdots + 0px_n + pc_1 \le 0 \\ \vdots \\ -a_{1n}py_1 - \cdots - a_{mn}py_m + 0px_1 + \cdots + 0px_n + pc_n \le 0 \\ \boldsymbol{b_1y_1 + by_2 + \cdots + b_my_m - c_1x_1 - c_2x_2 - \cdots - c_nx_n \le 0} \\ py_1, \dots, py_m \ge 0;\ px_1, \dots, px_n \ge 0, p > 0\end{array}\right\} \quad (7)$$

Thus, the pay-off matrix of the skew-symmetric game is as follows:

Player **B**

$$\text{Player} \quad \begin{bmatrix} 0 & \dots & 0 & a_{11} & \dots & a_{1n} & -b_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & a_{m1} & \dots & a_{mn} & -b_m \\ -a_{11} & \dots & -a_{m1} & 0 & \dots & 0 & c_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{1n} & \dots & -a_{mn} & 0 & \dots & 0 & c_n \\ b_1 & \dots & b_m & -c_1 & \dots & -c_n & 0 \end{bmatrix} \qquad (8)$$

Next, we seek to obtain the nonsymmetric game from a linear programming problem. Again, we examine the following basic linear programming problem of the maximization form:

$$\left.\begin{array}{l}\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2\end{array}\right\} \quad (9)$$

$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$x_1, x_2, \dots, x_n \geq 0$; and
$$c_j \geq 0, b_i \geq 0, a_{ij} \geq 0$$

We can rewrite LPP (9) above by dividing each constraint i by b_i, and substituting each term in the objective with a new variable, u_j (i.e., $u_j = c_j x_j$). Hence, each term on the L.H.S becomes:

$$\frac{a_{ij}x_j}{b_i} \tag{10}$$

and $\frac{a_{ij}x_j}{b_i} = \frac{a_{ij}u_j}{b_i c_j}$ (since $x_j = \frac{u_j}{c_j}$). Hence, every term $\frac{a_{ij}x_j}{b_i} = \frac{a_{ij}u_j}{b_i c_j} = A_{ij}u_j$ (where $A_{ij} = \frac{a_{ij}}{b_i c_j}$). Therefore, LPP (9) above can be written as

$$\left.\begin{array}{l} \text{Maximize } z = u_1 + u_2 + \cdots + u_n \\ \text{subject to} \\ A_{11}u_1 + A_{12}u_2 + \cdots + A_{1n}u_n \leq 1 \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2n}u_n \leq 1 \\ \qquad\vdots \\ A_{m1}u_1 + A_{m2}u_2 + \cdots + A_{mn}u_n \leq 1 \\ u_1, u_2, \dots, u_n \geq 0 \end{array}\right\} \tag{11}$$

The nonsymmetric game in general form is thus

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \tag{12}$$

RESULTS

i. Consider the following linear programming problem:

Maximize $z = 5x_1 + 2x_2$
subject to
$10x_1 - 6x_2 \leq 10$
$2x_1 + x_2 \leq 20$
$-15x_1 + 20x_2 \leq 300$
$x_1, x_2 \geq 0$

The dual of the LPP is given as

Minimize $z^* = 10y_1 + 20y_2 + 300y_3$
subject to
$10y_1 + 2y_2 - 15y_3 \geq 5$
$-6y_1 + y_2 + 20y_3 \geq 2$
$y_1, y_2, y_3 \geq 0$

As discussed, the next step in the conversion process is to convert this LPP to a maximization problem

Maximize $-z^* = -10y_1 - 20y_2 - 300y_3$

subject to
$-10y_1 - 2y_2 + 15y_3 \leq -5$
$6y_1 - y_2 - 20y_3 \leq -2$
$y_1, y_2, y_3 \geq 0$

The super LPP formed by combining this LPP and the original LPP (given on this illustration) is given as

Maximize $z - z^* = 5x_1 + 2x_2 - 10y_1 - 20y_2 - 300y_3$
subject to
$10x_1 - 6x_2 + 0y_1 + 0y_2 + 0y_3 \leq 10$
$2x_1 + x_2 + 0y_1 + 0y_2 + 0y_3 \leq 20$
$-15x_1 + 20x_2 + 0y_1 + 0y_2 + 0y_3 \leq 300$
$0x_1 + 0x_2 - 10y_1 - 2y_2 + 15y_3 \leq -5$
$0x_1 + 0x_2 + 6y_1 - y_2 - 20y_3 \leq -2$
$x_1, x_2, y_1, y_2, y_3 \geq 0$

We can re-assign new variable letters to the x_j's and y_j's for the sake of convenience and easy computation as follows: let $x_1 = t_1$, $x_2 = t_2$, $y_1 = t_3$, $y_2 = t_4$, $y_3 = t_5$ and $w = z - z^*$.

Therefore, the LPP can be rewritten as:

Maximize $w = 5t_1 + 2t_2 - 10t_3 - 20t_4 - 300t_5$

subject to
$10t_1 - 6t_2 + 0t_3 + 0t_4 + 0t_5 \leq 10$
$2t_1 + t_2 + 0t_3 + 0t_4 + 0t_5 \leq 20$
$-15t_1 + 20t_2 + 0t_3 + 0t_4 + 0t_5 \leq 300$
$0t_1 + 0t_2 - 10t_3 - 2t_4 + 15t_5 \leq -5$
$0t_1 + 0t_2 + 6t_3 - t_4 - 20t_5 \leq -2$
$t_1, t_2, t_3, t_4, t_5 \geq 0$

The skew-symmetric game pay-off matrix (in compact form) with payoff from player B to player A is as follows:

Player **B**

$$\text{Player } \mathbf{A} \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Where $A = \begin{bmatrix} -10 & -2 & 15 \\ 6 & -1 & -20 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$, $c = \begin{bmatrix} -10 \\ -20 \\ -300 \end{bmatrix}$

ii. We consider an LPP with the 3 special properties as highlighted in the section:

Maximize $z = x_1 + x_2$
subject to

$x_1 + 5x_2 \leq 5$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

To convert the above LPP to a nonsymmetric game; first, let each term, $c_j x_j$ in the objective function be u_j(i.e., let $u_j = c_j x_j$), then, divide each constraint i by b_i so that each term on the LHS of constraint i becomes: $\frac{a_{ij}x_j}{b_i}$.

Since $x_j = \frac{u_j}{c_j}$, $\frac{a_{ij}x_j}{b_i}$ becomes $\frac{a_{ij}u_j}{b_i c_j} = A_{ij}u_j$
where $A_{ij} = \frac{a_{ij}}{b_i c_j}$

Now, for the LPP given above, $A_{11} = \frac{1}{5(1)} = 0.2$, $A_{12} = \frac{5}{5(1)} = 1$, $A_{21} = \frac{2}{4(1)} = 0.5$, $A_{22} = \frac{1}{4(1)} = 0.25$

Therefore, the nonsymmetric game payoff matrix corresponding to the given LPP is:

$$\begin{bmatrix} 0.2 & 1 \\ 0.5 & 0.25 \end{bmatrix} \tag{14}$$

Note to the reader: The selection of the LPP for conversion to the nonsymmetric game was intentional and carefully considered. Our research has shown that, for the conversion to result in a nonsymmetric payoff matrix, the following condition must be met: the set of right-hand values of the linear constraints, b_i, the set of coefficients, c_j of the decision variables and the set of coefficients, a_{ij} in the linear constraints must each be of the same algebraic sign. This condition can be verified computationally.

CONCLUSION

We attempted and succeeded in making the conversion of an LPP to a skew-symmetric game, we first had to obtain a super LPP (which is skew-symmetric). Upon examination, the first appearance of our obtained super LPP did not fit the usual form of an LPP transformed from a two-person zero-sum game. The RHS of each of its constraints had different constant values, as opposed to just one constant value. And we could not just move the RHS values to the left since all the terms on the left must be an algebraic term (i.e., a variable with a coefficient). To correct this, we multiplied the constraints and the objective function with a positive constant, p. This enabled us to move the RHS to the left. The values of the RHS which were different, became one constant value i.e., zero. The process made it possible to identify the skew-symmetric game matrix. We also succeeded in converting an LPP to a nonsymmetric game.

In conclusion, we found that LP problems can be converted to skew-symmetric and nonsymmetric games. From published literature, the conversion of a linear programming problem to a symmetric game has rarely or scarcely been done. But in this research, the success achieved in converting a LP problem to a skew-symmetric game and also a nonsymmetric game problem adds to the scanty literature published on it.

Specifically, the conversion to nonsymmetric game problems has shown relevance in specific economic scenarios, such as in the comparative advantage problem and the minimum diet problem, where amounts of all limited resources and all prices are considered positive. Since, they meet the criteria of the LPP conversion to Nonsymmetric game, these problems can be studied as a nonsymmetric game problem.

Lastly, the ability to and the subsequent process of converting LPP to game availed us the benefit of decrypting some of the mathematical properties of the game, and understanding the structure of its corresponding LPP (for instance, during the conversion to Skew symmetric game, we encountered the Super LPP and its special sparse structure, which requires minimal computational resources and therefore, easy for a computer to solve. As a result of this sparse property, the computation of the solution to a skew symmetric game is highly efficient.)
It also enables us clearly see and understand the framework of these games (for instance, we can clearly see from the super LPP associated with the skew symmetric game, that the players have exactly, opposing strategies: given the LPP special structure, of zero diagonal and the upper diagonal elements negative of the lower diagonal.)

Expanding on these findings, future research could investigate the transformation of linear programming problems into non-zero-sum games and explore potential practical applications arising from any unique structural properties of the given linear programming problem.

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