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APPLICATION AND COMPUTATIONAL SIMULATIONS OF NONLINEAR DUFFING OSCILLATOR

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ABSTRACT

A duffing oscillator occurs due to the motion of a body being subjected to a nonlinear power, linear sticky damping, and periodic forcing. It reveals the oscillations of mechanical systems under the action of a periodic external force. This work studies the application of Duffing oscillators, especially in damping and chaos theory, and also develops an alternative computational method for simulating the Duffing equation. By applying the new homotopy perturbation method and computational method, the findings of this study extract key elements into a model to make it predictive and interpretative. The model is a system with one variable x. $\ddot{x}$ *is the inertia or the second time derivative of displacement,* ∂, β, α *are parameters,* $\dot{x}$ *is a small damping, The numerical simulation shows the phase plots and system time series.*

1. INTRODUCTION

Duffing oscillator is a nonlinear second-order differential equation and was named after a German Electrical Engineer called Georg Duffing (1861- 1944) in the year 1918. It recently received attention due to the variety of their applications in engineering. For example, magneto-elastic mechanical systems [15;1;3] nonlinear vibration of beams and plates and fluid flow-induced vibration [17] which the nonlinear Duffing equation can follow. The equation explains the motion of a damped Oscillator with a more complicated potential than in simple harmonic motion [14]. The Duffing oscillator models the behaviour of many practical problems arising in engineering, physics, and many real-world applications.

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Duffing oscillators with third and fifth power nonlinear terms can be found in the modelling of free vibrations of a restrained uniform beam with intermediate lumped mass, the nonlinear dynamics of slender elastic, the generalized pochhammer- chree (PC) equation, the generalized compound Kdv equation in nonlinear wave systems, among others [2].

Sunday [16] reported in his work that the Duffing oscillator occurs due to the motion of a body being subjected to a nonlinear power, linear sticky damped and periodic powering. It reveals the oscillations of mechanical systems under the action of a periodic external force. The problem of a nonlinear oscillator was first systematically tackled by Georg Duffing by examining the effects of quadratic and cubic stiffness nonlinearities [11]. Meiss [13] concluded that a dynamical system consists of an abstract phase space, whose coordinates describe the state at any instant and a dynamical rule that specifies the immediate future of all state variables given only the present values of those same state variables. Many mechanical systems involve nonlinearity of the Duffing type [12], Cveticanin [5] literature shows that the Duffing Oscillator is a dynamical system that exhibits chaotic behaviour, but the most recent advances in nonlinear science and theoretical physics have focused on the development of efficient methods, these methods include: homotopy perturbation method [18;7]. frequency–amplitude formulation [9], energy balance method [6;7], coupled homotopy-variational approach [10].

Some researchers considered damping when the amplitude of oscillation reduces over time, most analytical methods are unable to handle non-conservative oscillators. However, the new homotopy perturbation method will be introduced to solve nonlinear and nonhomogeneous differential equations that can solve the non-conservative Duffing oscillator problem. Also, this work studies the application of Duffing oscillators most especially in damping and chaos theory and similarly develops an alternative computational method for simulating the Duffing equation.

2. Methodology

The basic idea of the New Homotopy Perturbation method (NHPM)

The NHPM presented in [4] was used to solve the nonlinear Duffing oscillator equation. This method obtained the truncated series solution that coincides with the Maclaurin expansion of the exact solution. To improve the accuracy of the series solutions, the Laplace transformation and Pade approximant were used to produce the analytical approximate solution with high accuracy. Finally, the inverse Laplace transformation was used to get the exact analytical solution.

$$H(U(x), P) = L(U(x)) - u_0(x) + pu_0(x) + p[N(U(x)) - f(r(x))] = 0, \tag{1}$$

By applying G^{-1} to both sides of the equation, we have

$$U(x) = T(x) + G^{-1}(u_0(x)) + p\left[G^{-1}(f(r(x))) - G^{-1}(u_o(x)) - G^{-1}(N(U(x)))\right] \tag{2}$$

Where T incorporates the constants of integration and satisfies GT = 0.

Applying the NHPM,

$$u_0(x) = \sum_{n=0}^{\infty} a_n F_n(x), \tag{3}$$

Where $a_o, a_1, a_2 \cdots$ are unknown coefficients, and $F_0(x), F_1(x), F_2(x) \ldots$ are specific functions,

$$\sum_{n=0}^{\infty} p^n U_n(x) = T(x) + G^{-1}\left(\sum_{n=o}^{\infty} a_n F_n(x) + p\left[G^{-1}(f(r(x)) - G^{-1}\left(\sum_{n=0}^{\infty} a_n F_n(x)\right) - G^{-1}N\left(\sum_{n=0}^{\infty} p^n U_n(x)\right)\right]\right). \tag{4}$$

Comparing coefficients of terms with identical powers of p leads to

$$p^0 : U_0(x) = T(x) + G^{-1}\left(\sum_{n=0}^{\infty} a_n F_n(x)\right), \tag{5}$$

$$p1 : U_1(x) = G^{-1}(f(r(x))) - G^{-1}\left(\sum_{n=0}^{\infty} a_n F_n(x) - G^{-1}N(U_0(x)),\right) \tag{6}$$

$$P^2 : U_2(x) = -G^{-1}N(U_0(x), U_1(x)). \tag{7}$$

$$p^j : U_J(x) = -G^{-1}N\left(U_0(x), U_1(x), U_2(x) \ldots . U_{j-1}(x)\right), \tag{8}$$

Solving the equation in such a way that $U_1(x) = 0$

The Exact solution may be obtained as follows

$$u(x) = U_0(x) = T(x) + G^{-1}(\sum_{n=0}^{\infty} a_n F_n(x)) \tag{9}$$

Solving Nonlinear Duffing Equation

$$\ddot{x}(t) + \dot{x}(t) + \beta(t) + \alpha^3(t) = \gamma(\cos^3(\omega t) - \sin t), \tag{10}$$

With initial condition $x(0)$ and $\dot{x}(0) = 1$ and the exact solution $x(t) = \cos x$.

$$\ddot{x}(t) = x_0(t) - A\left[x_0(t) + \dot{x}(t) + \beta(t) + \alpha^3(t) - \gamma(\cos^3 \omega t + \sin t)\right] \tag{11}$$

Denoting $\frac{\partial^2}{\partial x^2}$ by L, and L^{-1} as two-fold integration, using the operator L, it becomes

$$Lx(t) = x_0(t) - A\left[x_0(t) + \dot{x}(t) + \beta(t) + \alpha^3(t) - \gamma(\cos^3(\omega t) + \sin t)\right]. \tag{12}$$

Applying the inverse operator L^{-1} to both sides of equation and using the initial conditions,

$$x(t) = 1 + L^{-1}[x_0(t)] - AL^{-1}\left[(x_0(t) + \dot{x}(t) + \beta(t) + \alpha^3(x) - \gamma(\cos^3(\omega t) + \sin t\right], \tag{13}$$

By replacing $x_0(t) = \sum_{n=0}^{\infty} a_n t^n$ in the above equation, we get;

$$x(t) = 1 + L^{-1}\left(\sum_{n=0}^{\infty} a_n t^n\right) - AL^{-1}\left[\left(\sum_{n=0}^{\infty} a_n t^n + \dot{x}(t) + \beta(t) + \alpha^3(t) - \gamma(\cos^3 \omega t + \sin t)\right)\right] \tag{14}$$

Substituting $x(t)=\sum_{i=0}^{\infty}A^{i}x_{i}(t)$ into the equation above and considering the Maclaurin series of the excitation term,

$$\text{Cos } \omega t \;;\; f(t)=\sum_{n=0}^{\infty}(-1)^{n}\frac{t^{2n}}{(2n)!} \tag{15}$$

$$f(t)=1-\frac{(\omega t)^{2}}{2!}+\frac{(\omega t)^{4}}{4!},\ldots\rightarrow\sum_{n=0}^{\infty}(-1)^{n}\frac{(\omega t)^{2n}}{(2n)!} \tag{16}$$

For $\sin(t)$, the Maclaurin series expansion of $\sin(t)$ is derived as

$$f(t)=\frac{1}{1!}t^{1}+\frac{-1}{3!}t^{3}+\frac{1}{5!}t^{5}+\ldots\Rightarrow\sum_{n=0}^{\infty}(-1)\frac{t^{2n+1}}{(2n+1)!} \tag{17}$$

Where the series for $\cos(3\omega t)$ simply follows from $\cos(t)$ series after substitution $x\rightarrow 3t$

$$\cos 3\omega t=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n)!}(3\omega t)^{2n}\ldots\ldots\ldots\Rightarrow\sum_{n=0}^{\infty}\frac{(-1)^{n}9^{n}}{(2n)!}\omega t^{2n}$$

$$\cos^{3}(\omega t)=\frac{3}{4}\cos(\omega t)+\frac{1}{4}\cos(3\omega t)$$

$$=\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)\omega t^{2n}}{(2n)!}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n}9^{n}(\omega t)^{2n}}{(2n)!}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{4}\frac{(3+9^{1})(-1)^{1}}{(2(1))!}\right)\omega t^{2n} \tag{18}$$

$$\cos^{3}\omega t-\sin t=1-t-\frac{3}{2\omega t^{2}}+\frac{1}{6\omega t^{3}}+\frac{7}{8\omega t^{4}}-\frac{1}{120\omega t^{5}} \tag{19}$$

And equating the terms with identical power of A gives;

$$A^{0}:\; x_{0}(t)=1+L^{-1}\sum_{n=0}^{\infty}a_{n}t^{n}$$

$$A^{1}:x_{1}(t)=-L^{-1}\left[\sum_{n=0}^{\infty}a_{n}t^{n}+\dot{x}_{0}(t)+\beta(t)+\alpha_{0}{}^{3}(t)-\left(1-t-\frac{3}{2\omega t^{2}}+\frac{1}{6\omega t^{3}}+\frac{7}{8\omega t^{4}}-\frac{1}{120\omega t^{5}}\right)\right]$$

$$A^{2}:\; x_{2}(t)=-L^{-1}\left[-x_{1}^{1}(t)+\beta_{1}(t)+3x_{0}{}^{2}(t)x_{1}(t)\right] \tag{20}$$

Solving the above equation for $y_{1}(x)$ leads to the result

$$x_1(t)=\frac{-1}{2}(1+a_0)t^2-\frac{1}{6}(1+a_0+a_1)t^3-\frac{1}{24}(3+4a_0+a_0+a_1+2a_2)t^4+\frac{1}{120}(1-4a_1-2a_2-6a_3)t^5+...$$

$$x(t)=\ x_0(t)=1\text{-}\frac{\omega t^2}{2!}+\frac{\omega t^4}{4!}+...+\frac{-1^n\,\omega t^{2n}}{2n!},n\rightarrow 0 \qquad (21)$$

Which is the partial sum of the Taylor series of the exact solution at $x=0$.

RESULTS AND DISCUSSION

Phase portraits

Figures 1-3 are the results of computational simulations of the nonlinear Duffing oscillator.

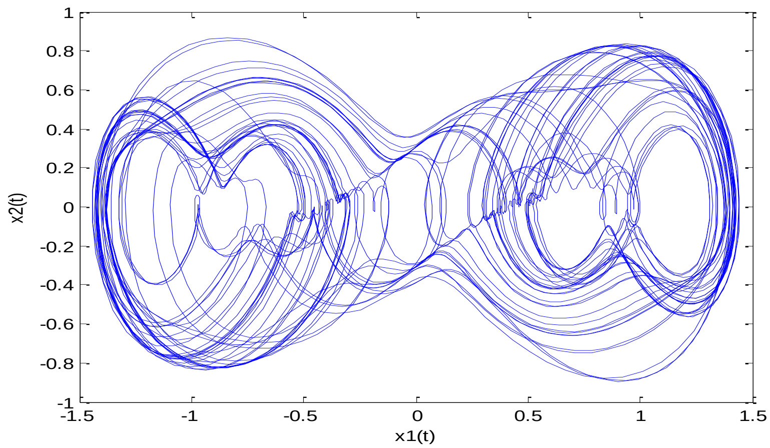

Figure 1: Depicts the phase plane solution to the nonlinear Duffing oscillator model for $\gamma=0.42$ and the range of time values $t\in[0,1000]$, with initial condition x1 = x2 =0.

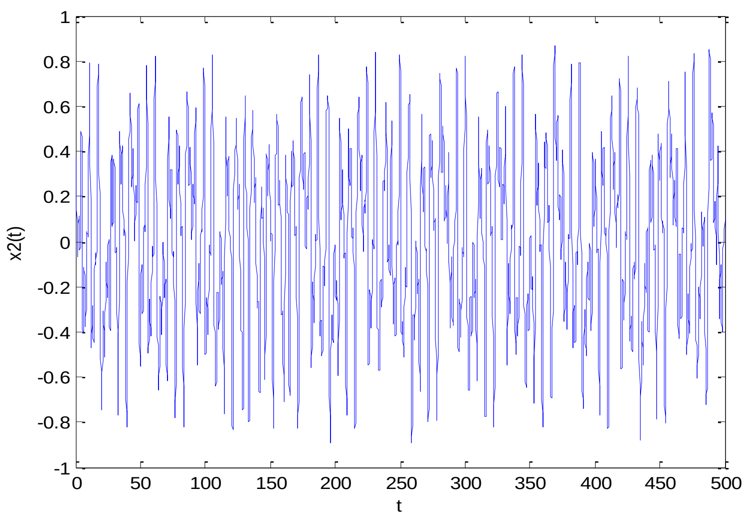

Figure 2: Depicts the time series solution to nonlinear Duffing oscillator model for $\gamma=0.42$

and the range of time values $t \in [0,500]$, with initial condition x1 = x2 = 0 and the range of time values $t \in [0,500]$, with initial condition x1 = x2 = 0.

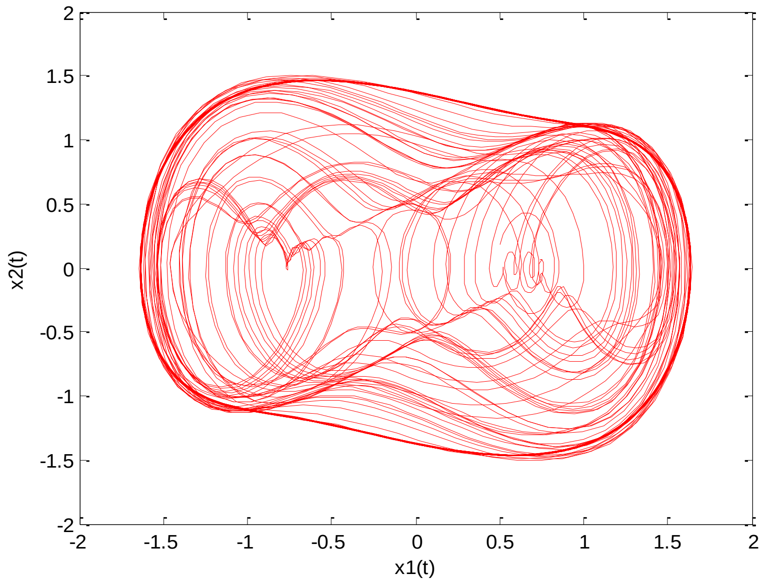

Figure 3: Depicts the time series solution to the nonlinear Duffing oscillator model for $\gamma = 0.70$ and the range of time values $t \in [0,1000]$, with initial condition x1 = x2= 0.

Table 1 Comparison of the present method with the NHPM solution, power series solution, and computed solution.

X_a	NHPM solution	PSM solution	Computed solution
0.1000000000	0.9950041653	0.9950041653	0.9950041653
0.2000000000	0.9800665778	0.9800665778	0.9800665778
0.3000000000	0.9553364891	0.9553364891	0.9553364891
0.4000000000	0.9210609940	0.9210609940	0.9210609940
0.5000000000	0.8775825622	0.8775825618	0.8775825618
0.6000000000	0.8253356166	0.8253356149	0.8253356148
0.7000000000	0.7648421950	0.7648421872	0.7648421872
0.8000000000	0.6967067388	0.6967067093	0.6967067092
0.9000000000	0.6216100638	0.6216099683	0.6216099681
1.0000000000	0.5403025794	0.5403023059	0.5403023057

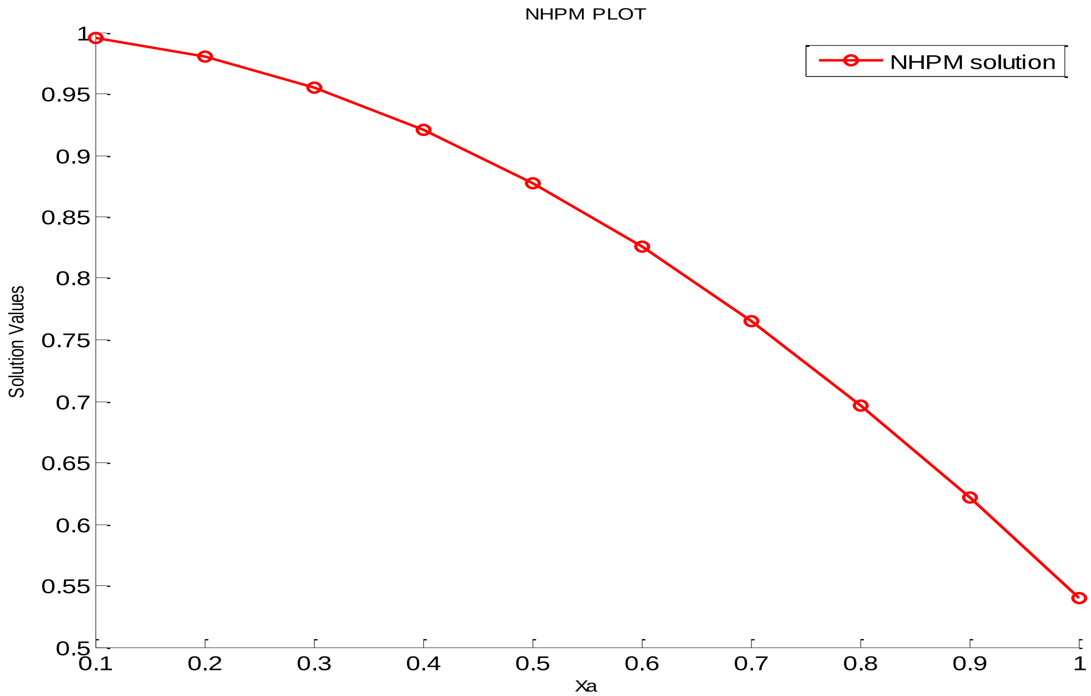

Figure 4: Depicts the graph of NHPM solution

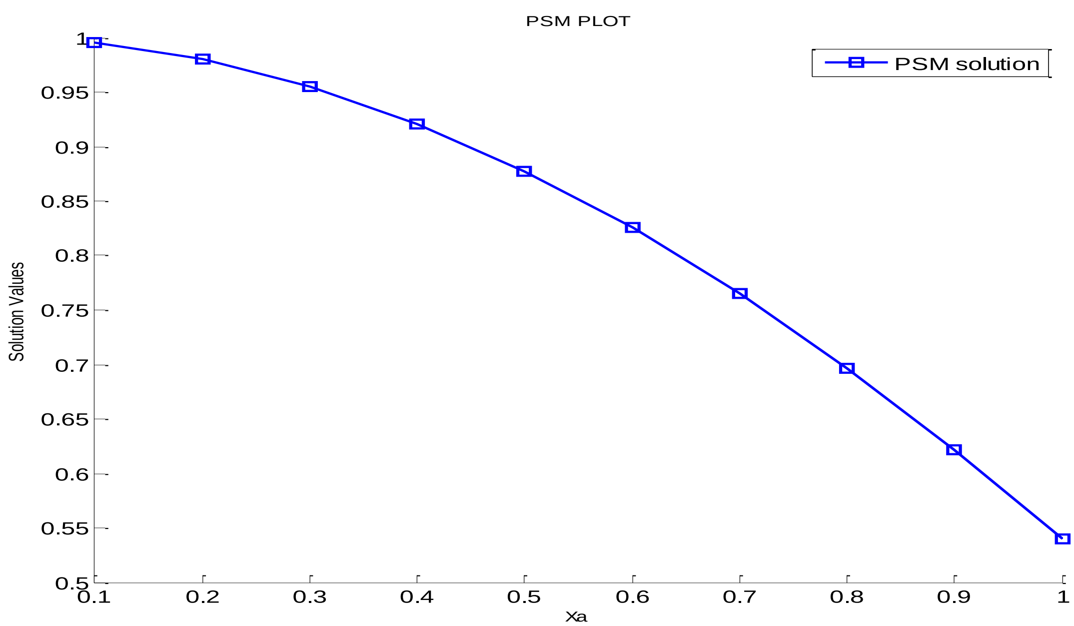

Figure 5: Depicts the graph of PSM solution

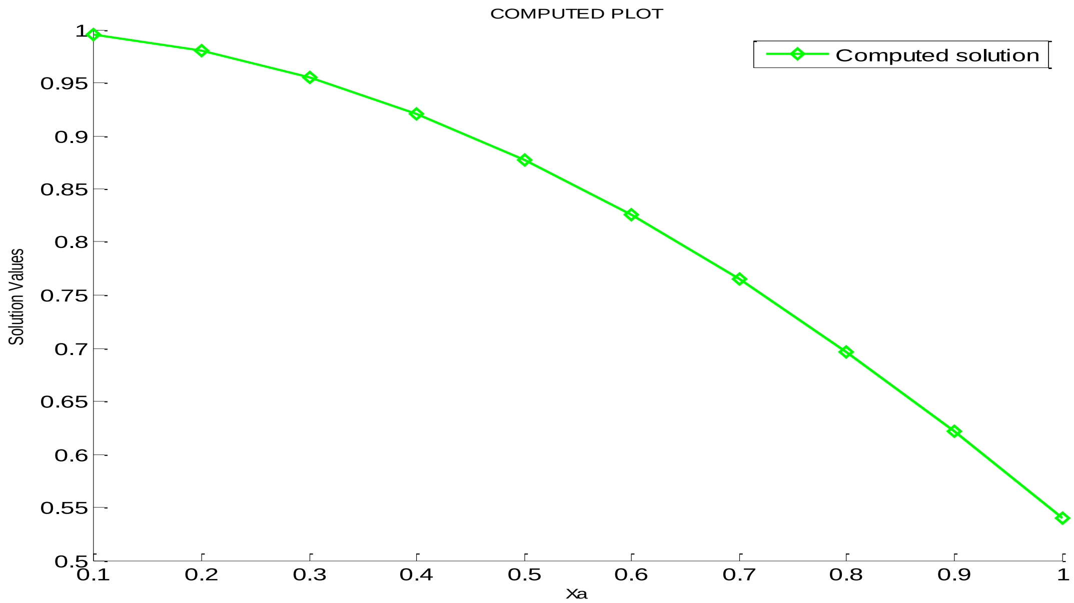

Figure 6: Depicts the graph of the COMPUTED solution

DISCUSSION

The investigation has been done to solve nonlinear problems with an analytical method which is called the New Homotopy Perturbation method to get a better solution in comparison with the Power Series method, The New Homotopy Perturbation Method and Computed procedure have been done differently by assuming different trial function as the solution of the problem which presents simplicity and acceptable accuracy of this method among other analytical methods. From Table 1, it is observed that the solution of the method used corresponds to the Computed solution and PSM solution with slight error as plotted in Figures 4, 5 and 6.

CONCLUSION

The New Homotopy Perturbation Method (NHPM) was used in this study to solve the analytical solution of the nonlinear Duffing oscillator equation and compare its results with the Power Series Method (PSM) and numerical method (Runge-Kutta, R_4). The phase plots, time series graph and the line graph have also been shown in the results. The Figures and the related Tables show that it is logical to say that NHPM is a very applicable and suitable approach for solving nonlinear differential equations with enough efficiency and acceptable accuracy in generating the exact solution.

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