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COLLOCATION TECHNIQUE FOR THE NUMERICAL SOLUTION OF THREE DIMENSIONAL VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS BY ORTHOGONAL BASIS FUNCTION

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ABSTRACT

In this paper, standard collocation approximation method is proposed for solving three-dimensional integral equations. New orthogonal polynomials were constructed and used as basis function that gives less rigorous works in terms of computational efforts and enhanced accuracy, Power series and Legendre Polynomials were used as alternative basis functions to checkmate the results of the new Orthogonal Polynomials constructed and used; therefore, three variants of basis functions were used in this work. The proposed methods changed the three-dimensional Volterra and Fredholm integral equations into algebraic system of equations with some unknown constants that were obtained by using MAPLE 18 software. Some numerical examples were solved and less computational works were achieved and the results obtained were in good agreements using three variants with the results available in the literature using different approaches.

1 INTRODUCTION

Three dimensional Volterra and Fredholm integral equations reformulated many varied problems in engineering and physics. It is being utilized as mathematical models for many different science applications such as plasma physics. So, it very important to get some information about the analytical solutions of these problems because these solutions give significant information about the character of the modeled event. But in some cases, it is more difficult to obtain analytical solutions of these models. These are usually difficult to solve analytically and in many cases their solutions must be approximated. To approximate the solutions of these models, in recent years several numerical approaches have been proposed.

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In this paper, we consider the three-dimensional Volterra and Fredholm integral equations of the form:

$$u(x,t,z) = f(x,t,z) + \int_0^x \int_0^t \int_0^z k(x,t,z,h,w,l)u(h,w,l)dhdwdl \tag{1}$$

and The functions $f(x,t,z)$ and $k(x,t,z,h,w,l)$ were assumed to be given smooth real valued functions on $(x,t,z) \in [a,b] \times [c,d] \times [e,f]$ and $d = (x,t,z,h,w,l); a \leq h \leq x \leq b,\ c \leq w \leq t \leq d$ and $e \leq l \leq z \leq f$ respectively and $u(x,t,z)$ is the unknown solution to be determined.

Three-dimensional Volterra and Fredholm integral equations often pose significant analytical challenges due to their complex nature, especially when involving nonlinearities or fractional orders. As a result, obtaining exact or closed-form solutions is rarely feasible. This has led to the development and application of various numerical methods to approximate their solutions with acceptable accuracy. Such equations arise frequently in mathematical modeling of physical, engineering, and biological systems, where multi-dimensional dependencies and memory effects must be considered. Numerous researchers have contributed to the numerical treatment of these integral equations. For instance, [1] introduced numerical methods for solving linear Fredholm-Volterra integral equations, which combine characteristics of both Fredholm and Volterra types. In another approach, [2] employed the moving least squares (MLS) method in conjunction with Chebyshev polynomials to address Volterra-Fredholm integral equations, enhancing computational efficiency and approximation accuracy. Furthermore, [3] focused on the numerical solution of nonlinear Volterra-Fredholm equations, emphasizing iterative and projection-based methods for improved convergence. More recently, [4] discussed the construction and application of orthogonal polynomials in solving fractional-order integro-Volterra-Fredholm differential equations, offering a framework to tackle the added complexity introduced by fractional derivatives. These studies highlight the growing importance of robust numerical techniques in addressing multi-dimensional integral equations in modern scientific and engineering problems.

The meshless methods have gained more attention, particularly moving least squares method, it has been applied in many branches of modern sciences, such as surface construction [5-6]. Recently, numerous approaches have been used to solve nonlinear computation of the three-dimensional Volterra and Fredholm integral equations such as [7] Beinstein approximation, [8] presented muchled homotopy perturbation method for solving the nonlinear mixed Volterra-Fredholm Integral equations, [9] proposed iterative method and convergence analysis for a kind of mixed non-linear Volterra-Fredholm Integral equations, [10] applied of Homotopy analysis method for solving a class of non-linear Volterra-Fredholm Integro-differential, [11] employed the petrov-Gerlerkin Method for numerical solution of stochastic Volterra Integral equations, and [12] applied shifted Chebyshev polynomials for solving three dimensional Volterra Integral equations of the second kind.

The aim of this paper is to develop an effective and efficient collocation schemes to solve three dimensional Volterra and Fredholm integral equations. One great advantage of our schemes is that it reduces the problems under consideration to system of algebraic equations by using the constructed orthogonal polynomials as basis function. However, in this paper, orthogonal collocation techniques were used as basis function to obtain the numerical solution of linear three-dimensional Volterra and Fredholm integral and the zeros of the constructed orthogonal polynomials were used as the collocation points. Thereafter, the results obtained were compared with the analytic solution where such exists.

Description of the numerical techniques
General problems considered

In this section, two types of three-dimensional integral equations are considered.
Type 1: The three-dimensional Volterra integral equation of the form;

$$u(x,t,z) = f(x,t,z) + \int_0^x \int_0^t \int_0^z k(x,t,z,h,w,l)u(h,w,l)dhdwdl \tag{3}$$

where $(x,t,z) \in D = \{O,X\} \times \{O,T\} \times \{O,Z\}$ and $u(x,t,z)$ is unknown function to be determined; $f(x,t,z)$ and $k(x,t,z,h,w,l)$ are given functions defined respectively on D.

Type 2: The three-dimensional Fredholm integral equation of the form;

$$u(x,t,z) = f(x,t,z) + \int_a^b \int_c^d \int_e^f k(x,t,z,h,w,t)\, u(h,w,l)dhdwdl \tag{4}$$

Where $(x,t,z) \in D = \{O,X\}\times\{O,T\}\times\{O,Z\}$, $u(x,t,z)$ is unknown function $f(x,t,z)$ and $k(x,t,z,h,w,l)$ are given functions defined respectively on D.

Construction of orthogonal polynomial (CP)

Let $Q_n(x)$ be a polynomial of exact degree n, then Q_n is said to be orthogonal with respect to a weight function $w(x)$ within the interval $[a,b] \in$ R with $a < b$ if

$$\int_a^b Q_n(x)Q_m(x)\, w(x)dx = \delta_{m,n} \tag{5}$$

With $\delta_{m,n}$ is the Kronecker symbol defined by

$$\delta_{m,n} = \begin{cases} 1, if & m = n \\ o, if & m \neq n \end{cases} \tag{6}$$

The weight function $w(x)$ is continuous and also positive on $[a,b]$ such that the moments

$$\int_a^b w(x)\, x^n dx; \quad n \in N \tag{7}$$

exist and finite, then

$$< Q_n, Q_m >= \int_a^b Q_n(x)Q_m(x)w(x)dx \tag{8}$$

defines the inner product of the polynomials Q_n and Q_m. We adopted the weight function $w(x) = 1-x^2$ in the interval [0,1]. Hence, we use the property below to construct our basis function

$$\begin{cases} Q_n(x) = \sum_{r=0}^{n} C_r{}^{(n)} x^r \\ < Q_n, Q_m >= 0 \\ Q_n = 1 \end{cases} \tag{9}$$

leading *terms* $x^r; r \geq 0$ for

$$Q_0(x) = \sum_{r=0}^{0} C_r^{(0)} x^r = C_0^{(0)}$$

$$Q_0(1) = 1 \Rightarrow C_0^{(0)} = 1$$

Therefore,

$$Q_0(x) = 1$$

For $Q_1(x)$, we have

$$Q_1(x) = \sum_{r=0}^{1} C_r^{(1)} x^r = C_0^{(1)} + C_1^{(1)} x$$

When $x = 1$, implies

$$Q_1(1) = C_0^{(1)} + C_1^{(1)} = 1 \quad (10)$$

$$< Q_0, Q_1 > = \int_0^1 (1 - x^2) Q_0(x) Q_1(x) dx = 0$$

$$\Rightarrow \int_0^1 (1 - x^2) \left(C_0^{(1)} + C_1^{(1)} x \right) dx = 0$$

$$\Rightarrow \frac{2}{3} C_0^{(1)} + \frac{1}{4} C_1^{(1)} = 0$$

(11)

Solving equation 10 and equation 11, we obtain

$$C_0^{(1)} = -\frac{3}{5}; \quad C_1^{(1)} = \frac{8}{5}$$

Hence,

$$Q_1(x) = \frac{1}{5}(8x - 3) \quad (12)$$

Similarly, for $Q_2(x)$, we have

$$Q_2(1) = C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \quad (13)$$

For $x = 1$, we obtain

$$Q_2(1) = C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1$$

and,

$$< Q_0, Q_2 > = \int_0^1 (1 - x^2) Q_0(x) Q_2(x) dx = 0$$

$$\Rightarrow \int_0^1 (1 - x^2) \left(C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \right) dx = 0$$

$$\Rightarrow \frac{2}{3} C_0^{(2)} + \frac{1}{4} C_1^{(2)} + \frac{2}{15} C_2^{(2)} = 0 \quad (14)$$

$$< Q_1, Q_2 > = \int_0^1 (1 - x^2) Q_1(x) Q_2(x) dx = 0$$

$$\Rightarrow \frac{1}{5} \int_0^1 (1 - x^2)(8x - 3) \left(C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \right) dx = 0$$

$$\Rightarrow \frac{19}{300} C_1^{(2)} + \frac{4}{75} C_2^{(2)} = 0$$

(15)

Solving equation (13), equation (14) and equation (15), we obtain

$$C_0^{(2)} = \frac{11}{26}; \quad C_1^{(2)} = -\frac{80}{26}; \quad C_2^{(2)} = \frac{95}{26}$$

Hence,

$$Q_2(x) = \frac{1}{26}\left(95x^2 - 80x + 11\right) \qquad (16)$$

Following the same procedure, the following orthogonal polynomials were obtained for $r \geq 3$.

$$\left.\begin{array}{l} Q_3^*(x) = \frac{1}{46}\left(448x^3 - 595x^2 + 208x - 15\right) \\ Q_4^*(x) = \frac{1}{743}\left(21042x^4 - 38304x^3 + 22232x^2 - 4424x + 197\right) \\ Q_5^*(x) = \frac{1}{4043}\left(352176x^5 - 815430x^4 + 6669x^3 - 229320x^2 + 29840x - 903\right) \\ Q_6^*(x) = \frac{1}{22180}\left(6180603x^6 - 17379648x^5 + 18440235x^4 - 9144960x^3 + 2116935x^2 - 195264x + 4279\right) \end{array}\right\} \qquad (17)$$

Standard collocation method by Constructed orthogonal polynomial (CP) basis function

The orthogonal polynomials were constructed by using the Gram-Schmidt principle as a basis function in the approximating solution in this section. Thus, the infinite series solution is of the form

$$u(x,t,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{w=0}^{\infty} a_{i,w,w}\, Q_i(x)Q_w(x)Q_w(z) \qquad (x,t,z) \in D \qquad (18)$$

The infinite series solution in equation (18) is truncated as finite series of the form

$$u(x,t,z) \approx u_N(x,t,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w}\, Q_i(x)Q_j(x)Q_w(z) \qquad (x,t,z) \in D \qquad (19)$$

Here, $a_{i,j,w}$ are the unknown to be determined and $Q_i(x)$ $(i \geq 0)$ are the orthogonal polynomials constructed above. N is the degree of approximant. Substituting equation (19) into equation (3) to obtain

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w}\, Q_i(x)Q_j(t)Q_w(z)$$
$$- \int_0^x\int_0^t\int_0^z k(x,t,z,h,w,l)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w}\right)\left(Q_i(h)Q_j(k)Q_w(l)\right)\, dhdwdl$$
$$= f(x,t,z) \qquad (20)$$

Thus, equation (20) is collocated using the zeroes of the orthogonal polynomials constructed at $x = x_r$, $t = t_q$ and $z = z_s$ to obtain (see [13])

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N}\Bigg(Q_i(x_w)Q_j(t_q)Q_w(z_m)$$
$$- \int_0^{x_w}\int_0^{t_w}\int_0^{z_w} k(x_r,t_q,z_s,h,w,l)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w}\right)\Bigg)$$
$$= f(x_r,t_q,z_s) \qquad (21)$$

Where

$$\left.\begin{array}{r} Q(x_r) = 0 \\ Q(t_q) = 0 \\ Q(z_s) = 0 \end{array}\right\} \qquad (22)$$

Using the collocation points in equation (21) gives system of linear algebraic equations contains $(N+1)^2$ equations in the same number of given constants. Solving this system of linear algebraic equations to obtain the values of the constants $a_{i,j,k}$ such that $i,j,k = 0,1,\cdots,N$.

Standard collocation method by power series (PS) basis function

The power series (PS) method is used in approximating the unknown function $u(x,t,z)$ as :

$$u(x,t,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{w=0}^{\infty} a_{i,j,w} x^i t^j z^w \quad ; (x,t,z) \in D$$

Where $a_{i,j,w}$ are constants to be determined. If the infinite series in equation (23) is truncated into finite series, then equation (23) is written as:

$$u(x,t,z) \approx u_N(x,t,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w} x^i\, t^j z^z \qquad (24)$$

Where N is any natural number. Substituting equation (24) into equation (3) to obtain

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{w=0}^{N} a_{i,j,w}\, x^i t^j z^w - \int_0^x\int_0^t\int_0^z k(x,t,z,h,w,l)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{N} a_{i,j,w}\right) h^i w^j l^w \, dhdwdl = f(x,t,z) \qquad (25)$$

Hence, the residual equation is defined as

$$R_N(x,t,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{i,j,w}\, x^i t^j z^w - \int_0^x\int_0^t\int_0^z k(x,t,z,h,k,l)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{N} a_{i,j,k}\right) h^i w^j l^k \, dhdkdl - f(x,t,z) = 0 \qquad (26)$$

Collocating equation (26) using the zeroes of the constructed orthogonal polynomials as collocation points as discussed by author [14], gives

$$\left.\begin{array}{r} Q(x_r) = 0 \\ Q(t_q) = 0 \\ Q(z_s) = 0 \end{array}\right\} \qquad (27)$$

Equation (26) is written as

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{i,j,w}\Bigg({x_r}^i {t_q}^j {z_s}^w - \int_0^{x_r}\int_0^{t_q}\int_0^{z_s} k\left({x_r}^i, {t_q}^j, {z_s}^k, h, k, l\right)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{N} a_{i,j,w}\right) h^i k^j l^k \, dhdkdl - f\left(x_r, t_q, z_s\right)\Bigg) = 0 \qquad (28)$$

Thus, equation (28) resulted to system of linear algebraic equations contain $(N + 1)^2$ equations in the same number as the unknowns. Solving this system of algebraic equations to obtain the values of the constants $a_{i,j,k}$ such that $i,j,k = 0,1,\cdots,N$

Standard collocation method by shifted Legendre Polynomial (LP) basis function

In this section, the infinite series solutions given in power series (PS) is replaced as

$$u(x,t,z) \approx u_N(x,t,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{i,j,k} L_i(x) L_j(t) L_k(z); \qquad (24)$$

Where $a_{i,j,k}$ are already defined and $L_i(x)$ $L_j(t)$ $L_k(z)$ are the Legendre polynomials and the infinite series in equation (24) is truncated, then equation (24) is written as

$$u(x,t,z) \approx u_N(x,t,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{i,j,k} L_i(x) L_j(t) L_k(z); \qquad (25)$$

Thus, substituting equation (25) into equation (3) to obtain

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{i,j,k} L_i(x) L_j(t) L_k(z) - \int_0^{x}\int_0^{t}\int_0^{z} k(x,t,z,h,k,l)\left(\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{N} a_{i,j,k}\right) L_i(h) L_j(k) L_k(l)\, dhdkdl = f(x,t,z) \qquad (26)$$

Hence, the same procedure used in case of power series basis function is then employed to get the appropriate approximate solution for various values of N.

Numerical Examples

In this section, we consider three examples to illustrate the efficiency of the proposed methods and absolute errors are obtained for all examples considered.

Example 1. Consider the following three-dimensional Volterra integral equation

$$u(x,t,z) + 24\int_0^{t}\int_0^{x}\int_0^{z} (x^2, t)\, u(h,w,l) dhdwdl = 4x^5t^3z + 4x^3t^3z^3 + 3x^4t^3z^2 + x^2t + z^2t + xtz$$

$$x,t,z \in [0,1] \qquad (27)$$

With the exact solution given as

$$u(x,t,z) = x^2t + z^2t + xtz \qquad (28)$$

Applying the proposed methods in section (2.3), (2.4), and (2.5), we present the numerical solutions obtained as follows:

Table 1: Approximate solutions Example 1 for $N = 4$

(x,t,z)	Exact	Constructed Polynomial CP	Legendre polynomial LP	Power Series PS
(0,0,0)	0.0000000	0.0000000	0.0000000	0.000000000
(0.1,0.1,0.1)	0.0030000	0.0030029560	0.0030029559	0.0030029561
(0.2,0.2,0.2)	0.0240000	0.0238079835	0.0238079831	0.0238079835
(0.3,0.3,0.3)	0.0810000	0.0817371413	0.0797371406	0.0797371413
(0.4,0.4,0.4)	0.1920000	0.1929874913	0.1889874905	0.1889874920
(0.5,0.5,0.5)	0.3750000	0.3751089704	0.3753089696	0.3753089720
(0.6,0.6,0.6)	0.6480000	0.6477203242	0.6477203242	0.6477203242
(0.7,0.7,0.7)	1.0290000	1.0288998997	1.0288998997	1.0288998997
(0.8,0.8,0.8)	1.5360000	1.5357176538	1.5297176538	1.5297176538
(0.9,0.9,0.9)	2.1870000	2.1865900000	2.1865900000	2.1865900000
(1.0,1.0,1.0)	3.0000000	3.0000000000	2.9962340828	2.9962340828

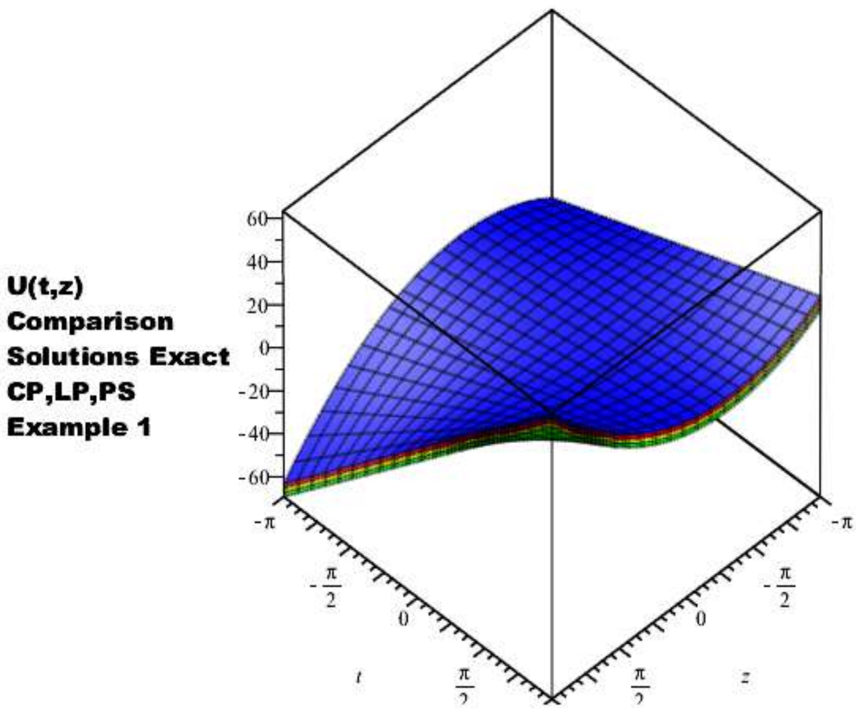

Figure 1: Depict 3D plots surface for the comparison between exact solutions (blue), Constructed Polynomial (red), Power Series (yellow), Legendre polynomial (green) at $x = 2$ on interval $z = -\pi \cdots \pi$, $t = -\pi \cdots \pi$ space domain of three-dimensional Volterra integral equation coordinates.

Example 2. Consider the following two-dimensional Fredholm integral equation

$$u(x,t,z) + \frac{1}{5}\int_0^1\int_0^1\int_0^1 xu\,(h,w,l)dhdwdl = \frac{1}{180} + xt(x-t) \qquad (29)$$

With the exact solution given as

$$u(x,t,z) = xt(z-t) \tag{30}$$

Applying the proposed methods in section (2.3), (2.4), and (2.5), we present the numerical solutions obtained as follows:

Table 2: Approximate solutions Example 2 for $N = 4$

(x,t,z) $z = 2$	Exact	Constructed Polynomial CP	Legendre polynomial LP	Power Series SP
(0,0)	0.00000000	0.000000000	0.000000000	0.000000000
(0.1,0.1)	0.01900000	0.019245000	0.01826000	0.018230000
(0.2,0.2)	0.07200000	0.071578000	0.07132000	0.071451000
(0.3,0.3)	0.15300000	0.153256000	0.15246000	0.152410000
(0.4,0.4)	0.25600000	0.25559000	0.25541000	0.255431000
(0.5,0.5)	0.37500000	0.37473000	0.37342000	0.374321000
(0.6,0.6)	0.50400000	0.50423000	0.50324000	0.503256000
(0.7,0.7)	0.63700000	0.63745000	0.63652000	0.636342000
(0.8,0.8)	0.76800000	0.76761000	0.76734000	0.767415000
(0.9,0.9)	0.89100000	0.89123000	0.89781000	0.882312000
(1.0,1.0)	1.00000000	1.00000000	1.89152000	1.861241000

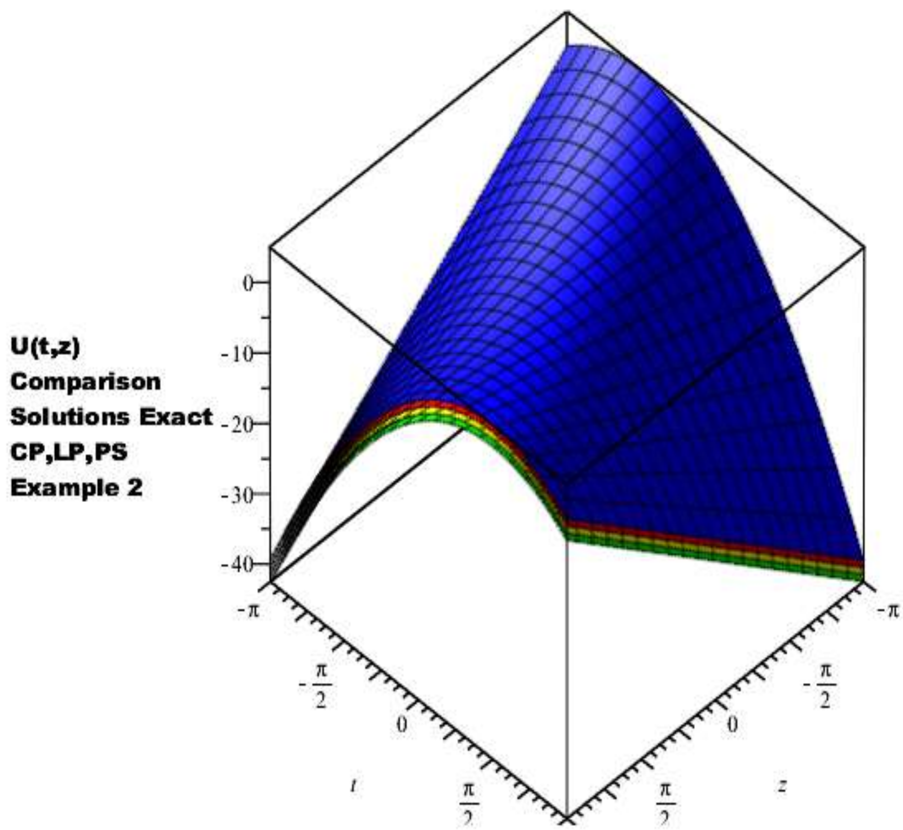

Figure 2: Depict 3D plots surface for the comparison between exact solutions (blue), Constructed Polynomial (red), Power Series (yellow), Legendre polynomial (green) at $x = 2$ on interval $z = -\pi \cdots \pi$, $t = -\pi \cdots \pi$ space domain of two-dimensional Fredholm integral equation coordinates.

Example 3. Consider the following three-dimensional Volterra integral equation

$$u(x,t,z)+\frac{1}{20}\int_0^x\int_0^t\int_0^z zu\,(h,w,l)dhdwdl-\frac{1}{20}\int_0^1\int_0^1\int_0^1(x+h)\,(h,w,l)dhdwdl$$
$$=-\frac{1}{5}(x+2x)sin\left(\frac{1}{2}\right)^2\sin(1)+\frac{1}{20}\,xz(\sin(t)+\sin(z)-\sin(t+z))$$
$$+\sin(t+z)\qquad x,t,z\ \in[0,1]\qquad (31)$$

with the exact solution given as

$$u(x,t,z)=sin(t+z)\qquad (32)$$

Applying the proposed methods in section (2.3), (2.4), and (2.5), we present the numerical solutions obtained as follows:

Table 3: Approximate solutions Example 3 for $N = 4$

(t,z)	Exact	Constructed Polynomial CP	Legendre polynomial LP	Power Series PS
(0,0)	0.000000000	0.000000000	0.000000000	0.000000000
(0.1,0.1)	0.198669330	0.198669330	0.198669330	0.198669330
(0.2,0.2)	0.389418342	0.389418342	0.389418342	0.389418342
(0.3,0.3)	0.564642423	0.564642423	0.564642423	0.564642423
(0.4,0.4)	0.717356093	0.717356093	0.717356093	0.717356093
(0.5,0.5)	0.841470921	0.841470921	0.841470921	0.841470921
(0.6,0.6)	0.932039083	0.932039083	0.932039083	0.932039083
(0.7,0.7)	0.98544971	0.98544971	0.98544971	0.98544971
(0.8,0.8)	0.99957360	0.99957360	0.99957360	0.99957360
(0.9,0.9)	0.97384762	0.97384762	0.97384762	0.97384762
(1.0,1.0)	0.90929742	0.90929742	0.90929742	0.90929742

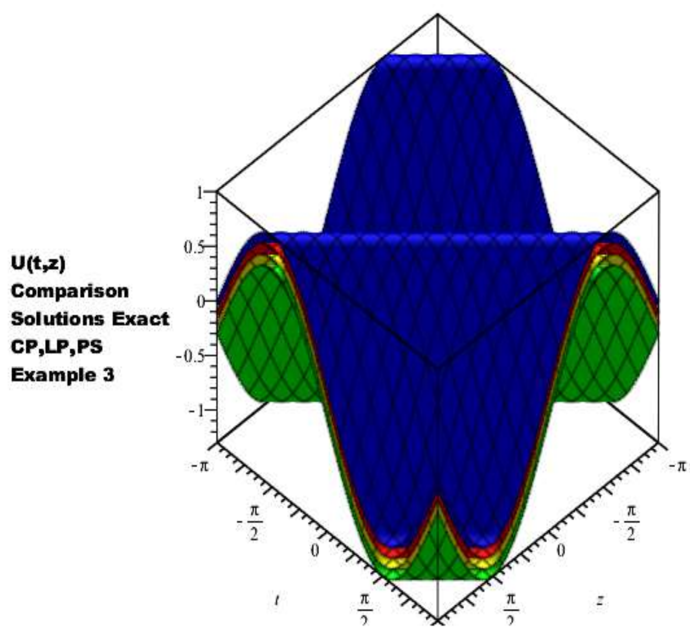

Figure 3: Depict 3D plots surface for the comparison between exact solutions (blue), Constructed Polynomial (red), Power Series (yellow), Legendre polynomial (green) on interval $z = -\pi \cdots \pi$, $t = -\pi \cdots \pi$ space domain three-dimensional Volterra integral equation coordinates.

CONCLUSION

In this study, we proposed and applied standard collocations techniques using three basic functions: Constructed polynomial (CP), Power series (PS), and shifted Legendre polynomial (LP) for the numerical solutions of the three-dimensional Volterra and Fredholm integral equations occur in applied mathematics and engineering sciences. From the three examples considered, the solutions are presented and compared with exact solutions which demonstrated the efficiency of the constructed polynomial (CP) as the closest to the exact solutions (red), followed by shifted Legendre polynomial (yellow) and least by Power series (PS) (green). Therefore, the present numerical techniques are proposed to serve as good tools to solve several problems in applied mathematics.

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