Journal of NAMP 71, (2025) 199-210

The Nigerian Association of Mathematical Physics

Journal homepage: https://nampjournals.org.ng

MATHEMATICAL MODEL ON DIMENSIONAL ANALYSIS OF STRATIFIED DEEP WATER EQUATIONS UNDER MODIFIED GRAVITY AND CORIOLIS EFFECT TO OBTAIN REYNOLDS NUMBER

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ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

Keywords:
Dimensionless Number;
Laminar Flow;
Turbulence;
Fluid Mechanics;
Modified Gravity.

ABSTRACT

The mathematical model of stratified deep water under modified gravity and Coriolis effect describes the behavior of fluid layers with different densities in a deep body of water. The model takes into account the effects of gravity, coriolis effect and other forces that can cause the fluid layers to move and interact with each other. The important aspect of the model is the effectiveness of dimension of the Reynolds numbers as the deep water continuously stratifies. Reynolds number is a dimensionless quantity that represents the ratio of inertial forces to viscous forces in the stratified deep water. Reynolds number can have a significant impact on the behavior of the fluid layers and equally affect the stability of the stratified deep water layers, with higher Reynolds numbers leading to more turbulent behavior. Overall, the mathematical model of stratified deep water and the effect of the Reynolds numbers provide valuable insights into the behavior of fluid layers in deep bodies of water and can be used to predict and understand various phenomena, such as ocean currents, waves, and tides.

1 INTRODUCTION

The impact of Reynolds number in stratified deep water under modified gravity [1] is a complex and multifaceted topic that has garnered significant attention in recent years [2]. In this introduction, we will provide an overview of the key concepts and ideas related to this topic [3], and highlight some of the important findings and insights that have emerged from our recent research findings [4].

To begin with, it is important to understand what Reynolds number is and how it relates to the behavior of fluid layers in deep bodies of water [5][6]. The Reynolds number is a dimensionless quantity that represents the ratio of inertial forces to viscous forces in the fluid [7]. In the context of stratified deep water, the Reynolds number can have a significant impact on the behavior of the fluid layers [8],[9].

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https://doi.org/10.60787/jnamp.vol71no.627

The model shows that the Reynolds number can also affect the stability of the fluid layers [10], with higher Reynolds numbers leading to more turbulent behavior [11]. One important role of Reynolds number in stratified deep water under modified gravity is to predict the turbulence in deep water regime [12] [13]. In the model, modified gravity acts as a restoring force [14] that helps to maintain the stratification of the fluid layers. However, in situations where the gravity is modified [15], such as in the case of a rotating fluid owing to the coriolis effect [16] or a fluid in a gravitational field [17] with a non-uniform strength [18], the behavior of the fluid layers can be altered significantly [19].

Additionally, in a rotating fluid [20], the Coriolis force can act as an additional restoring force [21] that helps to maintain the stratification of the fluid layers [22]. This can lead to the formation of stable, rotating fluid layers that are characterized by a well-defined axis of rotation [23]. In a gravitational field with a non-uniform strength [24], the fluid layers can become unstable and start to oscillate, leading to the formation of waves and other complex phenomena. Another important aspect of the impact of Reynolds number in stratified deep water under modified gravity is the effect of the fluid properties on the behavior of the fluid layers [28]. The viscosity and density of the fluid can have a significant impact on the stability of the fluid layers and the formation of waves and other phenomena [18][20]. In general, fluids with higher viscosity and lower density tend to be more stable and less prone to turbulence [30], while fluids with lower viscosity and higher density tend to be more unstable and more prone to turbulence [19][25].

2.0 ASSUMPTIONS OF THE MODEL

From the model equations we performed dimensional analysis for stratified deep water under modified gravity and the Coriolis effect to derive dimensionless numbers like Froude Reynolds number. This number is essential for understanding the dynamics of flow in stratified deep water environment. The assumptions:

i. The fluid is incompressible

ii. The fluid is stratified, meaning that the density varies with depth. This stratification is often modeled as a layered fluid with different density layers, or as a continuous density gradient.

iii. The gravity acceleration is not constant and may vary with depth or other factors.

iv. The Coriolis force is included in the equations of motion, which is important for large-scale deep water flows in rotating systems.

v. The flow is assumed to be laminar, which is a common assumption when deriving Reynolds numbers. But, in some cases, turbulent flow may be considered, but the Reynolds number is still used to characterize the flow regime.

vi. Viscous effects are neglected in the initial analysis, and the Reynolds number is derived based on the balance between inertial and viscous forces.

vii. The flow is considered to be driven by internal forces like gravity, Coriolis, and buoyancy with no external forces like wind, pressure gradients at the initial analysis.

MODEL DIAGRAM

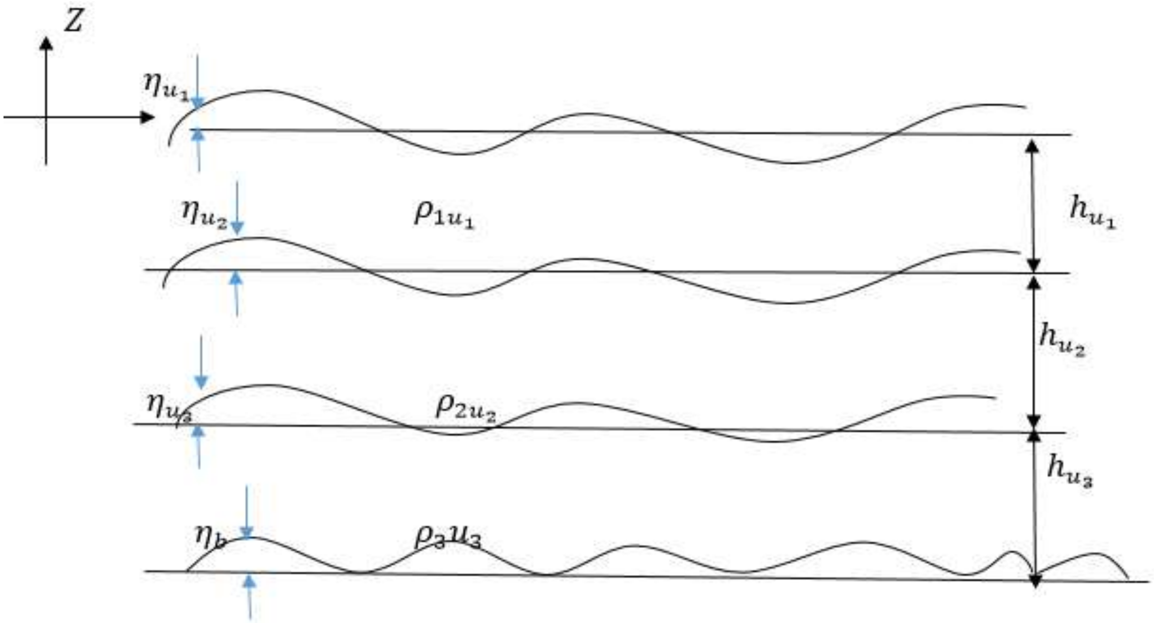

Figure 1. The geometry of stratified deep water flow under modified gravity and Coriolis effect

The stratification occurs at thermocline regime where the $H = h_1 + h_2 + h_3$ for the three strata and velocities u_1, u_2 and u_3 with varying densities, ρ_1, ρ_2, and ρ_3.

3.0 MODEL EQUATION

The model equations can be scaled using the scaling procedure of 2-D incompressible continuity and N.-S equations. The equations as obtained and subsequently applied [16] [19].

$$\frac{\partial(h_1u_1)}{\partial t} + \frac{\partial\left(h_1u_1^2 + \frac{g\frac{\rho_0-\rho_1}{\rho_1}h_1^2}{2}\right)}{\partial x} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} + fu_1 \qquad (a)$$

$$\frac{\partial(h_1v_1)}{\partial t} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} - fu_1 \qquad (b)$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1u_1)}{\partial x} = 0 \qquad (c)$$

$$\frac{\partial(h_2u_2)}{\partial t} + \frac{\partial\left(h_2u_2^2 + \frac{g\frac{\rho_2-\rho_1}{\rho_1}h_2^2}{2} + g\frac{\rho_2-\rho_1}{\rho_1}h_2h_1\right)}{\partial x} = -g\frac{\rho_1-\rho_2}{\rho_2}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_1-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial x} + fu_2 \qquad (d)$$

$$\frac{\partial(h_2v_2)}{\partial t} = -g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial y} - g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial y} - fu_2 \qquad (e)$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial(h_2u_2)}{\partial x} = 0 \qquad (f)$$

$$\frac{\partial(h_3u_3)}{\partial t} + \frac{\partial\left(h_3u_3^2 + \frac{g\frac{\rho_3-\rho_2}{\rho_2}h_2^2}{2} + g\frac{\rho_3-\rho_2}{\rho_2}h_3h_2\right)}{\partial x} = -g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_3}{\partial x} - g\frac{\rho_3-\rho_2}{\rho_2}h_3\frac{\partial(\xi)}{\partial x} + fu_3 \qquad (g)$$

$$\frac{\partial(h_3v_3)}{\partial t} = -g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_2}{\partial y} - g\frac{\rho_3-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial y} - fu_3 \qquad (h)$$

$$\frac{\partial h_3}{\partial t} + \frac{\partial(h_3u_3)}{\partial x} = 0 \qquad (i)$$

4.0 **MODEL ANALYSIS**

Equation (a-i) is the model equation [16] [19] and we wish to take scale analysis for the first layer equation to obtain some important quantities. Independent variables of this equations are

$x, y\ and\ t$, while the dependent variables are $u_1, v_1\ and\ p_1$. The height of the stratified layer at thermocline $= h$.

It is of interest to note that often the correct time scale can be obtained by combining the length and velocity scales. Using generalized dimensions we obtain velocity $as\ \frac{Length}{Time} \Longrightarrow u \backsim \frac{L}{T} \Longrightarrow t_s = \frac{H}{U_c}$, $x^* = {}^{x}/_{H}$, $y^* = {}^{y}/_{H}$, $t^* = {}^{t}/_{t_s}$

So $u^* = {}^{u}/_{U_c}$, $v^* = {}^{v}/_{V_c}$, $p^* = {}^{p}/_{P_s}$

Consider the continuity and momentum equations of the model at first stratification under modified gravity which is the gravity that ensures continuous stratification and Coriolis effect.

For the continuity equation:

$$\frac{\partial h_1}{\partial t} + \frac{\partial (h_1 u_1)}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial h_1}{\partial t} + h_1 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial h_1}{\partial x} = 0 \qquad (2)$$

Recall $\frac{DU}{Dt} = \frac{du_1}{dt} + u.\nabla U$

$$\Rightarrow \frac{\partial h_1}{\partial t} + u_1 \frac{\partial h_1}{\partial x} = -h_1 \frac{\partial u_1}{\partial x} \qquad (3)$$

From the momentum equation:

$$\frac{\partial (h_1 u_1)}{\partial t} + \frac{\partial (h_1 u_1^2 + g^{'} h_1^2/2)}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial (\varepsilon)}{\partial x} + f u_1 \qquad (4)$$

This gives

$$\frac{\partial (h_1 u_1)}{\partial t} + \frac{\partial (h_1 u_1^2)}{\partial x} + g^{'} \frac{\partial (h_1^2/2)}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \qquad (5)$$

Expanding equation (5)

$$u_1 \frac{\partial h_1}{\partial t} + h_1 \frac{\partial u_1}{\partial t} + 2 u_1 h_1 \frac{\partial u_1}{\partial x} + u_1^2 \frac{\partial h_1}{\partial x} + g^{'} \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \qquad (6)$$

Then

$$u_1 \left(\frac{\partial h_1}{\partial t} + u_1 \frac{\partial h_1}{\partial x} \right) + h_1 \left(\frac{\partial u_1}{\partial t} + 2 u_1 \frac{\partial u_1}{\partial x} \right) + g^{'} \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \qquad (7)$$

Substitute equation (3) into equation (7), which yields

$$-u_1 h_1 \frac{\partial u_1}{\partial x} + h_1 \frac{\partial u_1}{\partial t} + 2 u_1 h_1 \frac{\partial u_1}{\partial x} + g^{'} \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \qquad (8)$$

$$h_1 \frac{\partial u_1}{\partial t} + u_1 h_1 \frac{\partial u_1}{\partial x} + g^{'} h_1 \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1$$

$$h_1 \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} \right) + g^{'} h_1 \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \qquad (9)$$

But $\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{du_1}{dt}$, hence equation (9) becomes

$$h_1 \frac{du_1}{dt} + g^{'} h_1 \frac{\partial h_1}{\partial x} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \quad (10)$$

$$h_1 \frac{du_1}{dt} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + f u_1 \quad (11)$$

$$\frac{du_1}{dt} = -g^{'} h_1 \frac{\partial h_1}{\partial x} - g^{'} h_1 \frac{\partial \Sigma}{\partial x} + \frac{1}{h_1} f u_1 \quad (12)$$

Equation (12) which is the total acceleration in the first stratified column becomes

$$\frac{Du_1}{Dt} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} \quad (13)$$

5.0 **SCALING ANALYSIS OF THE MODEL EQUATION**

We can now take scaling analysis of equation (2) using generalized dimensions to obtain:

$$\frac{Length}{Time} \Longrightarrow u \backsim \frac{L}{T} \Longrightarrow t_s = \frac{H}{U_c}$$

$$x^* = {}^{x}/_{H} \ , \quad y^* = {}^{y}/_{H} \ , \quad t^* = {}^{t}/_{t_s}$$

So $\quad u^* = {}^{u}/_{U_c} \ , \quad v^* = {}^{v}/_{V_c} \ , \quad p^* = {}^{p}/_{P_s}$

$$\frac{\partial h_1}{\partial t} + u_1 \frac{\partial h_1}{\partial x} + h_1 \frac{\partial u_1}{\partial x} = 0 \quad (14)$$

$$\frac{\partial (H h_1^*)}{\partial (t^* t_s)} + U_C U_1^* \frac{\partial (H h_1^*)}{\partial (H x^*)} + H h_1^* \frac{\partial (U_c u_1^*)}{\partial (H x^*)} = 0 \quad (15)$$

$$\frac{H}{t_s} \frac{\partial h_1^*}{\partial t^*} + \frac{u_c H u_1^*}{H} \frac{\partial h_1^*}{\partial x^*} + \frac{H h_1^* U_c}{H} \frac{\partial u_1^*}{\partial x^*} = 0 \quad (16)$$

$$\frac{H}{t_s} \frac{\partial h_1^*}{\partial t^*} + U_c \, . u_1^* \frac{\partial h_1^*}{\partial x^*} + U_c \, . h_1^* \frac{\partial u_1^*}{\partial x^*} = 0 \quad (17)$$

But $t_s = \frac{H}{U_c}$ (18)

$$\therefore \quad U_c \frac{\partial h_1^*}{\partial t^*} + U_c \, . u_1^* \frac{\partial h_1^*}{\partial x^*} + U_c \, . h_1^* \frac{\partial u_1^*}{\partial x^*} = 0 \quad (19)$$

$$U_c \left(\frac{\partial h_1^*}{\partial t^*} + u_1^* \frac{\partial h_1^*}{\partial x^*} + h_1^* \frac{\partial u_1^*}{\partial x^*} \right) = 0 \quad (20)$$

$$\Rightarrow \frac{\partial h_1^*}{\partial t^*} + u_1^* \frac{\partial h_1^*}{\partial x^*} + h_1^* \frac{\partial u_1^*}{\partial x^*} \quad (21)$$

Equation 21 shows that scaled continuity equation for stratified deep water is identical to unscaled continuity equation (2) and equation (20) can be generally compared with equation (22) which is the scaled continuity equation of fluid flow.

$$U_c \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) = 0 \quad (22)$$

We can extend the procedure to the x – momentum equation (12), here the gravity is partially modified and $\frac{\partial^2}{\partial^2 x} = \frac{\partial}{\partial x}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial H x^*}(\frac{\partial}{\partial H x^*}) = \frac{1}{H} \frac{\partial}{\partial {x^*}^2}$

From equation (12),

$$\frac{du_1}{dt} = -g^{'}(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}) \quad (23)$$

Equation (23) is when the effect of coriolis is zero, since coriolis effect is fundamentally zero at the equator (0° $latitude$), here due to the vertical motion at the equator the deep water particles at the equator moves strictly north or south (parallel to the rotation axis), the horizontal coriolis force is zero. Again in the context of energy equations for waves in a rotating fluid, the coriolis terms perform zero work because the force acts perpendicular to the velocity of the fluid parcels.

Now scaling the momentum equation of stratified deep water yield

$$g^{'}\left(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}\right) = \frac{\partial u_1}{\partial t}(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}) \quad (24)$$

The $(*)$ quantites are all dimensionless. We substitute the dimensionless quantities into equation (24).

$$\left[g^{'}\left(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}\right)\right]^{*} = \frac{\partial (U_c u_1^{*})}{\partial (t^{*} t_s)} \frac{\partial (H h_1^{*}}{\partial (H x^{*})} + \frac{\partial H \varepsilon^{*}}{\partial (H x^{*})} \quad (25)$$

$$\left[g^{'}\left(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}\right)\right]^{*} = \frac{U_c}{t_s} \frac{\partial (u_1^{*})}{\partial (t^{*})} \left(\frac{H}{H} \frac{\partial (h_1^{*}}{\partial (x^{*})} + \frac{H}{H} \frac{\partial \varepsilon^{*}}{\partial (x^{*})}\right) \quad (26)$$

But $\frac{U_c}{t_s} = U_c / \frac{H}{U_c} = \frac{U_c}{H}$, then equation (26) becomes

$$\left[g^{'}\left(\frac{\partial h_1}{\partial x} + \frac{\partial \varepsilon}{\partial x}\right)\right]^{*} = \frac{U_c^2}{H} \frac{\partial (u_1^{*})}{\partial (t^{*})} \left(\frac{\partial (h_1^{*}}{\partial (x^{*})} + \frac{\partial \varepsilon^{*}}{\partial (x^{*})}\right) \quad (27)$$

Equation (26) shows that the scaled momentum equation under gravity modification is identical to equation (24). An important development was noticed during the scaling of momentum equation for stratified deep water, in equation (27), we noticed the emergence of the term $\frac{U_c}{H}$ which is the same with the typical scaling of momentum equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{-1}{\rho} \frac{\partial \rho}{\partial x} + v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (28)$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial y} = \frac{-1}{\rho} \frac{\partial \rho}{\partial y} + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g^{'} \quad (29)$$

Introducing the dimensionless quantities into equation (28), gives

$$U_c \frac{\partial u^{*}}{\partial t} + U_c^2 \, u^{*} \frac{\partial u^{*}}{\partial x} + U_c \, v^{*} \frac{\partial u^{*}}{\partial y} = \frac{-P_S}{\rho} \frac{\partial P}{\partial x} + \nu U_c \left(\frac{\partial^2 u^{*}}{\partial x^2} + \frac{\partial^2 u^{*}}{\partial y^2}\right) \quad (30)$$

Then the partially- scaled result of equation (28) becomes

$$\frac{U_c}{t_s} \frac{\partial u^{*}}{\partial t^{*}} + \frac{U_c^2}{H} u^{*} \frac{\partial u^{*}}{\partial x} + \frac{U_c^2}{H} v^{*} \frac{\partial u^{*}}{\partial y} = \frac{-P_s}{\rho H} \frac{\partial P^{*}}{\partial x^{*}} + \frac{\nu U_c}{H^2} \left(\frac{\partial^2 u^{*}}{\partial x^{*2}} + \frac{\partial^2 u^{*}}{\partial y^{*2}}\right) \quad (31)$$

Since $t_s = H/U_c$, the equation (31) becomes

$$\frac{U_c^2}{H} \left(\frac{\partial u^{*}}{\partial t^{*}} + \, u^{*} \frac{\partial u^{*}}{\partial x^{*}} + v^{*} \frac{\partial u^{*}}{\partial y^{*}}\right) = \frac{-P_s}{\rho H} \frac{\partial P^{*}}{\partial x^{*}} + \frac{\nu U_c}{H^2} \left(\frac{\partial^2 u^{*}}{\partial x^{*2}} + \frac{\partial^2 u^{*}}{\partial y^{*2}}\right) \quad (32)$$

By inspection we can clearly see that there is a common term in equation (27) and (32) which is $\frac{U_c^2}{H}$. Therefore, equation (32) becomes

$$\frac{\partial u^*}{\partial t^*}+ u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=\frac{-P_s}{\rho U_c^2}\frac{\partial P^*}{\partial x^*}+\frac{\nu}{U_c H}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right) \quad (33)$$

Observation from equation (33) shows that all quantities on the left- hand side are dimensionless, the same with all derivative terms on the right-hand side of equation (33). Hence, $\frac{P_s}{\rho U_c^2}$ $and\ \nu/(u_c H)$ are also dimensionless.

6.0 **DIMENSIONAL ANALYSIS OF OBTAINED QUANTITIES**

The dimensional analysis of quantities obtained in equation (33) which equally correspond with equation (27) can now be analyzed.

So ρu_c^2 is dimensionally equivalent to:

$\rho u_c^2 \backsim \frac{Mass}{(Length)^3}\left(\frac{Length}{Time}\right)^2 \backsim \frac{M}{L^2}\frac{L}{T^2}$ Where T = time, L = length and M = mass

$$\rho u_c^2 \backsim pressure \backsim \frac{force}{area} \backsim \frac{mass.acceleration}{area}$$

From this it is obvious that $P/(\rho u_c^2)$ will be unitless. Again, this is a very important quantity in deep water analysis and the value is twice the dynamic pressure. Hence as known in fluid dynamics;

Dynamic pressure = $\frac{1}{2}\rho U_c^2$

For generality the "c" subscript on the velocity can be suppress.

$$\therefore pd = \frac{1}{2}\rho U \quad (34)$$

Equation (34) shows the validity of the model and it is very useful in study of Bernoulli's equation which is important in deep water stratification because it helps to describe the relationship between pressure, velocity, and depth in deep water. It helps us to understand how different layers of water interact with each other.

Consider the coefficient of the second order partial differential equation in (33), $\nu/(U_c H)$.

The reciprocal of this dimensionless quantity is the Reynolds Number which is very important in studies of transition of turbulence in deep water regime. It is expressed as:

$$\frac{1}{\nu/HU_c} = R_e = \frac{U_c H}{\nu} \quad (35)$$

By suppressing the subscript "c" on the velocity and H becomes L for generality then equation (35) becomes;

$$R_e = \frac{UL}{\nu} \quad (36)$$

Equation (36) is Reynolds number, which is a dimensionless quantity used to predict the flow in stratified deep water regime. It is defined as the ratio of inertial forces to viscous forces in a fluid flow. The property parameters that Reynolds number contains are. The Reynolds number can be expressed as dimensionless quantity useful in predicting the flow regime in deep water. It is defined as the ratio of inertial forces to viscous forces in a fluid flow. The property of Reynolds number are:

- Fluid velocity: Reynolds number is proportional to the velocity of the fluid. Higher fluid velocities result in higher Reynolds numbers, indicating a greater tendency for turbulent flow.
- Fluid viscosity: Reynolds number is inversely proportional to the viscosity of the fluid. Lower viscosities result in higher Reynolds numbers, indicating a greater tendency for turbulent flow
- Fluid density: Reynolds number also depends on the density of the fluid. Higher fluid densities result in lower Reynolds numbers, indicating a greater tendency for laminar flow.

RESULTS AND DISCUSSION

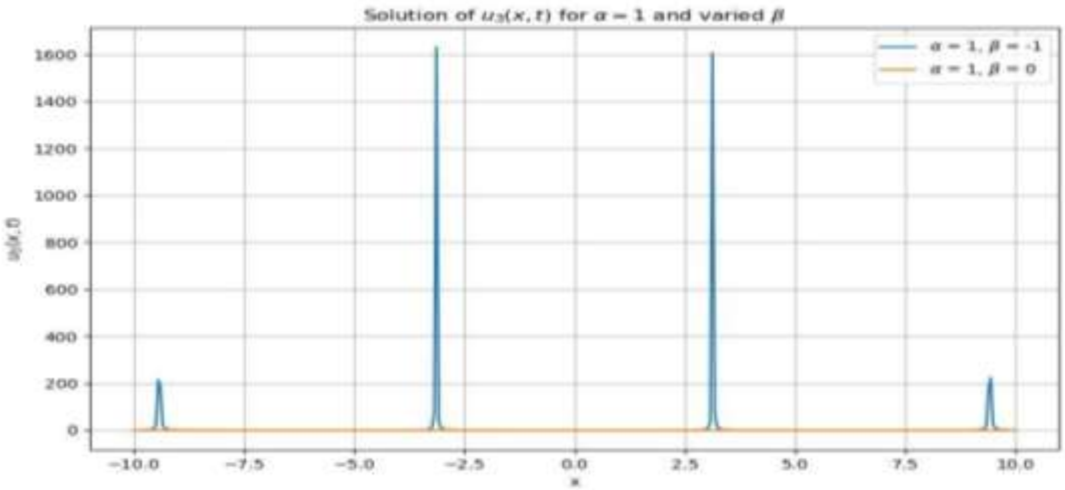

Figure 2: The effect of Reynolds number at resonance on stratified deep water under modified gravity.

From figure 2, shows the effect of resonance as the deep water stratifies leading to high amplitude of oscillation and this can lead to mixing and less turbulence owing to Reynolds number. There is symmetrical properties at this regime giving rise to dynamic pressure and the stratified deep water is in dynamic equilibrium.

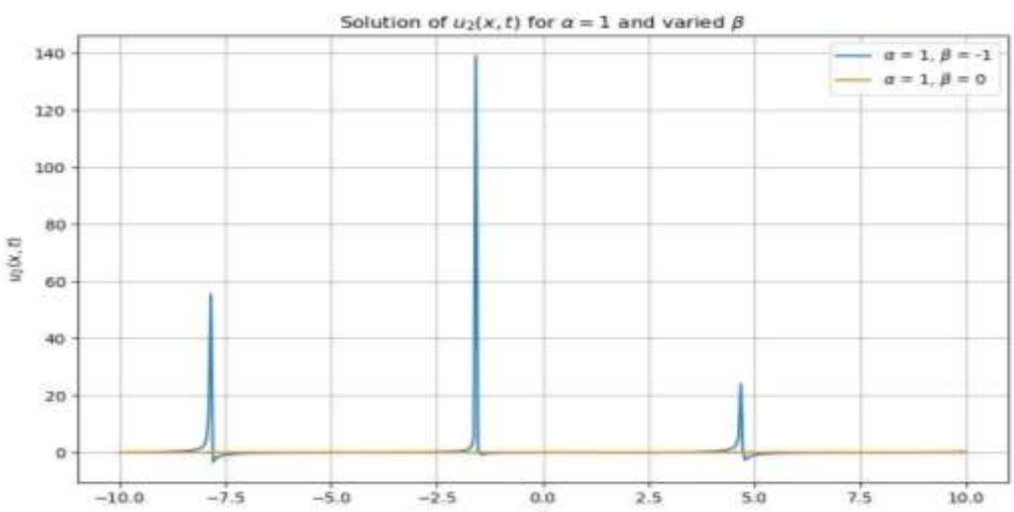

Figure 3: The simulation showing unstable and stable stratification at resonance owing to increase in mixing leading to turbulence due to Reynolds number.

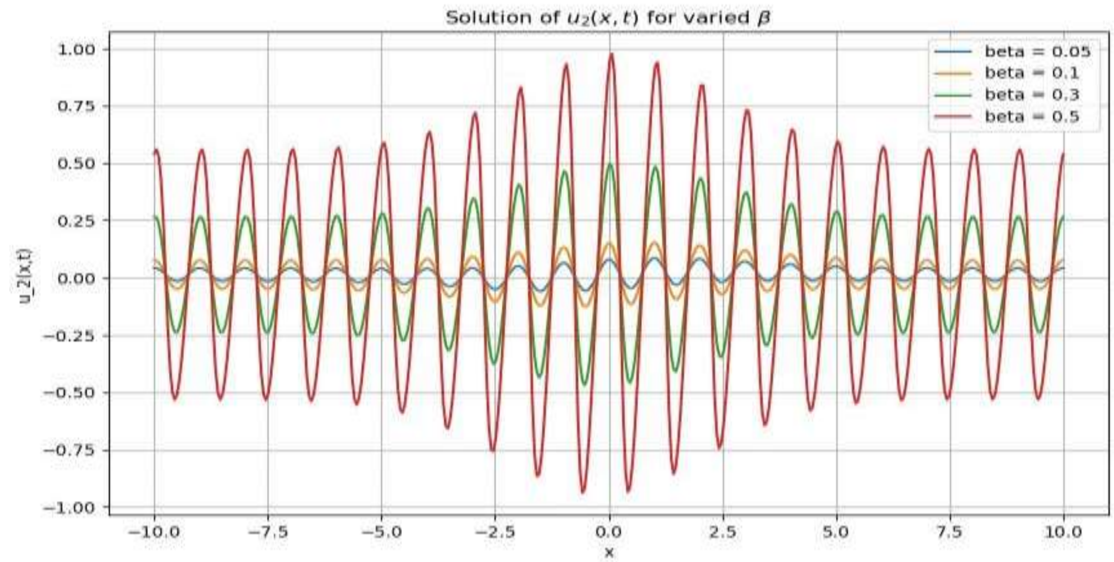

Figure 4: The effect of Reynolds number as the deep water stratifies at different velocities with varied values of β.The Froude number is an important dimensionless number in fluid dynamics and is used to describe the relationship between inertial and gravitational forces in a fluid flow. In the context of deep water stratification, the Froude number is used to determine the stability of the water column and the development of internal waves.

The analysis developed in this model demonstrates how dimensional scaling of the stratified deep water equations under modified gravity and Coriolis influence naturally yields key nondimensional quantities particularly the Reynolds number. This quantity encapsulates the balance between inertial and viscous forces and therefore determines whether stratified flow remains laminar, transitions, or evolves toward turbulent mixing. In a stratified oceanic or deep-water environment, this is essential because mixing, internal wave dynamics, and interlayer momentum exchange are highly Reynolds-number dependent.

Deriving Reynolds number in this context is significant because it provides a rigorous theoretical bridge between the governing momentum equations and the observable behavior of stratified layers. It allows the system to be characterized without requiring specific dimensional units, meaning the same framework can be applied across scales from laboratory tanks to real ocean basins. Moreover, its emergence from the scaled momentum equation confirms internal consistency of the model and supports its physical realism.

The figures presented in the manuscript illustrate how Reynolds number modifies system response, stratification strength, and mixing behavior. Figure 2 shows the effect of resonance under stratification, where elevated Reynolds values intensify oscillatory responses of the layers. This suggests enhanced dynamic pressure and temporary energetic equilibrium. Figure 3 demonstrates the evolution from stable to unstable stratification regimes as Reynolds number increases: higher Re produces stronger interfacial shear, enhanced mixing, and turbulence. Figure 4 complements this by showing how changing velocities influence Reynolds number and, consequently, the degree of stratification stability versus potential breakdown. Collectively, these figures visualize the progressive transition from orderly, laminar stratified motion to increasingly energetic, mixed, and turbulent states driven by Reynolds intensity.

The findings also relate directly to the stability of stratified flows. Stable stratification occurs when buoyancy forces dominate, suppressing vertical exchange; however, rising Reynolds number increases inertial forcing, promotes shear instabilities, and reduces stability. Thus, the analyses presented confirm that Reynolds number serves as a predictive stability indicator: low Re corresponds to stable, layered flow, while high Re aligns with instability, mixing, wave amplification, and turbulence consistent with modern oceanographic understanding of stratified flows.

CONCLUSION

This manuscript developed and analyzed a mathematical model of stratified deep water flow under modified gravity and Coriolis effects through dimensional and scaling analysis

The study successfully derived a key nondimensional parameter, the Reynolds number from the scaled momentum equations and demonstrated its fundamental role in characterizing stratified flow behavior. The numerical simulations and graphical results showed how Reynolds number governs dynamic pressure, mixing tendencies, turbulence generation, and the transition between stable and unstable stratification regimes. Overall, the work provides theoretical and interpretive understanding of how inertial viscous balance influences deep water stratification dynamics, offering insights applicable to geophysical fluid processes in oceanographic systems.

Conflict of interest: The authors declare no conflict of interest.

LIST OF SYMBOLS

λ	Wavelength
$u = (u, v, w)$	The velocity vector
ρ	The density of flow
g	The gravitational constant
g'	The modified gravity
f	The coriolis parameter
u	Velocity in the x-direction
v	The velocity in the y-direction
L	The length scale
h	Vertical length scale
x	x direction
y	y direction
t	The required time
$\frac{D}{Dt}$	The total material derivative
$\xi(x, y)$	Denotes the thermocline regime
h	The water height above each stratified column
U_c	Dimensional velocity
u_1	Velocity in the first layer in the $x-$ direction
u_2	Velocity in the second layer in the $x-$ direction
v_1	Velocity in the first layer in the $y-$ direction
v_2	Velocity in the second layer in the $y-$ direction
t^*	Dimensional time
t_s	Scaled time
p^*	Dimensional pressure

u_1^*	Dimensional velocity in the first layer
x^*	Scaled x
y^*	Scaled y
H	Dimensionless height
α	Measure of strength of the system
β	Measure of stability of the system
F	Sum of all forces
m	Mass in (Kg)
a	Acceleration in (m/s^2)
L	Length in meter
P	Pressure in (Kg/ms^2)

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