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LIE ALGEBRAIZATION OF DOUBLE CONFLUENT HEUN EQUATION

IDIONG U.S.

Department of Mathematics, Adeyemi Federal University of Education Ondo.

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ABSTRACT

In this paper, we present a new algebraization of the double confluent Heun equation (DCHE) by writing its operator as the linear combination of quadratic elements in the universal enveloping algebra of $sl(2,\mathbb{C})$. We also obtain a new quasi-exactly solvable potential using a gauge transformation. It was observed that DCHE is only quasi-exactly solvable and therefore admits only polynomial solutions.

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1. INTRODUCTION

The canonical Heun equation is a second order differential equation in the complex domain given by

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}\right)\frac{\mathrm{d}\psi}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}\psi=0, \tag{1.1}$$

with regular singularities at $z=0,1,a$, irregular singularity at ∞ and where, $\alpha,\beta,\gamma,\delta,\epsilon$ are complex numbers that obey the constraint equation $\gamma+\delta+\varepsilon=\alpha+\beta+1$.

The confluent forms of the Heun equation (1.1) arise when two or more of the regular singularities $z=0,1,a$ merge to form an irregular singularity (∞). The Doubly-confluent Heun Equation (DCHE) is given by

*Corresponding author: IDIONG U.S.
E-mail address: idiongus@afued.edu.ng
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$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z^2} + 1\right)\frac{\mathrm{d}\psi}{\mathrm{d}z} + \frac{\alpha z - q}{z^2}\psi = 0. \tag{1.2}$$

Equation (1.2) has irregular singularities at $z = 0$ and $z = \infty$ each of rank 1. The properties and connecting formulas of solutions of equation (3.1) have been discussed in [3, 8, 11, 6, 13].

The novelty in this work shall be well understood in Remark 3.1 which is a clear departure from the conventional spin dependent models. In what follows, Section 2 shall state in clear terms the mathematical formalism required for the work. Section 3 consists the main result as well as very vital deductions stated as remarks and Section 4 contains the conclusions drawn.

2. MATHEMATICAL FORMALISM

Let us consider a linear space of polynomials of degree not more than n, given by $\mathcal{P}_{n+1} = \langle 1, z, z^2, \dots z^n \rangle$ where n is a non-negative integer. It is known (see [9], § 2.1, p.10) that a linear differential operator of the kth order, $H_k(z, \frac{\mathrm{d}}{\mathrm{d}z})$, is called quasi-exactly solvable, if it preserves the linear space of polynomials $\mathcal{P}_{n+1}$, that is, $H_k\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right) : \mathcal{P}_{n+1} \longrightarrow \mathcal{P}_{n+1}$. In otherwords, a k^{th} order differential operator given by

$$H_k\psi = \lambda\psi,$$

where, $H_k = \sum_{j=0}^{k} a_j(z)\frac{\mathrm{d}^j}{\mathrm{d}z^j}$ is said to be *quasi-exactly solvable* if it can be written as quadratic combination of the generators of certain elements of the universal enveloping algebra of $sl(2,\mathbb{C})$ of the form

$$H_k = \sum_{a,b=0,\pm} c_{ab}\, J_a J_b + \sum_{a=0,\pm} c_a J_a, \tag{2.1}$$

where the number of free parameters $c_{ab} \neq 0$ is given by $par(H_k) = (k+1)^2$. When the number of free parameters $par(H_k) = \frac{(k+1)(k+2)}{2}$ then H_k is said to be exactly solvable using H.L. Krall's classification ([5]). The generators of the lie algebra $sl(2,\mathbb{C})$ have the form

$$J_+ := z^2\frac{\mathrm{d}}{\mathrm{d}z} - 2jz, \qquad J_0 := z\frac{\mathrm{d}}{\mathrm{d}z} - j, \qquad J_- := \frac{\mathrm{d}}{\mathrm{d}z} \tag{2.2}$$

which obey the commutator relation

$$[J_0, J_+] = J_+, \quad [J_-, J_+] = 2J_0, \quad [J_-, J_0] = J_- \quad . \tag{2.3}$$

The technique of computing (2.1) is what we call *algebraization*. The next preliminary result which can be found in the works of ([12], §4, p.2912) and ([4]) gives explicit information on algebraization. [[12], §4, p.2912] The standard quadratic expression for the Hamiltonian $-H_2$ of degree 2 as polynomials in terms of J_+, J_0 and J_- is given by

$$\begin{aligned} -H_2\psi = [&c_{++}(J_+)^2 + c_{+0}[J_+J_0 + J_0J_+] + c_{00}(J_0)^2 + c_{+-}[J_+J_- + J_-J_+] \\ &+ c_{0-}[J_0J_- + J_-J_0] + c_{--}(J_-)^2 + c_+J_+ + c_0J_0 + c_-J_- + c_*]\psi. \end{aligned} \tag{2.4}$$

(see [4]). Equation (2.4) in expanded form is given by

$$-H_2\psi = [c_{++}z^4 + 2c_{+0}z^3 + (c_{00} + 2c_{+-})z^2 + 2c_{0-}z + c_{--}]\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}$$
$$+[(2j-1)[2c_{++}z^3 + 3c_{+0}z^2 + (2c_{+-} + c_{00})z + c_{0-}] + c_+z^2 + c_0z + c_-]\frac{\mathrm{d}\psi}{\mathrm{d}z}$$
$$+[2j(2j-1)c_{++}z^2 + 2j(2j-1)c_{+0}z + c_{00}j^2 - 2jc_{+-} - j[2c_+z + c_0] + c_*]\psi. \tag{2.5}$$

Here, $j = \frac{n}{2}$, is the spin number, where $n \in \mathbb{Z}$. In what follows, we present the main result which investigates whether the DCHE satisfies the Krall's criterion for quasi-exact solvability. The technique of canonical polynomial is used to compute the eigenfunctions and eigenvalues are obtained. The hidden symmetry is obtained via the Lie structure metric and a new quasi-exact solvable potential is obtained using gauge transformation.

In the next section, the main results of this paper are presented.

3. MAIN RESULTS

In this section, we discuss the Lie algebraization of the DCHE, the generation of new QES potentials via gauge transformation and corresponding eigenfunctions and eigenvalues. The Doubly-confluent Heun Equation (DCHE) is given by

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z^2} + 1\right)\frac{\mathrm{d}\psi}{\mathrm{d}z} + \frac{\alpha z - q}{z^2}\psi = 0. \tag{3.1}$$

Equation (3.1) has irregular singularities at $z = 0$ and $z = \infty$ each of rank 1. Let equation (3.1) be re-written in another form by multiplying through equation (3.1) with z^2 to get

$$z^2\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + (z^2 + \gamma z + \delta)\frac{\mathrm{d}\Psi}{\mathrm{d}z} + (\alpha z - q)\Psi = 0. \tag{3.2}$$

3.1 Lie Algebraization

By using the connections between equations (2.4) and (2.5), the coefficients of the differential expressions in (2.5) and (3.2) are compared to obtain

$$\begin{array}{llll} c_{++} = 0 & c_{+0} = 0 & c_{+-} = \frac{1}{2} & c_+ = 1 \\ c_{0+} = 0 & c_{00} = 0 & c_{0-} = 0 & c_0 = \gamma - n + 1 \\ c_{-+} = \frac{1}{2} & c_{-0} = 0 & c_{--} = 0 & c_- = \delta \end{array}$$

Therefore, the structure metrics $\{c_{ab}\}$ of the DCHE is given by

$$g = \{c_{ab}\} = \begin{pmatrix} c_{++} & c_{+0} & c_{+-} \\ c_{0+} & c_{00} & c_{0-} \\ c_{-+} & c_{-0} & c_{--} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}. \qquad (3.3)$$

The matrix in (3.3) reveals the hidden symmetry of the Lie algebraic DCHE.
The norm of $g = \{c_{ab}\}$ is given as

$$\parallel g \parallel = \det[c_{ab}]_{a,b=\pm,0} = 0. \qquad (3.4)$$

ere, $\alpha = -n$ and the Casimir eigenvalue $c_* = -\frac{n}{2}(n - \gamma - 2) - q$. Next, with these coefficients, we write the DCHE in equation (3.2) in terms the quadratic polynomial of operators (2.2) as

$$H = \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ + (\gamma - n + 1)J_0 + J_+ + \delta J_- - \frac{n}{2}(n - \gamma - 2) - q. \qquad (3.5)$$

Equation (3.5) is an element in the center of $U(\mathcal{G})$. We observe here, that the free parameters for the DCHE H_2 are six (6) in number which confirms the differential equation (3.1) as an QES differential equation. The presence of the term in positive grading J_+ also confirms this fact.

Remark 3.1: The Hermitian operator in (3.5) exhibits several structural features that distinguish it from standard spin Hamiltonians commonly encountered in the literature, such as the Heisenberg, XXZ, or simple Zeeman models. The bilinear combination $\frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+$ which is proportional to $\mathrm{J}^2 - J_0^2$, is fully consistent with conventional $su(2)$-invariant constructions and represents the familiar quadratic spin interaction. In contrast, the presence of linear ladder-operator terms J_+ and δJ_- is non-standard. Such terms explicitly break conservation of the spin projection J_0 and do not appear in equilibrium spin models unless external driving, source terms, or algebraic deformations are introduced. The asymmetry between the coefficients of J_+ and J_- further emphasizes the departure from isotropic or time-reversal-symmetric settings typical of elementary models. The linear J_0 contribution, with coefficient $(\gamma - \mathrm{n} + 1)$, also differs from a conventional Zeeman term. Its explicit dependence on the representation label n, together with the parameter γ, suggests that this term is not associated with a physical magnetic field but instead encodes representation-dependent or algebraic information. A similar interpretation applies to the additive constant $-\frac{n}{2}(n - \gamma - 1) - q$, whose dependence on n and γ is unusual in basic spin Hamiltonians and is characteristic of spectrum-shifting terms in exactly or quasi-exactly solvable constructions. Taken together, these features indicate that H is best understood not as a conventional spin-interaction Hamiltonian, but as an algebraically engineered operator, potentially arising in driven, deformed, or quasi-exactly solvable models where representation-dependent terms and explicit ladder-operatorsources play a central role.

Now, the equation (3.5) can now be written using the expansion of the generators of universal enveloping algebra of $sl(2, \mathbb{C})$ in the form

$$H_{dch}^{QES}\Psi(z):=\left[z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\left(z^2-\left(\gamma-2(n-1)\right)z+\delta\right)\frac{\mathrm{d}}{\mathrm{d}z}-nz-\frac{n}{2}(n-\gamma-1)-q\right]\Psi(z)=0. \quad (3.6)$$

Equation (3.6) is the result of substituting the sl(2) generators into the algebraic form of the Hamiltonian. Having considered the algebraization of the double confluent Heun equation, we now consider the evaluation of its ground state solution, its new eigenfunction and its new quasi-exactly solvable potential.

3.2 Gauge Transformation

Now, from the differential equation (3.6), the coefficient functions $p(z), q(z)$ and $r(z)$ are given as

$$p(z)=z^2,\quad q(z)=-\left(z^2-\left(\gamma-2(n-1)\right)z+\delta\right),\quad r(z)=-nz-\frac{n}{2}(n-\gamma-1)-q. \quad (3.7)$$

The energy eigenvalue attached to (3.6) is

$$\Xi_{dch}=-\frac{n}{2}(n-\gamma-1)-q. \quad (3.8)$$

The presence of the term in positive grading makes H_{dch}^{QES} a quasi-exactly solvable operator. Thus, the approach adopted in this case allows the operator H_{dch}^{QES} to act on the space of monomials $z^{2j}, (j=0,1,2,\dots,n)$ (see [7, 9]).

In what follows, the gauge transformation of the DCHE is carried out to obtain an equivalent form, namely a Schrödinger equation $(\mathcal{S}-E_{dch}^{(n)})\Psi=0$ where $\mathcal{S}$ is given by the transformation

$$\mathcal{S}=\mu(z)^{-1}\cdot H_{dch}^{QES}\cdot\mu(z)=-\frac{\mathrm{d}^2}{\mathrm{d}w^2}+U(w),$$

such that the QES potential is

$$U(w)=\frac{12z^2-16z(2z-\gamma+2n-2)+4[z^2-(\gamma-(2n-2))z+\delta]^2}{16z^2}-(n+1)z-E_{dch}^{(n)}.$$

Quasi-exactly solvable (QES) potentials (which in this instance is $U(w)$) are significant in physics as they enable certain parts of the energy spectrum and eigenstates to be determined exactly in otherwise non-integrable systems. They provide reliable analytical models for quantum systems where perturbation theory is ineffective and serve as benchmarks for numerical and variational methods. QES potentials expose hidden algebraic structures and partial symmetries, illustrating that symmetry can act on a subspace of the Hilbert space. Many QES models are relevant to practical scenarios, such as anharmonic oscillators and double wells, with exactly known low-lying states influencing observable properties. Additionally, the parameter dependence in QES systems allows for the exploration of spectral transitions and instability mechanisms, effectively linking mathematical theory with physical relevance.

Here, the energy eigenvalue of the Schrödinger operator $\mathcal{S}$ is

$$E_{dch}^{(n)} = \frac{1}{2}[\gamma - 2n + 3] + \Xi_{dch}.$$

where w is given by

$$w = \int z\, u^{-1} \mathrm{d}u = \log_e z$$

and the gauge function for DCHE is

$$\mu(z) = \exp\left(\frac{1}{2}\int z \frac{q(u)}{p(u)} \mathrm{d}u\right) = z^{\gamma-2(n-1)} \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right). \tag{3.8}$$

and its eigenfunction is in terms of polynomial functions in the polynomial space $\mathcal{P}_{n+1}$.

Remark 3.2 The gauge function $\mu(z)$ in (3.8) is derived from functions $p(z)$ and $q(z)$ defined in (3.7), which come from the *algebralaized* form (3.6), not the original DCHE (3.2).

3.3 Polynomial Solutions and Eigen states of Lie Algebraic DCHE

Theorem 3.3 *Consider the DCHE,* H_{dch}^{QES}*. The eigenfunction* Ψ *that satisfies the equation* $H_{dch}^{QES}\Psi = 0$ *is given by*

$$\Psi(z) = \mathcal{P}_n(z) z^{\gamma-2(n-1)} \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right),$$

Where $\mathcal{P}_n(z) = \sum_{k=0}^{n} a_k z^k$ and a_k satisfies the recurrence relation

$$a_{k+1} = -\frac{[k[(k-1) + 2(n-1) - \gamma] + \Xi_{dch}]a_k + k\delta a_{k-1}}{(n+k)}, \quad k = 0, \dots, n-1.$$

Proof. Let us assume that the eigenfunction of the operator H_{dch}^{QES} be given by

$$\Psi(z) = \mu(z) \sum_{k=0}^{n} a_k z^k.$$

In what follows, the Jacobi tridiagonal matrix of the Lie algebraic operator

$$H_{dch}^{QES} z^k = \left\{ z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} - (z^2 - (\gamma - 2(n-1))z + \delta) \frac{\mathrm{d}}{\mathrm{d}z} - nz + \Xi_{dch}) \right\} z^k$$

$$= -(n+k)z^{k+1} + [k(k-1) - (\gamma - 2(n-1))k + \Xi_{dch}]z^k + k\delta z^{k-1} \tag{3.9}$$

Restricted to $\mathcal{P}_{n+1} = \langle z^k, k = 0, \dots, n \rangle$ is obtained. Here, by equation (3.9) the matrix entries are given by

$$t_{k,k+1} = -(n+k); \quad t_{k,k} = k[(k-1) + 2(n-1) - \gamma] + \Xi_{dch}; \quad t_{k,k-1} = k\delta.$$

It is now necessary to look at the nature of Jacobi tridiagonal matrices, their corresponding eigenvalues as well as eigenfunctions for each case.

Table 1: Values for QES DCHE Tridiagonal Matrix Entries $\tau_{k,k+1}, \tau_{k,k}, \tau_{k,k-1}, k = 0, 1, 2, 3$

k	$\tau_{k,k+1}$	$\tau_{k,k}$	$\tau_{k,k-1}$
0	$\tau_{0,1} = -n$	$\tau_{0,0} = \Xi_{dch}$	$\tau_{0,-1} = 0$
1	$\tau_{1,2} = -(n+1)$	$\tau_{1,1} = 2(n-1) - \gamma + \Xi_{dch}$	$\tau_{1,0} = \delta$
2	$\tau_{2,3} = -(n+2)$	$\tau_{2,2} = 2[2n - 1 - \gamma] + \Xi_{dch}$	$\tau_{2,1} = 2\delta$
3	$\tau_{3,4} = -(n+3)$	$\tau_{3,3} = 3[2n - \gamma] + \Xi_{dch}$	$\tau_{3,2} = 3\delta$

In this case, H_{dch}^{QES} possesses the invariant subspace $\mathcal{P}_1$ spanned by the basis $\{1\}$, thus, the function $\mathcal{P}_0(z) = a_0$. The matrix equation corresponding to a_0 is given in terms of 1×1 matrix $[a_0]$ as $[T_0][a_0] = 0$ This enables one to compute the accessory parameter q. Thus, since $a_0 \neq 0$

$$\Xi_{dch} a_0 = 0 q = -\frac{n}{2}(n - \gamma - 1).$$

By the gauge transformation in equation (3.8), we get the ground state eigenfunction

$$\Psi_0(z) = \mu(z)\mathcal{P}_0(z) = a_0 \mu(z) = a_0 z^{\gamma - 2(n-1)} \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right).$$

In this case, H_{dch}^{QES} possesses the invariant subspace $\mathcal{P}_2$ spanned by the basis $\{1, z\}$, thus, the function $\mathcal{P}_1(z) = a_0 + a_1 z$. The matrix equation corresponding a_0 is given by the 2×2 matrix equation given by

$$T_2 A_2 = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} \\ \tau_{1,0} & \tau_{1,1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \Xi_{dch} & -n \\ \delta & 2(n-1) - \gamma + \Xi_{dch} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0. \quad (3.10)$$

Here the eigenvalue is obtained by setting $\det(T_2) = 0$. Thus, applying Eq. (3.7),

$$q_{\pm} = -\frac{1}{2}(n - \gamma - 2)(n - 1) \mp \frac{1}{2}\sqrt{[2(n-1) - \gamma]^2 - 4n\delta}.$$

By solving equation (3.10), one gets

$$a_1 = -\frac{[\Xi_{dch} + \delta] a_0}{\Xi_{dch} + 2(n-1) - \gamma - n} = \eta_1 a_0. \quad (3.11)$$

Thus, the eigenfunction for this case is

$$\Psi_1(z) = (1 + \eta_1 z) a_0 z^{\gamma - 2(n-1)} \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right)$$

where

$$\eta_1 = -\frac{[\Xi_{dch} + \delta]a_0}{\Xi_{dch} + 2(n-1) - \gamma - n}.$$

In this case, H_{dch}^{QES} possesses the invariant subspace $\mathcal{P}_3$ spanned by the basis $\{1, z, z^2\}$, thus, the function $\mathcal{P}_2(z) = a_0 + a_1 z + a_2 z^2$. The matrix equation corresponding a_0 is given by the 3×3 matrix equation given by

$$T_3 A_3 = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} & \tau_{0,2} \\ \tau_{1,0} & \tau_{1,1} & \tau_{1,2} \\ \tau_{2,0} & \tau_{2,1} & \tau_{2,2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} \Xi_{dch} & -n & 0 \\ \delta & 2(n-1) - \gamma + \Xi_{dch} & -(n+1) \\ 0 & 2\delta & 2(n-1-\gamma) + \Xi_{dch} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0. \qquad (3.12)$$

The energy Ξ_{dch} is obtained by solving $\det(T_3) = 0$ in equation (3.12). The resulting equation is a cubic equation

$$\Xi_{dch}^3 + [2(2n-\gamma) - 3]\Xi_{dch}^2 + [4n^2 - (4n-3)\gamma + \gamma^2 + n\delta + 2]\Xi_{dch} + n(2n - \gamma - 1) = 0. \, (3.13)$$

By standard formula for solving cubic polynomials (cf: Abramowitz and Stegun (1972), S 3.8.2, p.17) $\Xi_{dch}^3 + b_2 \Xi_{dch}^2 + b_1 \Xi_{dch} + b_0 = 0$ has the roots

$$\Xi_{dch}^{(1)} = (s_+ + s_-) - \frac{b_2}{3},$$

$$\Xi_{dch}^{(2)} = -\frac{1}{2}(s_+ + s_-) - \frac{b_2}{3} + i\frac{\sqrt{3}}{2}(s_+ - s_-),$$

$$\Xi_{dch}^{(3)} = -\frac{1}{2}(s_+ + s_-) - \frac{b_2}{3} + i\frac{\sqrt{3}}{2}(s_+ - s_-).$$

where

$$s_{\pm} = [r \pm (t^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$$

$$t = \frac{1}{3}b_1 - \frac{1}{9}b_2^2$$

$$r = \frac{1}{6}(b_1 b_2 - b_0) - \frac{1}{27}b_2^3$$

and

$$b_2 = 2(2n - \gamma) - 3,$$

$$b_1 = 4n^2 - (4n-3)\gamma + \gamma^2 + n\delta + 2,$$

$$b_0 = n(2n - \gamma - 1).$$

Hence, the eigenfunction of DCHE for this case is

$$\Psi_2(z) = (1 + \eta_1 z + \eta_2 z^2) a_0 \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right),$$

where

$$\eta_1 = \frac{\Xi_{dch}}{n}$$

$$\eta_2 = \frac{n\delta + [2(n-1) - \gamma + 2\delta + \Xi_{dch}]\Xi_{dch}}{n[2(n-\gamma-1) - (n+1) + \Xi_{dch}]}.$$

In this case, the operator H_{dch}^{QES} has a finite-dimensional invariant subspace $\mathcal{P}_{n+1}$, which is spanned by the basis $\{z^k | k = 0,1,2, \dots, n\}$. By the finite polynomial

$$\mathcal{P}_n(z) = \sum_{k=0}^{n} a_k z^k$$

it is possible to obtain the tridiagonal Jacobi matrix equation

$$T_{n+1} A_{n+1} = 0, \tag{3.14}$$

where T_{n+1} and A_{n+1} are explicitly is given by

$$T_{n+1} = \begin{pmatrix} \Xi_{dch} & -n & 0 & 0 & \dots & 0 \\ \delta & 2(n-1) - \gamma + \Xi_{dch} & -(n+1) & 0 & \dots & \vdots \\ 0 & 2\delta & 2(n-1-\gamma) + \Xi_{dch} & -(n+2) & \dots & \\ 0 & 0 & 3\delta & 3[2n-\gamma] + \Xi_{dch} & \ddots & \\ \vdots & 0 & 0 & 4\delta & \ddots & 0 \\ 0 & \dots & \vdots & \ddots & \tau_{n-1,n-1} & \tau_{n-1,n} \\ 0 & \dots & & 0 & \tau_{n,n-1} & \tau_{n,n} \end{pmatrix}$$

and

$$A_{n+1} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \\ \\ a_{n-1} \\ a_n \end{pmatrix}$$

It is known (see [2], Example 3.2, p.25) that

$$D_{n+1} = \det(T_{n+1}) = 0, \tag{3.15}$$

where

$$D_{n+1} = \tau_{n,n} D_n + \tau_{n,n-1}\tau_{n-1,n} D_{n-1} \tag{3.16}$$

with initial conditions

$$D_{-1} = 0, \qquad D_0 = 1, \qquad D_1 = \Xi_{dch}. \tag{3.17}$$

The associated characteristic equation to (3.15) is given by

$$\lambda^2 - (3n - \gamma - 3 + \Xi_{dch})\lambda + n(2n-1)\delta = 0 \tag{3.18}$$

and in most physical applications (3.18) has two real distinct roots

$$\lambda = \frac{(3n - \gamma - 3 + \Xi_{dch}) \pm \sqrt{(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta}}{2}$$

since the discriminant $(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta > 0$. Hence, the general solution is

$$D_{n+1} = k_1 \left[\frac{(3n - \gamma - 3 + \Xi_{dch}) + \sqrt{(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta}}{2}\right]^{n+1}$$

$$+ k_2 \left[\frac{(3n - \gamma - 3 + \Xi_{dch}) - \sqrt{(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta}}{2}\right]^{n+1}.$$

We note here that $\Xi_{dch} = \Xi_{dch}^{(n)} = -\frac{n}{2}(n - \gamma - 1) - q$ so that

$$\Xi_{dch}^{(-1)} = -\frac{\gamma+2}{2} - q; \Xi_{dch}^{(0)} = -q; \Xi_{dch}^{(1)} = \frac{\gamma}{2} - q; \Xi_{dch}^{(2)} = \gamma - q - 1.$$

By using the initial conditions in equation (3.16), one gets

$$k_1 = \frac{\Xi_{dch}^{(0)}}{\Xi_{dch}^{(0)} - (\gamma+3)} = \frac{q}{q + \gamma + 3} \tag{3.19}$$

and

$$k_2 = 1 - \frac{q}{q+\gamma+3} = \frac{\gamma+3}{q+\gamma+3}. \tag{3.20}$$

By re-substituting equations (3.19) and (3.20) into (3.18) one gets

$$D_{n+1} = \frac{q}{q+\gamma+3}\left[\frac{(3n - \gamma - 3 + \Xi_{dch}) + \sqrt{(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta}}{2}\right]^{n+1}$$

$$+ \left(\frac{\gamma+3}{q+\gamma+3}\right)\left[\frac{(3n - \gamma - 3 + \Xi_{dch}) - \sqrt{(3n - \gamma - 3 + \Xi_{dch})^2 - 4n(2n-1)\delta}}{2}\right]^{n+1}.$$

By applying equation (3.19) to equation (3.14) one solves for the accessory parameter q in $(n+1)$ −times, in this case to get q. The eigenfunction of DCHE in the general case in terms of polynomial $\mathcal{P}_n(z) = \sum_{k=0}^{n} a_k z^k$ is thus given as

$$\Psi(z) = \mathcal{P}_n(z) z^{\gamma-2(n-1)} \exp\left(-\frac{z}{2} - \frac{\delta}{z}\right).$$

where a_k is satisfies the recurrence relation

$$-(n+k)a_{k+1} + [k[(k-1) + 2(n-1) - \gamma] + \Xi_{dch}]a_k + k\delta a_{k-1} = 0$$

and hence

$$a_{k+1} = -\frac{[k[(k-1) + 2(n-1) - \gamma] + \Xi_{dch}]a_k + k\delta a_{k-1}}{(n+k)}, \quad k \in \{0,1,2, \ldots, n\}.$$

Now let

$$\upsilon_k \equiv \upsilon_k(\Xi_{dch}) = k[(k-1) + 2(n-1) - \gamma] + \Xi_{dch}.$$

Thus, we obtain the coefficients

$$a_1 = -\frac{\Xi_{dch} a_0}{n} = -\frac{\upsilon_0}{n} a_0$$

$$a_2 = -\frac{[2(2n-1) - \gamma + \Xi_{dch}]\Xi_{dch} + n\delta}{n(n+1)} a_0 = \left(\frac{\upsilon_0 \upsilon_1}{n(n+1)} - \frac{\delta}{n+1}\right) a_0$$

$$a_3 = \left(-\frac{\upsilon_0 \upsilon_1 \upsilon_2}{n(n+1)(n+2)} + \frac{\delta \upsilon_2}{(n+1)(n+2)} + \frac{2\delta \upsilon_0}{n(n+2)}\right) a_0 \ldots$$

The pattern of nested products and summations grows with each step. In general, a_k can be expressed as $a_k = C_k a_0$ where C_k is a coefficient that depends on $n, \delta,$ and the values of υ_j for $j < k$. Therefore,

$$a_n = \left(-\frac{\upsilon_{n-1}}{2n-1} C_{n-1} - \frac{(n-1)\delta}{2n-1} C_{n-2}\right) a_0$$

where C_{n-1} and C_{n-2} are the complex coefficients found from the previous iterative steps.

CONCLUSION

In this work, we have presented a new algebraization of the DCHE by writing it as the linear combination of quadratic elements in the universal enveloping algebra of $sl(2,\mathbb{C})$. The presence of the term in positive grading in the double confluent Heun Hamiltonian is an indication that H_{dch}^{QES} is only quasi-exactly solvable. Eigenfunctions and eigenvalues of DCHE have been obtained using canonical monomials that generate block triangular matrix. The novelty in this work is clearly stated in Remark 3.1 above.

In further research we shall study the Lie algebraization of other related linear and nonlinear Fuchsian equations and physical models that have practical implications on science and technology.

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