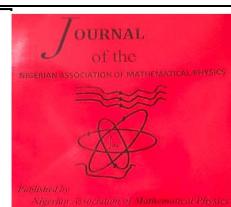


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## OPTIMIZATION OF TWO HYBRID POINTS IN A ONE STEP METHOD

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### ABSTRACT

Hybrid block methods have recently gained prominence in the numerical solution of first-order differential equations due to their enhanced stability and accuracy properties. This paper introduces an optimized one-step method that incorporates two hybrid points to improve computational efficiency and precision. By applying interpolation and collocation techniques, the block method is constructed by using a variable placeholder in place of a conventional fractional hybrid point, providing flexibility in optimizing the scheme. The optimal values of the hybrid points are obtained by equating the main method of the block to zero and solving the resulting equation. The derived method is of order four and is rigorously analyzed to be consistent, zero-stable, convergent, and A-stable, ensuring both reliability and long-term accuracy in practical computations. Results demonstrate that the proposed scheme consistently outperforms existing methods in the literature, including some of higher order, particularly in terms of accuracy and computational efficiency.

### 1. INTRODUCTION

Numerical methods are employed to obtain approximate solutions for equations that are analytically intractable. A differential equation, together with its initial or boundary conditions, is referred to as a differential problem. Some differential problems lack closed form analytical solutions, necessitating the use of numerical techniques to approximate their solutions. However, numerical methods introduce errors, which are defined as the difference between the analytical solution and the computed numerical approximation. Linear multistep methods (LMM) are commonly used numerical techniques for solving differential equations. Block methods for solving ordinary differential equations were introduced by Lambert <sup>1</sup>. A variation of these methods, known as hybrid block linear multistep methods (HBLMM), integrates additional points that are not part of the main grid points to enhance accuracy and robustness.

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The inclusion of hybrid points contributes to improved solution precision <sup>2-5</sup>. The optimization of three hybrid points in a single-step method by randomly choosing values for its hybrid points was carried out <sup>6</sup>. Oyelami optimized a second-derivative two-step method for solving first-order ordinary differential equations using Bhaskara points <sup>7</sup>. Several researchers have contributed significantly to the advancement of hybrid and block methods for solving differential equations. Different researchers have employed various hybrid points in their studies, each influencing the overall accuracy of the method <sup>8-11</sup>. Kuboye proposed a one-step second derivative block intra-step method tailored for stiff systems of ordinary differential equations, enhancing accuracy and stability <sup>12</sup>. Rageh introduced hybrid shifted orthonormal polynomial techniques for fractional differential equations, demonstrating improved approximation efficiency <sup>13</sup>. Ranocha developed multiderivative time integration methods that preserve nonlinear functionals via a relaxation approach, contributing to the field of structure-preserving algorithms <sup>14</sup>. Raymond constructed a four-step one hybrid block method specifically for solving fourth-order differential equations <sup>15</sup>. Adeyefa formulated an accurate four-step hybrid block method aimed at solving higher-order initial value problems, showcasing improved convergence properties <sup>16</sup>.

This paper focuses on the optimization of two hybrid points in a one-step method for improved accuracy when solving first order ordinary differential equations. The formulation of the method employs interpolation and collocation techniques, where the hybrid points are treated as variables. The block methods are then developed with respect to these variables. Subsequently, the error term of one of the main methods in the block is analyzed, leading to the determination of the optimized hybrid points, which are incorporate into the main block method. The evaluation of the proposed method covers key properties such as zero-stability, consistency, order of accuracy, convergence, and the region of stability.

### 1.1 Order and Error Constant

The order of a first order differential equation can be determined by using the linear operator considered as follows:

$$L[y(x_n); h] = \sum_i [\alpha_i y(x_n + ih) - h\gamma_i f(x_n + ih)] \tag{1.}$$

The function  $y(x)$  is an arbitrary test function that is continuously differentiable in the interval  $[a, b]$ . Expanding  $y(x_n + ih)$  and  $f(x_n + ih)$  in Taylors series about  $x_n$  and factoring the coefficients of  $h$ , gives

$$L[y(x_n); h] = C_0 y(x_n) + C_1 h y(x_n) + C_2 h^2 y^2(x_n) + \dots + C_p h^p y^p(x_n) + \dots \tag{2.}$$

Where  $c_i, i = 0,1,2, \dots$  are vectors.

$$\left. \begin{aligned} C_0 &= \sum_{j=0}^k \alpha_j, \\ C_1 &= \sum_{j=1}^k j\alpha_j - \sum_{j=0}^k j\beta_j \\ C_2 &= \frac{1}{2} \sum_{j=1}^k j^2\alpha_j - \sum_{j=1}^k j\beta_j \\ &\dots \\ C_q &= \frac{1}{q!} \sum_{j=1}^k j^q\alpha_j - \frac{1}{(q-1)!} \sum_{j=1}^k j^{q-1}\beta_j \end{aligned} \right\} \tag{3.}$$

A method is of order  $p$  if  $C_1 = C_2 = \dots = C_p = 0, C_{p+1} \neq 0$ . The nonzero coefficient  $C_{p+1}$  is referred to as the error constant.

**2. Derivation of the method**

The one step method with two hybrid points is derived using two hybrid points  $x_T$ , and  $x_{1-T}$  for solving first order ordinary differential equations. Where  $T > 0$  and  $T < 1$ , the approximate solution of an ordinary differential equation can be expressed by using a basis function of the form:

$$y(x) = \sum_{j=0}^4 a_j x^j \tag{4}$$

Differentiating Eq. (4) gives

$$y'(x) = f(x) = \sum_{j=1}^3 j a_j x^{j-1} \tag{5}$$

interpolating Eq. (4) at the points  $x_{n+1}$  and collocating Eq. (5) at the points  $x_{n+j}$ , where  $j = 0, T, 1 - T, 1$  yield the system of linear equations below

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+T} & 3x_{n+T}^2 & 4x_{n+T}^3 \\ 0 & 1 & 2x_{n+(1-T)} & 3x_{n+(1-T)}^2 & 4x_{n+(1-T)}^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ f_n \\ f_{n+T} \\ f_{n+(1-T)} \\ f_1 \end{bmatrix} \tag{6}$$

The  $a_i$ 's  $j = 0,1,2,3,4$  are obtained using the matrix inversion technique to solve Eq.(6).

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+T} & 3x_{n+T}^2 & 4x_{n+T}^3 \\ 0 & 1 & 2x_{n+(1-T)} & 3x_{n+(1-T)}^2 & 4x_{n+(1-T)}^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 3x_{n+1}^3 \end{bmatrix}^{-1} \begin{bmatrix} y_{n+1} \\ f_n \\ f_{n+T} \\ f_{n+(1-T)} \\ f_1 \end{bmatrix}$$

These  $a_i$  values are then substituted into Eq. (4) to derive the continuous method. Evaluating the continuous method at  $x = 0, T$  and  $1 - T$  results in the following hybrid block method:

$$y_n = \frac{h(-6T^2 + 6T - 1)f_{n+1}}{12T(-1+T)} + \frac{hf_{n+1-T}}{12T(-1+T)} + \frac{hf_{n+T}}{12T(-1+T)} + \frac{h(-6T^2 + 6T - 1)f_n}{12T(-1+T)} + y_{n+1} \tag{7}$$

$$y_{n+T} = y_{n+1} + \frac{h(6T^4 - T^3 - 11T^2 + 7T - 1)f_{n+1}}{24T^2 - 12T} + \frac{h(T^3 - T^2 - T + 1)f_{n+1-T}}{24T^2 - 12T} + \frac{h(7T^3 - 7T^2 - T + 1)f_{n+T}}{24T^2 - 12T} + \frac{h(-6T^4 + 17T^3 - 17T^2 + 7T - 1)f_n}{24T^2 - 12T} \tag{8}$$

$$y_{n+1-T} = \frac{(6T^4 - 23T^3 + 22T^2 - 6T)f_{n+1}}{24T^2 - 36T + 12} + \frac{(-7T^3 + 14T^2 - 6T)f_{n+1-T}}{24T^2 - 36T + 12} + \frac{(-T^3 + 2T^2)f_{n+T}}{24T^2 - 36T + 12} + \frac{(-6T^4 + 7T^3 - 2T^2)f_n}{24T^2 - 36T + 12} + y_{n+1} \tag{9}$$

In order to obtain an optimal value for  $T$ , the local truncation error in the formula Eq. (7) is equated to zero. Eq. (7) is chosen because it contains both  $y_n$  and  $y_{n+1}$  which are the main grid points.  $y_n$  is the initial condition and  $y_{n+1}$  is the next step forward. To obtain the optimized points the local truncation error of  $y_n$  is obtained.

$$L(y(x_n); h) = \left(-\frac{1}{144}T^2 + \frac{1}{144}T - \frac{1}{720}\right)h^5y^5(x_n) + \left(-\frac{1}{288}T^2 + \frac{1}{288}T - \frac{1}{1440}\right)h^6y^6(x_n) + \mathcal{O}(h^7) \quad (10)$$

Having obtained Eq. (10) the coefficient of  $h^5y^5(x_n)$  is equated zero to obtain the optimized points.

$$-\frac{1}{144}T^2 + \frac{1}{144}T - \frac{1}{720} = 0 \quad (11)$$

The solution to Eq. (11) are:

$$T = \frac{1}{2} - \frac{\sqrt{5}}{10} \text{ and } 1 - T = \frac{1}{2} + \frac{\sqrt{5}}{10} \quad (12)$$

The importance of Eq. (12) lies in the fact that it sets the coefficient of  $h^5y^5(x_n)$  in Eq. (10) to zero, thereby increasing the order of the method.

Substituting  $T$  into Eq (4-6) gives the main block methods:

$$y_n = y_{n+1} + \frac{\left(-f_n - f_{n+1} - 5f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} - 5f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}}\right)h}{12} \quad (13)$$

$$y_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} = y_{n+1} + \frac{\left(-(4\sqrt{5} - 15)f_{n+1} + (15 + 10\sqrt{5})f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} + (30 - 5\sqrt{5})f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} - \sqrt{5}f_n\right)h}{(-150 + 30\sqrt{5})} \quad (14)$$

$$y_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} = y_{n+1} + \frac{(-5 + \sqrt{5})(19 + 7\sqrt{5})f_{n+1} + (-5\sqrt{5} + 5)(11\sqrt{5} + 35)f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} + (-5 + \sqrt{5})^2(15\sqrt{5} + 5)f_{n+\frac{1}{2}+\frac{\sqrt{5}}{10}} - \left(\frac{5}{3} + \sqrt{5}\right)(-45 + 9\sqrt{5})^2f_n}{(600 + 120\sqrt{5})} \quad (15)$$

Eq. (13), (14) and (15) are the methods that form the optimised block method.

### 3. Analysis of the methods

#### 3.1 Order and Error Contant

After applying Eq. (3) to Eq. (13), Eq. (14) and Eq. (15), it is obtained that the methods are of order 6, 4, and 4 respectively with error constants:

$$\frac{1}{1512000}, \frac{1}{1500(-5+\sqrt{5})(\sqrt{5}+1)} \text{ and } \frac{-1}{1500(-5+\sqrt{5})(\sqrt{5}+1)}$$

### 3.2 Zero Stability

The one-step block methods incorporating two hybrid points can be generally expressed in the form of a matrix difference equation as follows:

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h[B^{(0)}F_{w-1} + B^{(1)}F_w] \tag{16}$$

And the matrices  $A^{(1)}, A^{(0)}, B^{(0)}$  and  $B^{(1)}$  are matrices whose entries correspond to the coefficients of the methods, with the first characteristic polynomial defined as

$$\rho(\lambda) = |\lambda A^{(1)} - A^{(0)}| = -\lambda^2(\lambda - 1) \tag{17}$$

This implies that its roots are  $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1$ . The zero-stability of the proposed one-step block hybrid method is confirmed because the condition  $\rho(R) = 0$  holds, with all roots  $\lambda_j$ , for  $j = 1, 2, 3$ , satisfying  $|\lambda_j| \leq 1$  and none of the roots having multiplicity greater than one.

### 3.3 Consistency

In order to establish the consistency of the block method the three following conditions has to meet.

1. The order of accuracy  $p \geq 1$
2.  $\sum_{j=0}^k \alpha_j = 0$
3.  $\rho'(1) = \sigma(1)$ ,  $\rho(r)$  is the first characteristic polynomial and  $\sigma(r)$  is second characteristic polynomials of the block.

Condition (1) has been met, the order of the methods is greater than 1. Condition (2) is also satisfied since  $C_0 = \sum_{j=0}^k \alpha_j = 0$

To verify condition (3), the characteristic polynomials are evaluated, yielding the parameters presented in Table 1 for each method. Based on the values in Table 1, condition (3) is confirmed to be satisfied.

**Table 1: Parameters for Determining Consistency when  $m = 2$**

Equation	Order p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
7	6	0	-1	-1
8	4	0	$\frac{1}{2} - \frac{\sqrt{5}}{10}$	$\frac{1}{2} - \frac{\sqrt{5}}{10}$
9	4	0	$\frac{1}{2} + \frac{\sqrt{5}}{10}$	$\frac{1}{2} + \frac{\sqrt{5}}{10}$

### 3.4 Convergence

A linear multistep method is convergent if and only if it is both consistent and zero-stable<sup>17</sup>. Based on this theorem, all the block methods developed are therefore convergent.

### 3.5 Region of Absolute Stability

The numerical schemes can be reduced into one-step methods of the form

$$y_{n+1} = \phi(h)y_n \tag{18}$$

By solving for  $y_{n+1}$  in the matrix equations methods. To study the stability of the resulting scheme it is straightforward to consider stability using the linear test equation

$$\dot{y} = \lambda y \tag{19}$$

Where  $\lambda$  is a complex constant. The stability polynomial for the proposed block method is

$$\lambda_k = \frac{z^3 + 12z^2 + 60z + 120}{z^3 - 12z^2 + 60z - 120} \tag{20}$$

Fig 1 illustrates the stability region of the proposed method, derived from the stability polynomial given in Eq. (20). The region indicates that the method is A-stable, as it encompasses the entire left half of the complex plane.

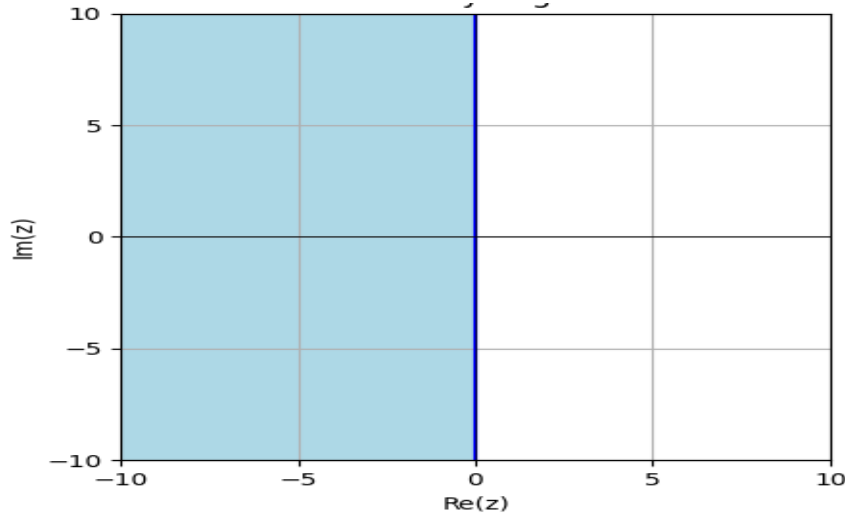


Figure 1: Stability region of the proposed method

#### 4.0 Numerical Results

This section presents several examples to demonstrate the efficiency of the proposed method. For comparison, the results are evaluated alongside those obtained from existing methods in the literature, the same step size  $h$  used by the referenced researchers is adopted for consistency.

##### Example 1.

Consider the first order differential equation

$$y'(t) = \frac{1}{2}(1 - y(t)), y(0) = \frac{1}{2}, t \in [0,1] \tag{21}$$

The exact solution is given by:

$$y = 1 - \frac{1}{2}e^{-\frac{1}{2}t} \tag{22}$$

Table 2: Errors for Example 1

$t$	Error in Proposed Method	Error in <sup>17</sup>	Error in <sup>20</sup>	Error in <sup>18</sup>
0.0	0	0	0	0
0.5	3.68 E-15	6.10E-14	1.17E-14	6.78013E-13
1.0	7.013E-15	3.19E-14	3.26E-14	6.35936E-13

1.5	1.00E-14	1.19E-13	4.65E-14	6.38045E-13
2.0	1.26E-14	2.77E-13	5.90E-14	1.18994E-12
2.5	1.50E-14	4.30E-13	7.01E-14	1.12410E-12
3.0	1.72E-14	5.58E-13	8.01E-14	1.09901E-12
3.5	1.91E-14	6.77E-13	8.89E-14	1.54798E-12
4.0	2.07E-14	7.80E-13	9.66E-14	1.46805E-12
4.5	2.22E-14	8.68E-13	1.03E-14	1.41909E-12
5.0	2.35E-14	9.59E-13	1.09E-14	1.78202E-12

**Example 2.**

Consider the first order differential equation

$$y' = -10(y - 1)^2, y(0) = 2, t \in [0,0.1] \tag{23}$$

The exact solution is given by:

$$y = 1 + \frac{1}{1+10t} \tag{24}$$

The errors for this example 2 are displayed in Table 3.

**Table 3: Errors for Example 2**

$t$	Error in Proposed Method	Error in <sup>6</sup>	Error in <sup>19</sup>	Error in <sup>20</sup>
0	0	0	0	0
0.01	3.42E-16	1.16E-10	2.82E-07	1.55E-06
0.02	4.03E-16	2.14E-10	4.04E-07	2.39E-06
0.03	3.86E-16	2.22E-10	4.47E-07	2.83E-06
0.04	3.50E-16	2.14E-10	4.50E-07	3.02E-06
0.05	3.12E-16	1.99E-10	4.35E-07	3.06E-06
0.06	2.77E-16	1.82E-10	4.11E-07	3.03E-06
0.07	2.47E-16	1.65E-10	3.84E-07	2.95E-06
0.08	2.21E-16	1.50E-10	3.57E-07	2.84E-06
0.09	1.99E-16	1.37E-10	3.30E-07	2.72E-06
0.1	1.80E-16	1.25E-10	3.05E-07	2.59E-06

**Example 3.**

Consider the following first order differential equation:

$$y' = ty, y(0) = 1, t \in [0,1] \tag{25}$$

The exact solution is

$$y = e^{\frac{t^2}{2}} \tag{26}$$

The errors for this example 3 are displayed in Table 4.

**Table 4: Errors for Example 3**

$t$	<b>Error in Proposed Method</b>	Error in <sup>9</sup>	Error in <sup>21</sup>
0	0	0	0
0.1	1.05E-12	1.16E-9	3.36E-10
0.2	4.28E-12	4.77E-9	1.27E-09
0.3	9.94E-12	1.12E-8	2.28E-09
0.4	1.85E-11	2.11E-8	1.38E-09
0.5	3.05E-11	3.55E-8	5.23E-09
0.6	4.69E-11	5.59E-8	2.40E-08
0.7	6.87E-11	8.45E-8	6.53E-08
0.8	9.72E-11	1.24E-7	1.45E-08
0.9	1.34E-10	1.80E-7	2.91E-07
1	1.80E-10	2.58E-7	5.43E-07

**Example 4**

The chaotic system known as the Arneodo-Coulet system is defined by the following set of equations:

$$y_1' = y_2 \tag{27}$$

$$y_2' = y_3 \tag{28}$$

$$y_3' = ay_1 - by_2 - y_3 - y_1^3 \tag{29}$$

With initial conditions  $y_1(0) = 0.21, y_2(0) = 0.22$  and  $y_3(0) = 0.61$ . The constants are  $a = 5, b = 3.8$ .

The solution of example 4 is displayed in the Figures below. The solution agrees with those found in literature <sup>21</sup>. Fig 2 shown the plot of  $y_1, y_2$  and  $y_3$  against  $t$

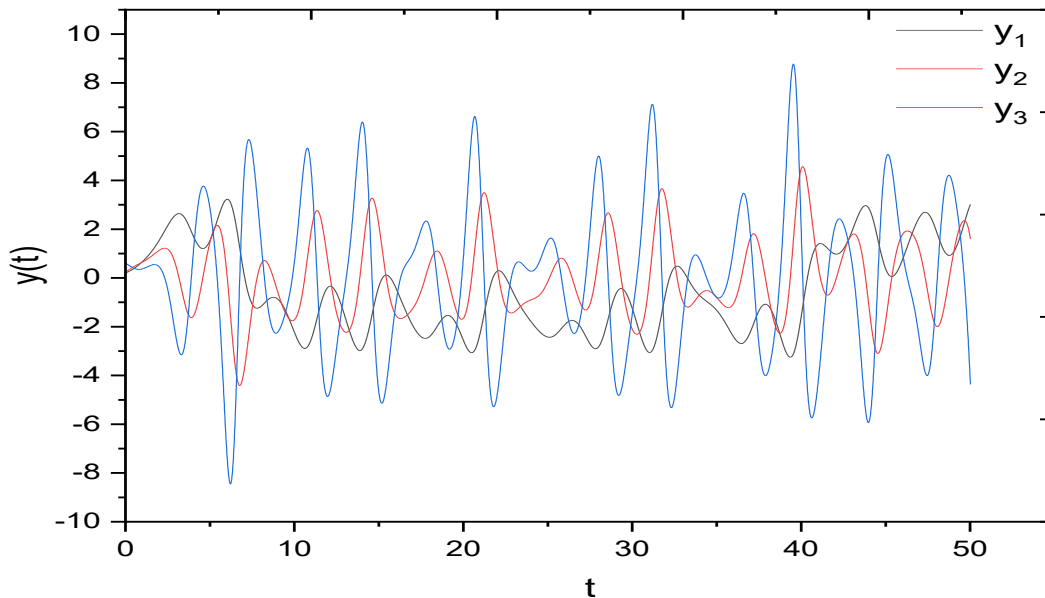


Figure 2:  $y(t)$  against  $t$

Fig 3 shows the phase portrait is a graphical representation of the system's dynamics in the  $y_1 - y_2$  plane. The complex structure in the phase portrait, characterized by folded loops and spiralling trajectories, is a hallmark of chaotic systems.

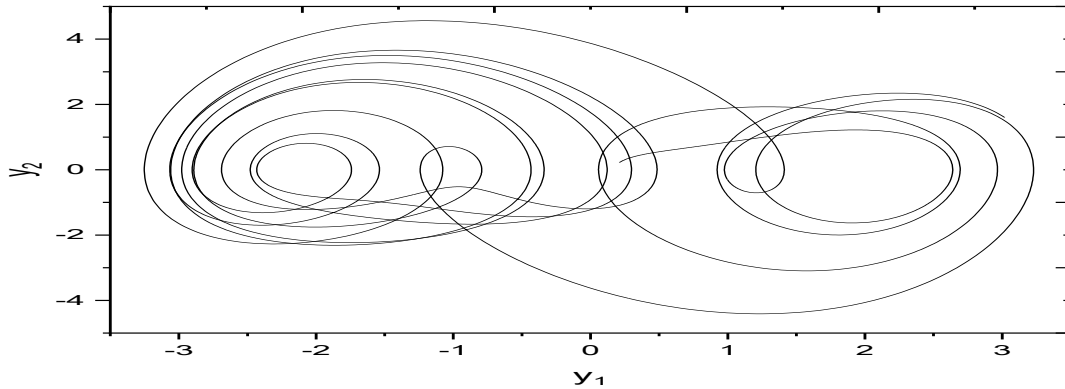


Figure 3: Phase portrait of  $y_2$  against  $y_1$

Fig 4 shows the plot of  $y_3$  against  $y_1$ . The trajectories in the phase portrait exhibit sensitive dependence on initial conditions, meaning that slight differences in the starting point can lead to dramatically different long-term behaviour.

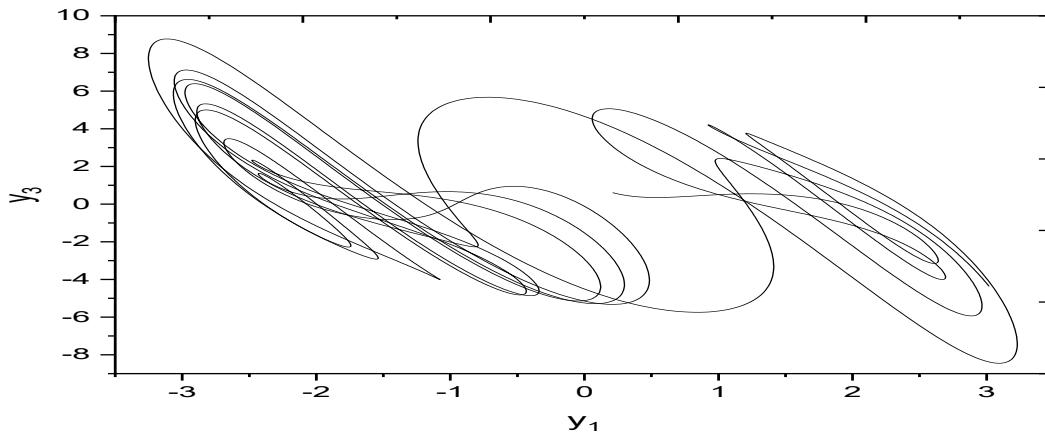


Figure 4: Phase portrait of  $y_3$  against  $y_1$

Fig 5 shows the plot of  $y_3$  against  $y_2$ .

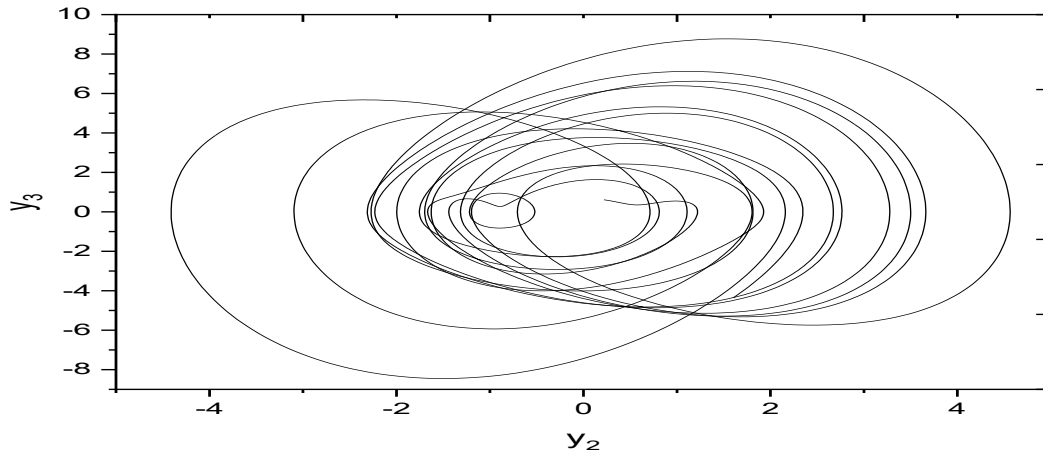


Figure 5: Phase portrait of  $y_3$  against  $y_2$

Fig 6 shows the plot of  $y_3$  against  $y_2$  against  $y_1$ . The phase space in Fig 6 is a three-dimensional space where each axis represents one of the variables in the system ( $y_1, y_2$ , and  $y_3$ ).

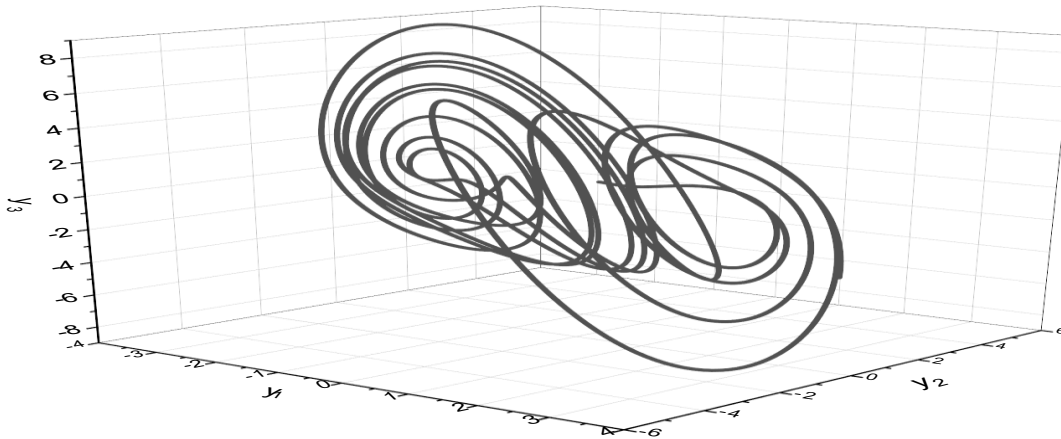


Figure 6: Phase portrait of example 4

The phase space in Fig 6 is a three-dimensional space where each axis represents one of the variables in the system ( $y_1, y_2$ , and  $y_3$ ). The plot shows the complex, folded structure known as the Arneodo-Coulet attractor. The trajectories in the phase space exhibit sensitive dependence on initial conditions, meaning that slight differences in the starting point can lead to dramatically different long-term behaviour.

### Discussion of Results

The discussion highlights the superior accuracy and robustness of the proposed method. Error analysis from Examples 1 to 3 shows consistently lower errors when compared to existing methods. In Example 4, the chaotic Arneodo-Coulet system is effectively analysed using phase portraits and time-series plots, confirming the method's reliability. Example 4 agrees with the results obtained in the literature. Additionally, all the methods we compared with are of higher order, this demonstrate the accuracy and the efficiency of the proposed block methods. The results demonstrate that the proposed approach is highly accurate and well-suited for high-precision numerical simulations.

### CONCLUSION

This paper explored the optimization of a one-step method incorporating two hybrid points. The block method was formulated using interpolation and collocation technique, where a variable was introduced as a placeholder in place of a fraction to construct the block methods. The optimized hybrid points were obtained by setting the local truncation error containing the coefficients of the main method to zero and solving for the solution. The solution is used as the hybrid points for the method. The solution was then substituted into the continuous method to derive the block method. It is worthy to note the optimised hybrid points are exactly the Gauss-Legendre quadrature nodes on  $[0,1]$  (shifted). The resulting block method is of order 4 and is proven to be consistent, zero-stable, and A-stable. The findings show that the proposed method outperforms the benchmark methods, even though they are of higher order. Numerical results further confirm its efficiency and accuracy compared to other methods in the literature.

References

- [1] Lambert JD, Watson IA. Symmetric multistep methods for periodic initial value problems. *IMA Journal of Applied Mathematics*. 1976;**18**(2):189–202.
- [2] Ramos H, Rufai MA. A two-step hybrid block method with fourth derivatives for solving third-order boundary value problems. *Journal of Computational and Applied Mathematics*. 2022;**404**:113419.
- [3] Adeyefa EO, James AA, Olagunju AS, Joseph FL. Development of a block method for solving multiple order ODEs. *International Journal of Mathematics in Operational Research*. 2024;**29**(1):51–65.
- [4] Abolarin OE, Ogunware BG. New hybrid method for direct numerical solution of nonlinear second, third and fourth orders ordinary differential equations. *International Journal of Mathematics in Operational Research*. 2022;**23**(3):285–315.
- [5] Emmanuel A, Adeyemi SO. Hybrid block method for direct integration of first, second, and third order IVPs. *Cankaya University Journal of Humanities and Social Sciences*. 2021;**18**(1):1–8.
- [6] Kashkari BSH, Syam MI. Optimization of one-step block method with three hybrid points for solving first-order ordinary differential equations. *Results in Physics*. 2019;**12**:592–6.
- [7] Oyelami OA, Mohammed U, Mohammed J, Nyor N. Development and application of multi-derivative hybrid block methods for solving nonlinear first-order ordinary differential equations. *Transactions of the Nigerian Association of Mathematical Physics*. 2025;**21**:13–28.
- [8] Gottlieb S, Grant ZJ, Hu J, Shu R. High order strong stability preserving multi-derivative implicit and IMEX Runge–Kutta methods with asymptotic preserving properties. *Applied Numerical Mathematics*. 2022;**182**:351–379. <https://doi.org/10.1016/j.apnum.2022.06.002>
- [9] Hussen I, Mebrate B. Semi-implicit scheme of Fisher equation based on Crank-Nicolson method and method of lagging. *Palestina Journal of Mathematics*. 2022;**11**(3):536–48.
- [10] Kuboye JO, Adeyefa EO. New developed numerical formula for solution of first and higher order ordinary differential equations. *Journal of Interdisciplinary Mathematics*. 2021;**25**(2):395–414.
- [11] Pirmohabbati P, Refahi Sheikhani AH, Abdolazadeh Ziabari A. Numerical Solution of Nonlinear Fractional Bratu Equation with Hybrid Method. *International Journal of Applied and Computational Mathematics*. 2020;**6**:162.
- [12] Mohammed U, Garba J, Semenov ME. One-step second derivative block intra-step method for stiff system of ordinary differential equations. *Journal of the Nigerian Mathematical Society*. 2021;**40**(1):47–57.
- [13] Rageh AAM, Hadhoud AR. Hybrid shifted orthonormal polynomial methods for fractional differential equations. *Mathematical Methods in the Applied Sciences*. 2024;**2024**:1–18.

- [14] Ranocha H, Schütz J. Multiderivative time integration methods preserving nonlinear functionals via relaxation [Preprint]. arXiv:2311.03883. 2024.
- [15] Raymond D, Skawme Y, Adiku L. Four-step one hybrid block methods for solution of fourth derivative ordinary differential equations. *Journal of Advances in Mathematics and Computer Science*. 2021;**36**(3):1–10.
- [16] Adeyefa EO, Olanegan OO. Accurate four-step hybrid block method for solving higher-order initial value problems. *Baghdad Science Journal*. 2022;**19**(4):78–87.
- [17] Soomro H, Daud H, Zainuddin N. Convergence of the 3-point block backward differentiation formulas with off-step point for stiff ODEs. *Journal of Physics: Conference Series*. 2021;**1943**(1):012137.
- [18] Soomro H, Zainuddin N, Daud H, Sunday J. Optimized hybrid block Adams method for solving first order ordinary differential equations. *Computers, Materials & Continua*. 2022;**72**(2):2947–2961.
- [19] Fotta AU, Alabi TJ. Block method with one hybrid point for the solution of first-order initial value problems of ordinary differential equations. *International Journal of Pure and Applied Mathematics*. 2015;**103**(3):511–21.
- [20] Rufai MA, Duromola MK, Ganiyu AA. Derivation of one-sixth hybrid block method for solving general first-order ordinary differential equations. *IOSR Journal of Mathematics*. 2016;**12**(5):20–7.
- [21] Abdelrahim R, Omar Z, Kuboye JO. New Hybrid Block Method with Three Off-Step Points for Solving First Order Ordinary Differential Equations. *American Journal of Applied Sciences*. 2016;**13**(2):209–212.