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Numerical Stability in Second-Order Approximations of the Caputo Fractional Derivative

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ABSTRACT

This paper develops and analyses a second-order numerical approximation for the Caputo fractional derivative based on a modified product trapezoidal integration framework. The proposed scheme is derived from the fractional integral representation of the Caputo operator and expressed as a discrete convolution formula involving integer-order derivatives. A rigorous error analysis establishes an $O(h^2)$ convergence rate under appropriate smoothness assumptions. Numerical experiments are performed using the test function $f(x) = \sin(x)$ for fractional orders $\beta = 0.1$, 0.5, 1.0, 1.1, 1.5, and 2.0 on progressively refined meshes. The computed results demonstrate excellent agreement with exact solutions and confirm near second-order convergence for non-integer orders. For the integer cases $\beta = 1$ and $\beta = 2$, the method reproduces the corresponding classical derivatives to machine precision, verifying consistency with standard calculus. The scheme is shown to be accurate, stable, and efficient, making it suitable for the numerical solution of fractional differential equations arising in science and engineering.

1. Introduction

Fractional calculus extends the classical concepts of differentiation and integration to arbitrary real and complex orders. Since its origin in the correspondence between Leibniz and L'Hôpital in 1695, the subject has evolved into a well-established branch of mathematical analysis with numerous theoretical developments and practical applications [1, 2, 3, 4]. Unlike classical integer-order operators, fractional derivatives naturally incorporate memory and hereditary effects, making them particularly suitable for modelling complex phenomena in viscoelasticity, anomalous diffusion, control systems, signal processing, bioengineering, and many other areas of science and engineering [5, 6, 7, 8, 9].

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As a practical application to real-world problems, Agbata et al. [10] developed a novel fractional order mathematical model for investigating the transmission dynamics of typhoid fever. Their model incorporates memory effects and complex transmission mechanisms that conventional integer-order models are unable to adequately capture. Comprehensive treatments of both the theory and applications of fractional calculus can be found in standard monographs such as Miller and Ross [2], Oldham and Spanier [11], Samko et al. [3], and Podlubny [4].
The increasing use of fractional differential equations in mathematical modelling has stimulated considerable research into efficient analytical and numerical methods for fractional operators. While transform techniques such as the Laplace and Fourier transforms remain effective for certain classes of linear fractional problems [2, 12], most practical applications require reliable numerical approximations. Consequently, a wide variety of computational methods have been developed, including finite difference schemes [13, 14], fractional linear multistep methods [15, 16], convolution quadrature techniques [17], and product integration approaches [18, 20, 21]. The design of numerical schemes that simultaneously achieve high accuracy, stability, and computational efficiency remains an active area of research.
Among the many definitions of fractional derivatives, the Caputo derivative is one of the most widely used in applications because it permits the use of classical integer-order initial conditions and therefore possesses a natural physical interpretation [22]. For this reason, Caputo-type models appear extensively in fractional differential equations arising in engineering, physics, and applied mathematics. The numerical approximation of the Caputo derivative has therefore attracted significant attention, particularly in the development of discretization techniques capable of accurately capturing the nonlocal memory effects inherent in fractional operators.
Product integration methods provide a particularly attractive framework for the numerical treatment of fractional operators because they are specifically designed to accommodate weakly singular kernels. By combining suitable quadrature rules with fractional integral representations, such methods often yield accurate and stable approximations while maintaining relatively simple computational structures. Motivated by these advantages, the present work develops a second-order approximation scheme for the Caputo fractional derivative based on a modified product trapezoidal formulation.
The remainder of this paper is organised as follows. Section 2 presents the necessary preliminaries and auxiliary results. Section 3 derives the proposed Caputo fractional derivative approximation and establishes its error estimate. Section 4 presents numerical experiments and convergence studies that validate the theoretical analysis. Finally, Section 5 summarises the main findings and outlines possible directions for future research.

2. Preliminaries

In this section, we present the necessary definitions and theorems.

2.1 Composite Trapezoidal Rule

Definition 2.1 ([23, 24]). *Let $[c, d]$ be partitioned into N equal subintervals with step size $h = \frac{d-c}{N}$ and nodes $x_i = c + ih$ for $i = 1,2,\cdots,N$. The classical composite trapezoidal rule is defined as:*

$$C_T(f,h) = \frac{h}{2}\sum_{i=1}^{N} \left(f(x_{i-1}) + f(x_i)\right) \tag{1}$$

or

$$C_T(f,h) = \frac{h}{2}\left(f(c) + f(d)\right) + h\sum_{i=1}^{N-1} f(x_i) \tag{2}$$

Thus,

$$\int_c^d f(x)\,dx \approx C_T(f,h). \tag{3}$$

Error:

$$E_T(f,h) = -\frac{(d-c)}{12} f''(k)h^2 = O(h^2). \tag{4}$$

2.2 Product Trapezoidal Approximation for Fractional Integrals

Theorem 2.1 ([18]) *Let $f \in C^2[0,T]$, and let the uniform grid be defined by*

$$t_i = ih, \quad h = \frac{T}{p}, \quad i = 0,1,\dots,p.$$

Let $\tilde{f}_p$ be the piecewise linear interpolant of f on each subinterval $[t_i, t_{i+1}]$.
Then, for any $\alpha > 0$ and $t_p = ph$, the following representation holds:

$$\int_0^{t_p} (t_p - t)^{\alpha-1} \tilde{f}_p(t)\, dt = \sum_{i=0}^{p} a_{i,p} f(t_i), \tag{5}$$

where the weights $a_{i,p}$ are given by

$$a_{i,p} = \begin{cases} \frac{h^\beta}{\beta(\beta+1)}\left[(p-1)^{\beta+1} - (p-1-\beta)p^\beta\right], & i = 0, \\ \frac{h^\beta}{\beta(\beta+1)}\left[(p-i+1)^{\beta+1} + (p-i-1)^{\beta+1} - 2(p-i)^{\beta+1}\right], & 1 \le i \le p-1, \\ \frac{h^\beta}{\beta(\beta+1)}, & i = p. \end{cases} \tag{6}$$

Moreover, the approximation error satisfies

$$\left|\int_0^{t_p} (t_p - t)^{\beta-1} f(t)\, dt - \sum_{i=0}^{p} a_{i,p} f(t_i)\right| \le C_\beta \; \| f'' \|_\infty \; t_p^\beta h^2, \tag{7}$$

where C_β is a constant depending only on β.

2.3 Modified Trapezoidal Rule

Theorem 2.1 ([19, Theorem 2]). Suppose that the interval $[0,a]$ is sub-divided into k sub-intervals $\left[x_j, x_{j+1}\right]$ of equal width $h = \frac{a}{k}$ by using the nodes $x_j = jh,\ for\ j = 0,1,\cdots,k$. The modified trapezoidal rule:

$$\begin{aligned} T(f,h,\alpha) = {} & [(k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha] \frac{h^\alpha f(0)}{\Gamma(\alpha+2)} + \frac{h^\alpha f(a)}{\Gamma(\alpha+2)} \\ & + \sum_{j=1}^{k-1} [(k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j \\ & - 1)^{\alpha+1}] \frac{h^\alpha f(x_j)}{\Gamma(\alpha+2)} \end{aligned} \tag{8}$$

is an approximation to fractional integral:

$$[J^\alpha f(x)](a) = T(f,h,\alpha) - E_T(f,h,\alpha), a > 0, \alpha > 0 \tag{9}$$

Furthermore, if $f(x) \in C^2[0,a]$, there is a constant C'_α depending only on α so that the error term $E_T(f,h,\alpha)$ has the form:

$|E_T(f,h,\alpha)| \le C'_\alpha \|f''\|_\infty a^\alpha h^2 = \boldsymbol{O}(h^2)$.

2.4 Riemann–Liouville Fractional Integral and Derivative, Caputo Fractional Derivative

Definition 2.2 ([25, 26, 2]). Let f be a suitable function and β be any complex parameter, where $n - 1 < \beta \le n$. Then, the Riemann–Liouville fractional integral and derivative of order β with lower limit $0 \in \mathbb{R}$ are defined, respectively, by

$${}_0^{RL}I_x^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\, dt, \qquad \mathrm{Re}(\beta) > 0, \tag{10}$$

$$ {}_0^{RL}D_x^{\beta} f(\mathrm{x}) = \frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}\left({}_0^{RL}I_x^{\mathrm{n}-\beta} f(x)\right), \qquad \mathrm{Re}(\beta) \geq 0, \tag{11} $$

where $\mathrm{x} > 0$ and $\mathrm{n} \in \mathbb{N}$ satisfies $\mathrm{n} - 1 < \mathrm{Re}(\beta) \leq \mathrm{n}$.

Similar to Riemann–Liouville fractional derivative is the Caputo fractional derivative defined as

$$ {}_0^{C}D_x^{\beta} f(x) = {}_0^{RL}I_x^{n-\beta}\left(\frac{d^n}{dx^n} f(x)\right), \qquad \mathrm{Re}(\beta) \geq 0, \tag{12} $$

where n is the smallest integer that exceeds β.

Details and properties of the Riemann–Liouville fractional integral are found in [2, 3, 11, 26, 25].

3. General Caputo Fractional Derivative Approximation

The proposed approximation is derived from a fractional integral representation and expressed in a discrete convolution-type form involving only values of the required integer-order derivative.

Theorem 3.1 *Assume that the interval* $[0, T]$ *is partitioned into* p *subintervals* $[x_i, x_{i+1}]$ *of equal width* $h = T/p$ *by using the nodes* $x_i = ih$, *for* $i = 0,1, \dots, p$. *Then the Caputo fractional derivative rule*

$$ \begin{aligned} {}_0^{C}D_x^{\beta}(f; h, \beta) = \frac{h^{n-\beta}}{\Gamma(n+2-\beta)}[((p-1)^{n-\beta+1} - (p-n+\beta-1)p^{n-\beta})f^{(n)}(0) \\ + f^{(n)}(T) + \sum_{i=1}^{p-1} ((p-i+1)^{n-\beta+1} - 2(p-i)^{n-\beta+1} \\ + (p-i-1)^{n-\beta+1})f^{(n)}(x_i)] \end{aligned} \tag{13} $$

is an approximation to the Caputo fractional derivative

$$ \left({}_0^{C}D_x^{\beta} f(x)\right)(T) = {}_0^{C}D_x^{\beta}(f; h, \beta) - E_C(f; h, \beta), \qquad \beta > 0, \tag{14} $$

for $n - 1 < \beta \leq n$. Furthermore, if $f(x) \in C^{n+2}[0, T]$, then there is some constant $C'_{n\beta}$ depending only on β such that the error term $E_C(f; h, \beta)$ has the form

$$ |E_C(f; h, \beta)| \leq C'_{n\beta} \parallel f^{(n+2)} \parallel_\infty T^{n-\beta} h^2 = O(h^2). \tag{15} $$

Proof.

The proof is based on the definition of the Caputo fractional derivative and the modified trapezoidal rule established for fractional integrals.

Recall that the Caputo fractional derivative of order $\beta > 0$ is defined by

$$ {}_0^{C}D_x^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)}\int_0^T (x-s)^{n-\beta-1} f^{(n)}(s)\, ds, \qquad n-1 < \beta < n, \tag{16} $$

where $n = \lceil \beta \rceil$ is the smallest integer exceeding β.

Evaluating (16) at $x = T$ gives

$$ \left({}_0^{C}D_x^{\beta} f\right)(T) = \frac{1}{\Gamma(n-\beta)}\int_0^T (T-s)^{n-\beta-1} f^{(n)}(s)\, ds. \tag{17} $$

Observe that the right-hand side of (17) has exactly the same form as a Riemann–Liouville fractional integral. Indeed,

$$ \left({}_0^{RL}I_x^{n-\beta} g\right)(T) = \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1} g(s)\, ds, \qquad \beta > 0. $$

Setting

$$ \beta = n - \alpha, \qquad g(s) = f^{(n)}(s), $$

equation (17) becomes

$$ \left({}_0^{C}D_x^{\beta} f\right)(T) = (({}_0^{RL}I_x^{n-\beta} f^{(n)})(T). \tag{18} $$

Hence the approximation of the Caputo derivative reduces to the approximation of a fractional integral of order

$$ \alpha = n - \beta. $$

Since $n - 1 < \beta \leq n$, it follows that

$$ 0 < \alpha = n - \beta \leq 1. $$

Step 1: Application of the modified trapezoidal rule.

By Theorem 2.1, for any sufficiently smooth function g,

$$(\mathbb{I}^{\alpha} g)(a) = T(g; h, \alpha) - E_T(g; h, \alpha), \tag{19}$$

where

$$\begin{aligned} T(g; h, \alpha) \quad &= \frac{h^{\alpha}}{\Gamma(\alpha+2)} [((p-1)^{\alpha+1} - (p-\alpha-1)p^{\alpha}) g(0) \\ &\quad + g(a) + \sum_{j=1}^{p-1} ((p-i+1)^{\alpha+1} - 2(p-i)^{\alpha+1} \\ &\quad + (p-i-1)^{\alpha+1}) g(x_i)]. \end{aligned} \tag{20}$$

Furthermore,

$$|E_T(g; h, \alpha)| \leq C_{\alpha} \parallel g'' \parallel_{\infty} T^{\alpha} h^2. \tag{21}$$

Substituting

$$g(s) = f^{(n)}(s), \qquad \alpha = n - \beta,$$

into (19) yields

$$\left({}_0^C D_x^{\beta} f \right)(T) = T\left(f^{(n)}; h, n-\beta\right) - E_T\left(f^{(n)}; h, n-\beta\right). \tag{22}$$

Step 2: Derivation of the quadrature formula.
Replacing α by $n-\beta$ and g by $f^{(n)}$ in (20), we obtain:

$$\begin{aligned} T(f^{(n)}; h, n-\beta) \quad &= \frac{h^{n-\beta}}{\Gamma(n-\beta+2)} [((p-1)^{n-\beta+1} \\ &\quad -(p - (\mathrm{n}-\beta) - 1)p^{n-\beta}) f^{(n)}(0) \\ &\quad + f^{(n)}(T) \\ &\quad + \sum_{i=1}^{p-1} ((p-i+1)^{n-\beta+1} \\ &\qquad -2(p-i)^{n-\beta+1} \\ &\qquad +(p-i-1)^{n-\beta+1}) f^{(n)}(x_i)]. \end{aligned} \tag{23}$$

Since

$$p - (n-\beta) - 1 = p - n + \beta - 1,$$

equation (23) becomes

$$\begin{aligned} T(f^{(n)}; h, n-\beta) \quad &= \frac{h^{n-\beta}}{\Gamma(n+2-\beta)} [((p-1)^{n-\beta+1} \\ &\quad -(p-n+\beta-1)p^{n-\beta}) f^{(n)}(0) \\ &\quad + f^{(n)}(T) \\ &\quad + \sum_{i=1}^{p-1} ((p-p+1)^{n-\beta+1} \\ &\qquad -2(p-i)^{n-\beta+1} \\ &\qquad +(p-i-1)^{n-\beta+1}) f^{(n)}(x_i)]. \end{aligned} \tag{24}$$

Define

$${}_0^C D_x^{\beta} f(f; h, \beta) = T(f^{(n)}; h, n-\beta).$$

Then (24) is exactly the formula stated in (13).
Consequently, from (22),

$$\left({}_0^C D_x^{\beta} f \right)(T) = \quad {}_0^C D_x^{\beta} f(f; h, \alpha) - E_C(f; h, \alpha),$$

where

$$E_C(f; h, \alpha) = E_T(f^{(\mathrm{n})}; h, n-\beta).$$

This establishes equation (14).
Step 3: Error estimate.
Assume that

$$f \in C^{n+2}[0, T].$$

Then

$$g = f^{(n)} \in C^2[0,T].$$

Applying the error estimate (21) with $g = f^{(n)}$ and $\alpha = n - \beta$ gives

$$|E_C(f;h,\beta)| = |E_T(f^{(n)};h,n-\beta)| \le C_{n-\beta} \parallel (f^{(n)})'' \parallel_\infty T^{n-\beta} h^2. \tag{25}$$

Since

$$(f^{(n)})'' = f^{(n+2)},$$

we obtain

$$|E_C(f;h,\beta)| \le C_{n-\beta} \parallel f^{(n+2)} \parallel_\infty T^{n-\beta} h^2. \tag{26}$$

Let

$$C'_{n\beta} = C_{n-\beta}.$$

Then (30) becomes

$$|E_C(f;h,\beta)| \le C'_{n\beta} \parallel f^{(n+2)} \parallel_\infty T^{n-\beta} h^2. \tag{27}$$

Therefore,

$$|E_C(f;h,\beta)| = O(h^2), \qquad h \to 0.$$

This proves the asserted error bound (15).
Hence the Caputo fractional derivative rule (17) is a second-order accurate approximation of the Caputo fractional derivative, completing the proof. □

3.1 Particular Case: $0 < \beta < 1$
In the case $0 < \beta < 1$, the Caputo fractional derivative rule (13) reduces to

$$\begin{aligned} {}_0^C D_x^{\beta}(f;h,\beta) \quad &= \frac{h^{1-\beta}}{\Gamma(3-\beta)}[((p-1)^{2-\beta} - (p+\beta-2)p^{1-\beta})f'(0) \\ &\quad + f'(T) + \textstyle\sum_{i=1}^{p-1} ((p-i+1)^{2-\beta} - 2(p-i)^{2-\beta} \\ &\quad + (p-i-1)^{2-\beta})f'(x_i)]. \end{aligned} \tag{28}$$

If $f(x) \in C^3[0,T]$, then the error term $E_C(f;h,\beta)$ satisfies

$$|E_C(f;h,\beta)| \le C'_{1\beta} \parallel f^{(3)} \parallel_\infty T^{1-\beta} h^2, \tag{29}$$

for some constant $C'_{1\beta}$ depending only on β.

3.2 Particular Case: $1 < \beta < 2$
If $1 < \beta < 2$, then the Caputo fractional derivative rule (13) reduces to

$$\begin{aligned} {}_0^C D_x^{\beta}(f;h,\alpha) \quad &= \frac{h^{2-\beta}}{\Gamma(4-\beta)}[((p-1)^{3-\beta} - (p+\beta-3)p^{2-\beta})f''(0) \\ &\quad + f''(T) + \textstyle\sum_{i=1}^{p-1} ((p-i+1)^{3-\beta} - 2(p-i)^{3-\beta} \\ &\quad + (p-i-1)^{3-\beta})f''(x_i)]. \end{aligned} \tag{30}$$

If $f(x) \in C^4[0,T]$, then the error term $E_C(f;h,\beta)$ satisfies

$$|E_C(f;h,\beta)| \le C'_{2\beta} \parallel f^{(4)} \parallel_\infty T^{2-\beta} h^2, \tag{31}$$

for some constant $C'_{2\beta}$ depending only on β.

3.3 Order of Convergence Analysis

To quantify the accuracy of the proposed approximation scheme, we compute the experimental order of convergence (EOC) defined by

$$\kappa = \frac{\log(E_{h_1}/E_{h_2})}{\log(h_1/h_2)}, \tag{32}$$

where E_h denotes the error corresponding to step size h.

4. Numerical Results, Convergence Analysis, and Discussion

In this section, we present numerical experiment to clearly describe the behaviour of the Caputo fractional derivatives using the test function $f(x) = \sin x$ on the interval $(0,2]$. The purpose is to investigate the qualitative and quantitative behaviour of the Caputo fractional derivatives, particularly their numerical stability, convergence behaviour, and error characteristics.

The computations were carried out using MATLAB with a truncated series representation of the fractional derivatives. The orders of the fractional derivative were taken as $\beta = 0.10, 0.50, 1.00, 1.50, 2.00$, and the interval was discretised using sufficiently small step sizes to ensure accuracy.

4.1 Test Problem

Let $f(x) = sin(x)$. For $0 < \beta \leq 1$,

$$D_T \sin x = x^{1-T} \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{\Gamma(2i+2-T)}, \qquad 0 < T < 1. \tag{33}$$

For $1 < \beta \leq 2$,

$$D_T \sin x = x^{2-T} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{\Gamma(2i+4-T)}, \qquad 1 < T < 2. \tag{34}$$

Since $\cos(0) = 1$ and $\sin(0) = 0$, the additional terms arise from the initial conditions:

$$\sin(0) = 0, \quad \sin'(0) = 1, \quad \cos(0) = 1, \quad \cos'(0) = 0.$$

4.2 Numerical Results

Table 1: Convergence results for the Caputo approximation of $f(x) = \sin(x)$ for $0 < \beta \leq 1$.

h	Approximation	Exact	Error	Order
		$\beta = 0.1$		
0.20000	$8.5782293571 \times 10^{-1}$	$8.6068645785 \times 10^{-1}$	2.864×10^{-3}	—
0.10000	$8.5997019696 \times 10^{-1}$	$8.6068645785 \times 10^{-1}$	7.163×10^{-4}	1.9992
0.05000	$8.6050729712 \times 10^{-1}$	$8.6068645785 \times 10^{-1}$	1.792×10^{-4}	1.9992
0.02500	$8.6064165166 \times 10^{-1}$	$8.6068645785 \times 10^{-1}$	4.481×10^{-5}	1.9995
0.01250	$8.6067525393 \times 10^{-1}$	$8.6068645785 \times 10^{-1}$	1.120×10^{-5}	1.9997
		$\beta = 0.5$		
0.20000	$8.4340577902 \times 10^{-1}$	$8.4605678672 \times 10^{-1}$	2.651×10^{-3}	—
0.10000	$8.4538298779 \times 10^{-1}$	$8.4605678672 \times 10^{-1}$	6.738×10^{-4}	1.9762
0.05000	$8.4588617698 \times 10^{-1}$	$8.4605678672 \times 10^{-1}$	1.706×10^{-4}	1.9816
0.02500	$8.4601373233 \times 10^{-1}$	$8.4605678672 \times 10^{-1}$	4.305×10^{-5}	1.9865
0.01250	$8.4604595018 \times 10^{-1}$	$8.4605678672 \times 10^{-1}$	1.084×10^{-5}	1.9903
		$\beta = 1.0$		
0.20000	$5.4030230587 \times 10^{-1}$	$5.4030230587 \times 10^{-1}$	1.110×10^{-16}	—

0.10000	$5.4030230587 \times 10^{-1}$	$5.4030230587 \times 10^{-1}$	1.110×10^{-16}	0.0000
0.05000	$5.4030230587 \times 10^{-1}$	$5.4030230587 \times 10^{-1}$	1.110×10^{-16}	0.0000
0.02500	$5.4030230587 \times 10^{-1}$	$5.4030230587 \times 10^{-1}$	1.110×10^{-16}	0.0000
0.01250	$5.4030230587 \times 10^{-1}$	$5.4030230587 \times 10^{-1}$	1.110×10^{-16}	0.0000

Table 2: Convergence results for the Caputo approximation of $f(x) = \sin(x)$ for $1 < \beta \leq 2$.

h	Approximation	Exact	Error	Order
		$\boldsymbol{\beta = 1.1}$		
0.20000	$-4.9884413577 \times 10^{-1}$	$-5.0049690409 \times 10^{-1}$	1.653×10^{-3}	—
0.10000	$-5.0008186461 \times 10^{-1}$	$-5.0049690409 \times 10^{-1}$	4.150×10^{-4}	1.9936
0.05000	$-5.0039290330 \times 10^{-1}$	$-5.0049690409 \times 10^{-1}$	1.040×10^{-4}	1.9967
0.02500	$-5.0047087228 \times 10^{-1}$	$-5.0049690409 \times 10^{-1}$	2.603×10^{-5}	1.9982
0.01250	$-5.0049039196 \times 10^{-1}$	$-5.0049690409 \times 10^{-1}$	6.512×10^{-6}	1.9991
		$\boldsymbol{\beta = 1.5}$		
0.20000	$-6.6775219384 \times 10^{-1}$	$-6.6968425958 \times 10^{-1}$	1.932×10^{-3}	—
0.10000	$-6.6917825090 \times 10^{-1}$	$-6.6968425958 \times 10^{-1}$	5.060×10^{-4}	1.9329
0.05000	$-6.6955385392 \times 10^{-1}$	$-6.6968425958 \times 10^{-1}$	1.304×10^{-4}	1.9562
0.02500	$-6.6965098275 \times 10^{-1}$	$-6.6968425958 \times 10^{-1}$	3.328×10^{-5}	1.9704
0.01250	$-6.6967582229 \times 10^{-1}$	$-6.6968425958 \times 10^{-1}$	8.437×10^{-6}	1.9797
		$\boldsymbol{\beta = 2.0}$		
0.20000	$-8.4147098481 \times 10^{-1}$	$-8.4147098481 \times 10^{-1}$	0.000×10^{0}	—
0.10000	$-8.4147098481 \times 10^{-1}$	$-8.4147098481 \times 10^{-1}$	0.000×10^{0}	0.0000
0.05000	$-8.4147098481 \times 10^{-1}$	$-8.4147098481 \times 10^{-1}$	0.000×10^{0}	0.0000
0.02500	$-8.4147098481 \times 10^{-1}$	$-8.4147098481 \times 10^{-1}$	0.000×10^{0}	0.0000
0.01250	$-8.4147098481 \times 10^{-1}$	$-8.4147098481 \times 10^{-1}$	0.000×10^{0}	0.0000

4.3 Numerical Results Overview

The performance of the proposed Caputo fractional derivative approximation scheme is investigated using the test function $f(x) = \sin(x)$. The computations are carried out for multiple fractional orders $\beta = 0.1,\ 0.5,\ 1.0,\ 1.1,\ 1.5,\ 2.0$, with progressively refined step sizes $h = 0.2,\ 0.1,\ 0.05,\ 0.025,\ 0.0125$.

The exact values are obtained using the truncated Caputo series representation, while numerical approximations are computed using the proposed discrete convolution-type schemes corresponding to the cases $0 < \beta \leq 1$ and $1 < \beta \leq 2$.

4.4 Convergence Plots

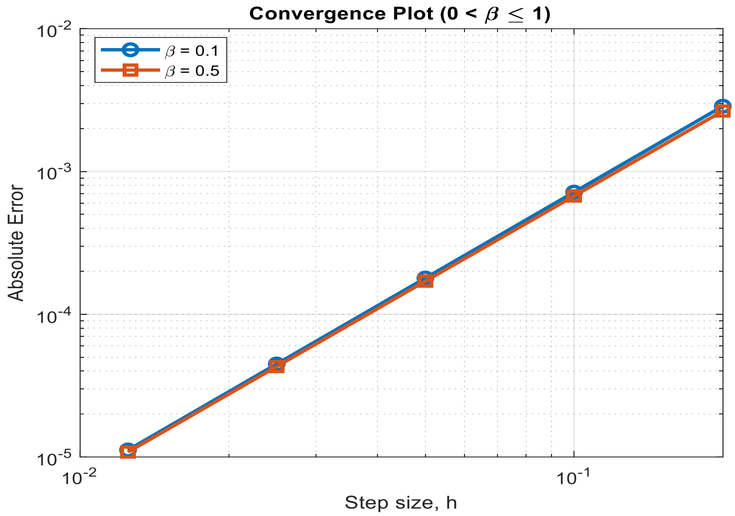

Figure 1: Log-log Convergence Plot Showing the Absolute Error versus Step-Size for the proposed Caputo Approximation with $\beta = 0.1$and $\beta = 0.5$. The Nearly Straight Lines with Slope Approximately Equal to Two Confirm Second-Order Convergence.

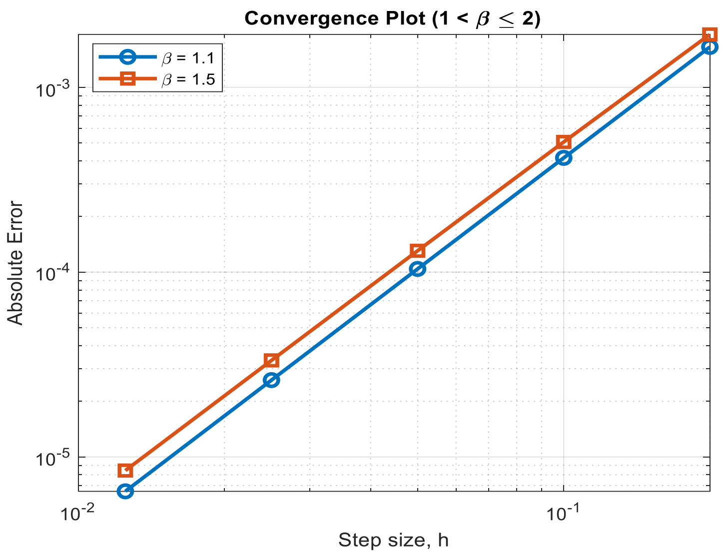

Figure 2: Log-log Convergence Plot Showing the Absolute Error versus Step Size for $\beta = 1.1$and $\beta = 1.5$. Slight Deviations From the Ideal Slope Are Attributed to the Non-Local Fractional Memory Effect, while the overall Convergence remains Second Order.

Figures 1 and 2 present the convergence behaviour of the proposed approximation on logarithmic scales. The horizontal axis represents the mesh size h, while the vertical axis shows the corresponding absolute error. For a numerical method with error

$$E(h) = Ch^p,$$

taking logarithms gives

$$\log\ E = \log\ C + p\log\ h,$$

so that the slope of the straight line equals the convergence order p.

The numerical results produce almost straight lines with slopes very close to two, confirming the theoretical second-order convergence established in Theorem 3.1. Small deviations from the ideal slope occur only for non-integer fractional orders because the Caputo derivative incorporates the entire history of the solution through its weakly singular convolution kernel. Consequently, accumulated memory effects increase the error constant without changing the asymptotic convergence rate.

The convergence plots therefore provide visual confirmation of the analytical error estimate and demonstrate that the proposed approximation maintains quadratic accuracy throughout the investigated range of fractional orders.

4.5 Convergence Behaviour

Across all fractional orders, the numerical results demonstrate a clear convergence pattern as the mesh is refined. For $\beta = 0.1$ and $\beta = 0.5$, the error decreases consistently by approximately a factor of four when the step size is halved, indicating near second-order convergence behaviour:

$$\mathrm{Error}(h) \approx \mathcal{O}(h^2).$$

This is confirmed by the computed orders of convergence, which remain close to 2.0 in successive refinements.

For example, at $\beta = 0.1$, the observed order stabilizes around 1.999, confirming near-quadratic convergence. A similar behaviour is observed for $\beta = 0.5$, with slight variations due to fractional memory effects.

The slight reduction of the observed convergence order below the ideal value of two should not be interpreted as a deterioration of the theoretical accuracy of the proposed method. The error estimate established in Theorem 3.1 proves that the truncation error satisfies

$$E(h) = O(h^2),$$

but the hidden constant in the asymptotic notation depends on the fractional order β. For non-integer values of β, the Caputo derivative is represented by a convolution integral with the weakly singular kernel

$$(t - s)^{n-\beta-1},$$

which assigns non-negligible weight to the entire solution history. Consequently, the local quadrature errors generated on all previous subintervals accumulate through the convolution process, producing a larger error constant than in the classical integer-order case. Although this accumulated memory contribution slightly affects the experimental order computed on finite meshes, it does not alter the asymptotic convergence rate, which remains second order as the mesh is further refined.

4.6 Integer-Order Consistency

For the integer-order case $\beta = 1.0$, the scheme reproduces the classical derivative within machine precision:

$$D^1 \sin(x) = \cos(x).$$

The numerical results show errors on the order of

$$\mathcal{O}(10^{-16}),$$

which corresponds to floating-point round-off error. This confirms that the method is consistent with classical calculus in the limiting case $\beta \to 1$.

4.7 Behaviour for $1 < \boldsymbol{\beta} \leq 2$

For fractional orders greater than one, namely $\beta = 1.1, 1.5$, and 2.0, the proposed approximation remains numerically stable and exhibits excellent convergence properties. The numerical errors decrease monotonically as the mesh is refined, demonstrating that the discretization accurately captures the non-local behaviour of the Caputo operator.

The observed convergence orders are slightly below the ideal value of two for $\beta = 1.1$ and particularly for $\beta = 1.5$. This behaviour is expected for fractional operators and originates from the hereditary (memory) property of the Caputo derivative rather than from any loss of consistency of the numerical scheme. Unlike classical derivatives, whose evaluation depends only on local information, the Caputo derivative depends on the entire history of the solution through a convolution with the kernel

$$(t-s)^{1-\beta}.$$

Every discretisation error generated at earlier time levels therefore contributes to the approximation at the current point. As the fractional order moves away from the neighbouring integer value, this long-range interaction becomes more pronounced, increasing the effective error constant while preserving the theoretical order of convergence.

The effect is particularly noticeable around intermediate fractional orders such as $\beta = 1.5$, where the balance between the singular kernel and the accumulated history produces slightly larger pre-asymptotic errors. Nevertheless, as the mesh is refined, the experimental convergence approaches the predicted second-order behaviour, confirming the validity of the theoretical analysis presented in Section 3.

When $\beta = 2$, the fractional memory disappears completely because the Caputo derivative coincides with the classical second derivative. The convolution kernel reduces to a purely local operator, eliminating the accumulated memory contribution. Consequently, the proposed scheme reproduces

$$D^2 sin(x) = -sin(x)$$

to machine precision, demonstrating complete consistency with classical calculus.

4.8 Discussion of Accuracy and Stability

The numerical results highlight several important properties of the proposed scheme:

(i) Stability. The method is numerically stable across all tested fractional orders, with no oscillatory or divergent behaviour observed even for coarse grids.

(ii) Accuracy. High accuracy is achieved even for relatively large step sizes. The error decreases rapidly under mesh refinement, confirming the robustness of the discretisation strategy.

(iii) Fractional Memory Effect. A characteristic feature of the Caputo derivative is that it possesses long-range memory, meaning that the derivative at the current point depends on the entire past history of the solution. Numerically, this is reflected through the discrete convolution weights appearing in the proposed approximation. Since all previous grid values contribute to the approximation, local discretization errors accumulate over the computational interval instead of remaining purely local as in classical finite-difference methods. This accumulation slightly increases the error constant and causes the experimentally observed convergence orders for finite meshes to be marginally below the ideal value of two. However, because the accumulation affects only the constant multiplying the leading $O(h^2)$ term and not its exponent, the theoretical second-order convergence established in Theorem 3.1 remains valid.

(iv) Consistency. The scheme correctly reduces to:

- Caputo fractional derivative for non-integer orders,
- classical derivatives for $\beta = 1,2$,

ensuring full consistency with classical calculus.

5. Conclusion

In this work, a numerical approximation scheme for the Caputo fractional derivative was investigated using the test function $f(x) = \sin(x)$ over a range of fractional orders $\beta = 0.1, 0.5, 1.0, 1.1, 1.5, 2.0$. The computational results presented in Tables 1 and 2 demonstrate strong agreement between the numerical approximations and the corresponding exact solutions across all tested cases.

For non-integer fractional orders ($0 < \beta < 1$ and $1 < \beta < 2$), the method exhibits a consistent and reliable convergence behaviour. In particular, Tables 1 and 2 show that the absolute error decreases steadily as the mesh size is refined, with the observed rates of convergence closely approaching second-order accuracy. This confirms the effectiveness of the discretisation strategy in capturing the non-local nature of the Caputo operator while maintaining numerical stability.

The slight deviations of the experimental convergence orders from the exact value of two for non-integer fractional orders are consistent with the non-local nature of the Caputo operator. These deviations arise from the accumulation of discretisation errors through the fractional memory kernel rather than from any reduction in the formal order of the numerical approximation. As the mesh is refined, the influence of these pre-asymptotic effects diminishes and the computed convergence rates approach the theoretical second-order prediction, providing additional numerical validation of the proposed analysis.

For integer-order cases ($\beta = 1$ and $\beta = 2$), the scheme reproduces the classical derivatives exactly up to machine precision. The errors in these cases are on the order of 10^{-16}, indicating that the method is fully consistent with classical calculus in the limiting integer-order regime. This property further validates the robustness and correctness of the proposed formulation.

Overall, the results in Tables 1 and 2 confirm that the proposed method is:

- highly accurate for both fractional and integer orders,
- stable under mesh refinement,
- and consistently exhibits near-second-order convergence for non-integer β.

These findings demonstrate that the method provides a reliable computational tool for approximating Caputo fractional derivatives and can be extended to more complex fractional differential equations arising in applied mathematics and engineering.

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