Journal of NAMP 73 (2026) 89-98

The Nigerian Association of Mathematical Physics

Journal homepage: https://nampjournals.org.ng

IMPACT OF STATE SIZE ON CONVERGENCE RATE OF MARKOV CHAIN WITH APPLICATION TO RAINFALL VARIABILITY IN MAKURDI. NIGERIA

V. Adah[1*] , I. O. Agada[2] and M. M. Chigbo[3]

[1,3]Department of Statistics, Joseph Sarwuan Tarka University, Makurdi, Nigeria

[2]Department of Physics, Joseph Sarwuan Tarka University, Makurdi, Nigeria

ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

Keywords:
Rainfall, Markov chain, Stationary distribution, Mixing time, Total variation distance

ABSTRACT

Rainfall variability is an important component of the climate system and can be effectively modelled using Markov chains. This study compares the convergence behaviour of two-state and three-state Markov chain models using daily rainfall data from Makurdi, Nigeria. Both models were found to be ergodic, ensuring unique stationary distributions. Mixing time, measured using Total Variation Distance (TVD), was employed to assess the speed of convergence under a uniform initial distribution. The two-state model converged to its stationary distribution after 24 days, while the three-state model required 31 days, indicating slower convergence with increased state complexity. Both models showed a higher long-run probability of dry conditions, with the two-state model estimating stationary probabilities of 57% for dry days and 43% for wet days. By comparing mixing times across different state-space models, this study provides insights into rainfall persistence and supports improved agricultural planning, drought assessment, and rainfall forecasting.

1. Introduction

Rainfall is a key component of the hydrological cycle and plays an essential role in weather, climate, and water resource management. It influences ecosystems, agriculture, disaster preparedness, and sustainable development [8]. Rainfall also affects tourism, particularly in areas where water-based activities depend on seasonal rainfall [13], and is vital for the survival and well-being of human populations [4]. Modeling and predicting rainfall are crucial for generating data as well as for providing information, which can then be used in a variety of applications, such as water resource management, hydrology, and agriculture [5]. The Markov chain is a stochastic process with discrete parameters that meet the nature of Markov. They are relatively simple to implement, where the next state depends only on the current state, making them straightforward to work with by reducing the complexity of the system. also, they often require less computational power compared to other complex models; with this efficiency, it is particularly valuable

*Corresponding author: **V. Adah**

E-mail address: adahvictoria14@yahoo.com,
https://doi.org/10.60787/jnamp.vol73no.730

when dealing with large datasets or predicting steady-state distributions [17]. Several studies have applied Markov chain models to rainfall analysis. A four-state model was used in Bangladesh [11] and West Java, Indonesia [3], while a three-state model was applied in the Kurdistan Region of Iraq [5] and southwestern Nigeria [10]. Two-state Markov chain models have been used for Makurdi rainfall and crop growth [1], rainfall variability in southeastern Ghana [14], and climate change assessment in Makurdi [2]. Other studies examined rainfall variability and seasonal characteristics across West Africa [20–22], while [19] also modeled rainfall distribution in Makurdi. These studies highlight the usefulness of Markov chain models for characterizing rainfall occurrence, persistence, and long-term behaviour in tropical climates. This review examines the existing research in terms of rainfall variability and applications with the aid of the Markov chain. The stationary distributions estimated by the authors did not tell anything about the convergence rate to stationarity. that is, how fast the convergence will be; this gap in literature the paper intends to address with the aid of the mixing times. The mixing time of a Markov chain is a measure of how quickly the chain converges to its stationary distribution (or equilibrium). Specifically, it is the number of steps required for the distribution of the chain to be within a certain distance (usually Total Variation Distance) from the stationary distribution, starting from any worst initial state [16]. Mixing time, although traditionally studied in probability theory and statistical physics, has also become useful in environmental and meteorological modelling for assessing the rate at which stochastic systems approach equilibrium. In climate and hydrological applications, mixing time provides a measure of persistence and memory in processes such as rainfall occurrence, temperature fluctuations, and atmospheric variability. [6] and [15] provide the theoretical foundation, while its application to rainfall occurrence modelling remains relatively limited, motivating the present study.

2. Methods

2.1 Data description: The daily data on rainfall (mm) for Makurdi, Nigeria used for the study is obtained from the National Aeronautics and Space Administration (NASA) (Modern Era Retrospective Analysis Version 2 (MERRA-2) for the period of 38 years (1985-2023).

2.2 Data discretization: Continuous rainfall data were discretized into two-state and three-state categories following the approach of [10], which is suitable for the tropical climate characterized by dry and rainy seasons. All statistical analyses were performed using Python 3.12.

For two states:

$$R_k = \begin{cases} \text{Dry day} & \text{if } R_k \leq 2.5\text{mm} \\ Wet\ day & otherwise \end{cases} \tag{1}$$

For three states:

$$R_k = \begin{cases} \text{Dry day} & if\ R_k \leq 2.5\text{mm} \\ \text{Wet day} & if\ 2.52 \leq R_k \leq 5.0\text{mm} \\ \text{Rainy day} & otherwise \end{cases} \tag{2}$$

2.3 Markov Chain

Markov chain with state space S and transition matrix P is a sequence of random variables $(\mathrm{X}_0\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n},)$ if for all x, y ∈ S, all t ≥ 1, and all events,

$$\mathrm{H}_{t-1} = \cap_{i=0}^{t-1} \{X_i = x_i\}$$

satisfying

$$\mathrm{P}(\mathrm{H}_{t-1} \cap \{X_i = x_i\}) > 0,$$

We have

$$\mathrm{P}\{\mathrm{X}_{\mathrm{t+1}} = \mathrm{y} \mid \mathrm{H}_{\mathrm{t-1}} \cap \{X_i = x_i\}\}$$
$$\mathrm{P}\{\mathrm{X}_{\mathrm{t+1}} = \mathrm{y} \mid \mathrm{X}_{\mathrm{t+1}} = \mathrm{x}_{\mathrm{t}}\} = P(x, y) \tag{3}$$

Equation (3), known as the Markov property, states that the probability of moving from state xxx to state y depends only on the current state x and not on the previous sequence of states.
The number of times a system changes states is known as the Markov chain transition count[18]. Mathematically;
Let P denote a transition probability matrix P_{xy} so that:

$$\hat{P}_{xy} = \frac{f_{xy}}{\sum_y^S f_{xy}} = \frac{f_{xy}}{f_{x\,.}}$$

The matrix is stochastic (that is, nonnegative and all rows sum to one) since for all x:

$$\sum_{y=1}^{n} P_{xy} = \sum_{y=1}^{n} P(X_1 = s_y | X_0 = s_x)$$
$$P(X_1 \in S | X_0 = s_x)$$

Where F_{xy} is the number of transitions made by system from state x to y at time(t)

2.4 Chi-Square Test for Markov Chain Dependency

To determine whether a Markov chain adequately represents the dependency structure of a sequence, a chi-square test of independence is commonly used. According to [12], the null hypothesis states that successive events are independent, while the alternative hypothesis states that successive events are dependent. For each transition pair, the observed and expected frequencies are defined as:

$$\chi^2 = \sum_{xy} \frac{O(x,y) - E(x,y)}{E(x,y)} \tag{4}$$

where $O(x, y)$ is the frequency observed weather state of transitioning from x to y, and $E(x, y)$ is the expected frequency given by:

$$E(x, y) = N(\, P_{xy}) \tag{5}$$

2.5 Stationary distribution of Markov chain

This is the long-term, steady-state distribution of the system, where the probabilities do not change after further transitions is the stationary distribution of the Markov chain. A distribution π is said to be a stationary or invariant distribution if $\forall_y \in S$,

$$\sum_{x \in S} \pi(x)\, P_{xy} = \pi(y), \tag{6}$$

which can be written in a compact form using matrix notation as

$$\pi = \pi \mathrm{P} \tag{7}$$

For a Markov chain with transition matrix P, the vector π with non-negative components is a stationary distribution.

2.6 Convergence to Stationarity

Convergence to stationarity requires the notion of distance between distributions. This distance can be measured with the classical total variation distance.

Theorem 1: Suppose $\{X_t, \mathrm{t} \geq 0\}$ is an irreducible and aperiodic Markov chain with a finite state space S and a stationary distribution π. Then for all x, y,
$P^t(x,y) \to \pi(x)\ as\ t \to \infty$.
The theorem above does not tell anything about the rate of the convergence to stationarity. that is, how fast the convergence will be. The total variation distance can measure the distance between P^t and π. The idea of distance is extended to define a notion of distance to stationarity of a Markov chain at a given time in its evolution [16].

2.7 Total variation distance between probability measures

Total variation distance is a metric that measures the difference between two probability distributions. Let ν ν_1 denote the distribution of the Markov chain after t steps and ν_2 the stationary distribution. The total variation distance between $P^t$and π, is given by:

$$d(P^t, \pi) = \|P^t - \pi\|_{\boldsymbol{TV}} = \max_{x \subseteq S} |P^t(x) - \pi(x)| \tag{8}$$

$P^t$and π are the probability values of event x under the distribution , respectively. [9].
This distance measures how far the distribution at step n is from the stationary distribution. This distance measure sees distributions (or probability measures) as maps from the set of all events to [0, 1], i.e $0 \leq d(P^t, \pi) \leq 1$ and measures the maximum deviation between the two distributions over all events.
For $n \in N$ the total variation distance from stationarity is

$$d(P^n, \pi) = \max_{x \subseteq S} |\sigma_x P^n - \pi| \tag{9}$$

Here, σ_x could be any initial distribution of the process.

$$\max_{x \subseteq S} |\sigma_x P^t - \pi| = \frac{1}{2} \sum_{x \subseteq S} |\sigma_x P^t - \pi| \tag{10}$$

If S is a countable finite state space
Let $\bar{d}(t)$ denote the variation distance between two Markov chain random variables $X_t \sim P^t(x,.)$ and $Y_t \sim P^t(y,.)$. That is

$$\bar{d}(t) = \max_{x,y \in S} |P^t(x,.) - P^t(y,.)| \tag{11}$$

Theorem 2:(Convergence theorem): Suppose $\{X_t, \mathrm{t} \geq 0\}$ is an irreducible and aperiodic Markov chain with a finite state space S and a stationary distribution π, then there exist constants $C > 0\ and\ \alpha \in (0,1)$such that:

$$\max_{x \subseteq S} |\sigma_x P^t - \pi| \leq C\alpha^t, \qquad t \in N \tag{12}$$

According to this theorem, the Markov chains with these characteristics will eventually converge. In addition to providing the chain's convergence, this theorem also calculates how long it takes the Markov chain to reach its stationary state.

2.8 Mixing Time of a Markov Chain

The Markov chain's mixing time, t_{mix}, is the number of time steps needed to get the chain within a predetermined threshold, ε, of its stationary distribution, even if the process starts from the worst possible initial distributions[6].

$$t_{mix}(\varepsilon) \coloneqq min\left\{t \in N : \max_{x \subseteq S}|\sigma_x P^t - \pi| < \varepsilon, for\ all\ x \in S\right\} \quad (13)$$

the parameter ε is called the mixing tolerance. it specifies how close the Markov chain distribution must be to the stationary distribution before it is declared that the chain has mixed. A common way to define the standard mixing time is:

$$t_{mix} \coloneqq t_{mix}(1/4) \quad (14)$$

An ergodic Markov chain is irreducible and aperiodic, ensuring a unique stationary distribution [7]. Its mixing time is the number of steps required to reach the stationary distribution within a specified accuracy, regardless of the initial state [15].

2.8.1 Initial Distribution

Consider a Markov chain with m states:$S = 1,2, \dots, m$ The uniform initial distribution assigns equal probability to every state:

$$\pi^{(0)} = \left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right) \quad (15)$$

for the three-state Markov chain, the initial distribution is

$$\pi^{(0)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

The uniform distribution provides an unbiased starting point because it does not favour any particular state.

2.9 Trend Analysis Comparing Early vs. Later Periods

The data is divided into periods, early period: 1985–2004 and later period: 2005–2023

N_D: Number of Dry observations, N_W: Number of Wet observations, and N: Total observations

$$F_i = \frac{N_i}{N} \times 100, \quad i = D, W \quad (16)$$

3.Results and Discussion

The Chi-square test of dependence (Table 1) yielded $\chi^2 = 2223.61$(p<0.001) for the two-state chain and yielded $\chi^2 = 2917{,}14$ (p<0.001) for the three-state chain. These results indicate significant dependence between successive rainfall states, confirming that the data satisfy the Markov property.

Table 1: Dependency of Sequence of Chains

State	Chi-square Statistic	p-value
Dry, Wet	2223.6120	0.0
Dry, Wet, Rainy	2917.1462	0.0

Matrix 1: Transition Count (2-state)

$$\begin{array}{ll} & \text{Dry day} \quad Wet\ day \\ \begin{array}{l} Dry\ day \\ Wet\ day \end{array} & \begin{pmatrix} 7066 & 948 \\ 4888 & 947 \end{pmatrix} \end{array}$$

Matrix 2: Transition Probabilities (2-state)

Dry day $\quad Wet\ day$

$$\begin{array}{l} Dry\ day \\ Wet\ day \end{array} \begin{pmatrix} 0.881707 & 0.118293 \\ 0.162296 & 0.837704 \end{pmatrix}$$

Matrix 1 and 2 show the transition count and probabilities of the sequence of dry day and wet day**;** the rows represent the current state (from), and the columns represent the following state (to). It can be seen from matrix 1 that, from the dry day there were 7066 transitions to the dry day and 948 transitions to the wet day. Matrix 2 gives the probability information: if it is dry today, there's an 88% chance that tomorrow will also be dry a 12% chance it will be wet. If it is wet today, there's a 16% chance that tomorrow will also be dry, an 84% chance it will be wet.

Matrix 3: Stationary Distribution (2-state)

$$\begin{array}{cc} \text{Dry day} & Wet\ day \\ (0.57841184 & 0.42158816) \end{array}$$

The stationary distribution in Matrix 3 means that in the long run, there will be 58% chance that the weather will be dry and 42% chance that it will be wet day. This implies that over a long period, the proportion of dry and wet days will stabilize at 57.8% and 42.2%, respectively. Knowing the stationary distribution tells us, on average, there are more dry days than wet days (in the long run, even though the weather can still vary day-to-day.

Matrix 4: Transition Count (3-state)

$$\begin{array}{cc} & \begin{array}{ccc} Dry\ day & Wet\ day & Rainy\ day \end{array} \\ \begin{array}{l} Dry\ day \\ Wet\ day \\ Rainy\ day \end{array} & \begin{pmatrix} 7066 & 328 & 620 \\ 2523 & 740 & 370 \\ 783 & 577 & 842 \end{pmatrix} \end{array}$$

From dry day, there were 7066 transitions to dry day, 328 transitions wet day, and 620 transitions to rainy day

Matrix 5: Transition Probabilities (3-state)

$$\begin{array}{cc} & \begin{array}{ccc} Dry\ day & Wet\ day & Rainy\ day \end{array} \\ \begin{array}{l} Dry\ day \\ Wet\ day \\ Rainy\ day \end{array} & \begin{pmatrix} 0.881707 & 0.040928 & 0.077365 \\ 0.1018445 & 0.694467 & 0.203688 \\ 0.262035 & 0.355587 & 0.382378 \end{pmatrix} \end{array}$$

Matrix 4 shows the transition count for the three-state chain, it can be seen that from the dry day, there were 7066 transitions to dry day, 328 transitions to wet day, and 620 transitions to rainy day. The transition probabilities of the sequence of dry, wet and rainy days captured in Matrix 5, The transition matrix shows that the probability of remaining in a dry state is 0.88, indicating strong persistence of dry conditions. Wet and rainy states have lower persistence probabilities of 0.69 and 0.38, respectively. The high probability of remaining dry suggests that Makurdi experiences predominantly dry conditions, consistent with the findings of [2], who reported that the area is becoming drier.

Matrix 6: Stationary Distribution (3-state)

$$\begin{array}{ccc} Dry\ day & Wet\ day & Rainy\ day \\ (0.57837541 & 0.26257841 & 0.15904618) \end{array}$$

The stationary distribution in matrix 6 reflects the long-term equilibrium of the system. This means that, over a long period of time, the weather system will tend to spend about 57% of the time in the **dry** state, 26% in the **wet** state, and 16% in the **rainy** state. This implies that no matter where the system starts (whether it's dry, wets, or rainy), it will eventually reach this distribution, since the Markov chain is **ergodic** (i.e., it is irreducible and aperiodic). These findings agree with results of [2] and [19]

Table 2: Mixing Time of the Sequence of Dry day and Wet day

Initial Distribution	Number of Steps
1.0, 0.0	30
0.0, 1.0	30
0.5, 0.5	24

Table 3: Mixing Time of the Sequence of Dry, Wet and Rainy day

Initial Distribution	Number of Steps
0.0, 1.0, 0.0	34
0.0, 0.0, 1.0	32
1/3, 1/3, 1/3	31

Tables 2 and 3 present the mixing times for different initial distributions. In both the two-state and three-state models, the uniform initial distribution converges faster to the stationary distribution than the non-uniform initial distributions. For the two-state model, the uniform distribution reaches stationarity in 24 days, compared with 31 days for the non-uniform distributions. A similar pattern is observed for the three-state model.

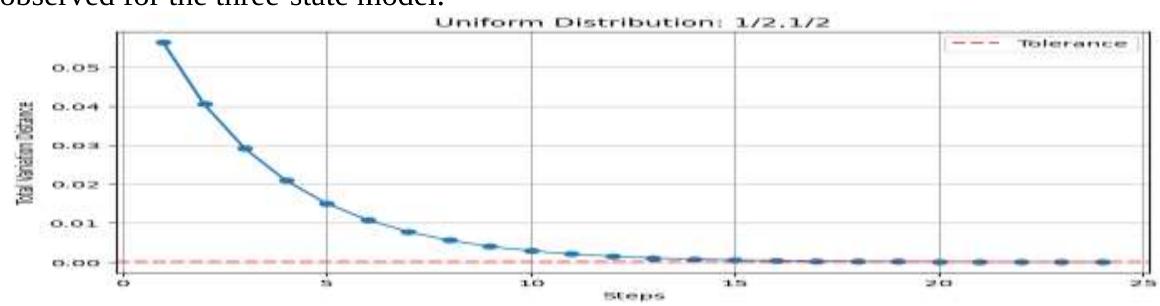

Figure 1: Mixing Behaviour of the Two-State Rainfall Markov Chain under a Uniform Initial Distribution

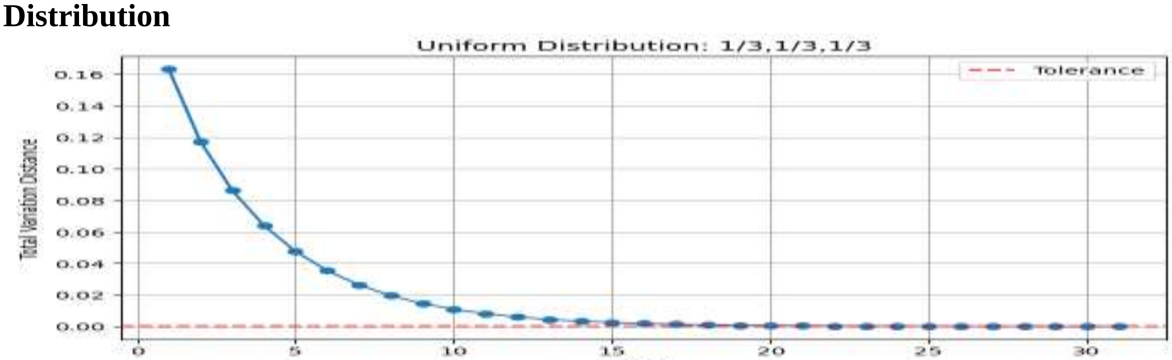

Figure 2 : Mixing Behaviour of the Three-State Rainfall Markov Chain under a Uniform Initial Distribution

Figures 1 and 2 show the total variation distance and mixing time for the two-state and three-state Markov chains under a uniform initial distribution. The two-state chain converges faster, with a mixing time of 24 days, compared with 31 days for the three-state chain. This indicates that increasing the state space slows convergence, consistent with [7]. The 7-day difference in mixing time (24 vs. 31 days) indicates a difference in the speed of convergence to the stationary distribution. While its practical impact on long-term agricultural and water resource planning is limited, it provides useful insight into rainfall persistence, climatic stability, and short-term decision-making.

Table 4: Dry-State Frequency Between Early and Later Periods (1985–2023)

Period	Dry Frequency (%)
1985–2004	46.8305
2005–2023	53.1695

The dry-state frequency increased from 46.83% in 1985–2004 to 53.17% in 2005–2023, indicating a greater proportion of dry conditions in the later period. This observed increase supports the Markov chain results by suggesting a shift toward higher dry-state occurrence.

Table 5: Comparison of AIC and BIC Values for Two-State and Three-State Models

Model	Number of States	Parameters (k)	Log-likelihood	AIC	BIC
Two-state Markov Chain	2	2	-4542.56	9089.12	9104.19
Three-state Markov Chain	3	6	-10148.6	20309.24	20354.46

The three-state Markov chain provides a more detailed representation of rainfall by distinguishing rainy conditions. Although it has higher AIC and BIC values than the two-state model, indicating a less parsimonious fit, it better captures rainfall variability when distinguishing between wet and heavy rainfall is important.

Table 6: Standard Errors and 95% Confidence Intervals for the Two-State Rainfall Model

Rainfall State	Stationary Distribution	Standard Error	95% Confidence Interval
Dry day	0.5784	0.0042	(0.5694, 0.5865)
Wet day	0.4216	0.0042	(0.4135, 0.4307)

The standard errors for both states are small, SE=0.0042, indicating high precision in the stationary probability estimates. The 95% confidence intervals are narrow, with the probability of dry days estimated between 56.94% and 58.65%, and wet days between 41.35% and 43.07%. These results confirm the stability and reliability of the estimated equilibrium probabilities

Table 7: Standard Errors and 95% Confidence Intervals for Three-State Rainfall Model

Rainfall State	Stationary Probability	Standard Error	95% Confidence Interval
Dry day	0.5784	0.0042	(0.5694, 0.5874)
Wet day	0.2626	0.0037	(0.2558, 0.2693)
Rainy day	0.159	0.0031	(0.1529, 0.1652)

The estimated standard errors are relatively small (0.0031–0.0042), indicating low uncertainty and high precision in the stationary probability estimates. The narrow 95% confidence intervals further confirm the stability of the estimates. For example, the probability of a dry day is expected to lie between 56.94% and 58.74%, while rainy days are expected to occur between 15.29% and 16.52% under the assumed Markov process

Conclusion

The mixing times of the two-state and three-state Markov chain models were compared to assess the effect of rainfall-state classification on convergence. The two-state model converged faster than the three-state model, indicating that increasing the number of rainfall states increases transition complexity and slows convergence. This highlights the trade-off between faster convergence and a more detailed representation of rainfall dynamics. Future studies could consider time-inhomogeneous Markov chains or models that account for uncertainty in transition probability estimates.

References

[1] Adah, V., and Agada, P. O. (2019). A Probabilistic Modelling Approach to Quantify and Assess the Impact of Climate Change on the Growth of Yam in Makurdi, Nigeria. *Fudma Journal of Sciences-issn: 2616-1370*, ***3***(4), 352-359

[2] Agada, P. O., Imande, M. T. and Ahmedu, M. O. (2018). Statistical Indicators of Climate change in Makurdi Metropolis: An Implication to crop production in the Area.The Journal *of the Mathematical Association of Nigeria (Abacus),* **45** (1): 198- 213.

[3] Azizah, A., Welastika, R., NurFalah, A., Ruchjana. B. N. and Abdullah, A. S. (2019) An Application of Markov Chain for Predicting Rainfall Data at West Java using Data Mining Approach. IOP Conference Series: Earth and Environmental Science. doi:10.1088/1755-1315/303/1/012026

[4] Dhritikesh, C. (2019). Significance of Change of Rainfall: Confidence Interval of Annual Total Rainfall. *Journal of Sciences,* **9**(3):151-166.

[5] Hajani, E. and Sarma, G. (2023). Generation of rainfall data series by using the Markov Chain model in three selected sites in the Kurdistan Region, Iraq. *AI in Civil Engineering*, 2(5):1-13 https://doi.org/10.1007/s43503-023-00014-2

[6] Hsu, D., Kontorovich, A., Levin, D. A., Peres, Y., Szepesvári , C., and Wolfe, G. (2019). Mixing Time Estimation in Reversible Markov Chains from a Single Sample Path. The Annals of Applied Probability,**29**(4):2439-2480

[7] Levin, D. A., Peres, Y. and Wilmer, E. L. (2009). Markov Chains and Mixing Times. Amer. Math. Soc., Providence, RI. MR2466937

[8] Miftahuddin, M., Maulidawani, Setiawan, I., IIhamsyah, Y. and Syamsuddin, F. (2020). Rainfall Analysis in the Indian Ocean by Using 6-States Markov Chain Model. *Journal of Earth and Environmental Science,* **429**(1): 012-012. DOI: 10.1088/1755-1315/429/1/012012

[9] Makur, A., _Mertzanidis, M., Psomas, A. and Terzoglou, A.(2023). On the Robustness of Mechanism Design under Total Variation Distance, https://doi.org/10.48550/arXiv.2310.07809

[10] Nuga, O. and Adekola L. (2018). A Markov chain Analysis of Rainfall Distributions in Three Southwestern Cities of Nigeria. *Research Journal of Physical Sciences,* **6**(4):1-5

[11] Pranto, M., Aziz, U., Das, L. C., Ghosh, S. and Islam, A. (2023). Forecasting Precipitation Using a Markov Chain Model in the Coastal Region in Bangladesh. Nature Environment and Pollution Technology, **23**(4): .2201-2209, DOI: 10.46488/NEPT.2024.v23i04.024

[12] Skuriat-Olechnowska, M. (2005). Statistical inference and hypothesis testing for Markov chains with Interval Censoring Application to bridge condition data in the Netherlands. A thesis submitted to the Delft University of Technology in conformity with the requirements for the degree of Master of Science

[13] Sneha, R. and, Uma, G. (2023). Analyzing the impact of Rainfall Patterns on Agriculture, Economy and Tourism in India. *International Journal of Environment and Climate Change,* **13**(11):4626-4637

[14] Tettey, M., Oduro, F. T., Adedia, D. and Abaye, D. A. (2017). Markov Chain Analysis of the Rainfall Patterns of Five Geographical Locations in the South Eastern Coast of Ghana. *Earth Perspectives*,**4**(6):1-11. DOI 10.1186/s40322-017-0042-6

[15] Wolfer, G. and Kontorovich, A. (2019). Estimating the Mixing Time of Ergodic Markov Chains. In *Conference on Learning Theory*, **99(**40): 3120-3159.

[16] Zhang, J. (2020). Markov Chains, Mixing Times and Coupling Methods with an Application in Social Learning. Senior Thesis submitted to Northwestern University

[17] Stidsen, T. and Glenstrup, A.J. (2005), Quantifying optimal mesh and ring design costs. Naval Research Logistics, 52: 150-158. https://doi.org/10.1002/nav.20008

[18] Huang, M., Tsai, C., Tsui, S., & Lin, W. (2023). Combining Data Discretization and Missing Value Imputation for Incomplete Medical Datasets. *Research Article*, **18**(11):295-032.

[19] Adah, V., Ogbuagu, E. M., & James, S. (2026). Optimal order selection in Markov chain modeling of rainfall occurrence in a tropical climate: Evidence from Makurdi, Nigeria. *Journal of Physical and Engineering Sciences, 1*(2), 45–50.

[20] Rauch, M., Bliefernicht, J., Maranan, M., Fink, A. H., & Kunstmann, H. (2025). Interannual rainfall variability in West Africa: Reconstruction based on atmospheric circulation patterns. International Journal of Climatology, 45(14),

[21] Mendes, O., Correia, E., & Fragoso, M. (2025). Variability and trends of the rainy season in West Africa with a special focus on Guinea-Bissau. *Theoretical and Applied Climatology, 156*(5), Article 242. https://doi.org/10.1007/s00704-025-05471-6

[22] Odunmorayo, M. T., & Adefisan, E. A. (2025). Evaluating the performance of coupled model intercomparison project phase 6 (CMIP6) models in simulating rainfall characteristics over West Africa. *Discover Environment, 3,* Article 221.