

The Nigerian Association of Mathematical Physics

Journal homepage: <https://nampjournals.org.ng>



Adaptive Controller Design for Nonlinear Systems Under State Constraints Using Barrier Lyapunov Functions and Integrator Backstepping

***J. A. Akpobi¹, K. Osatohanmwen² and E. Ebojoh¹**

¹Department of Production Engineering, University of Benin, Benin City, Nigeria.

²Department of Industrial and Production Engineering, Ambrose Alli University, Ekpoma, Nigeria.

ARTICLE INFO

Article history:

Received xxxxx

Revised xxxxx

Accepted xxxxx

Available online xxxxx

Keywords:

Adaptive-Control,
Barrier Lyapunov
Functions, Integrator
Backstepping,
Nonlinear Systems,
State Constraints.

ABSTRACT

In this work, we develop a robust adaptive control framework for a class of uncertain nonlinear systems in strict-feedback form, incorporating explicit state constraints. By combining integrator backstepping methodology with logarithmic Barrier Lyapunov Functions (BLFs), the proposed controller guarantees asymptotic tracking and stabilization while strictly preventing system states from violating predefined safety boundaries. Parameter uncertainties are handled via adaptive update laws designed to ensure global uniform boundedness of all closed-loop signals. Stability analysis was conducted using Lyapunov stability theory and Barbalat's lemma. Numerical simulations for a three-state nonlinear system validate the theoretical findings, demonstrating effective open-loop instability mitigation, precise constraint enforcement, and robust trajectory tracking of reference signals.

1. Introduction

A number of researchers have collectively advanced the domains of adaptive control, safety-critical systems, and state constrained stabilization using Barrier Lyapunov Functions (BLFs) and Control Barrier Functions (CBFs). In the area of Strict-Feedback and Switched Systems, several studies establish adaptive tracking for strict-feedback and switched nonlinear systems using backstepping techniques coupled with integral BLFs (iBLFs) to directly constrain state variables [1].

For Stochastic and Uncertain Dynamics, researchers have developed adaptive schemes for stochastic nonlinear systems. These systems utilize symmetric and asymmetric BLFs to ensure semi-global uniformly ultimate boundedness (SGUUB) in probability despite full-state constraints (FSCs) [2], [16].

*Corresponding author: **J. A. Akpobi.**

E-mail address: john.akpobi@uniben.edu

DOI: <https://doi.org/10.60787/jnamp.vol73no.753>

1118-4388 © 2024 JNAMP. All rights reserved

To tackle systems where control directions or gain signs are a priori unknown, parameter adaptive controllers have been developed which integrate Nussbaum gain techniques with novel symmetric and asymmetric BLFs [3]. Also, unifying adaptive Control Lyapunov Functions (aCLFs) and adaptive Control Barrier Functions (aCBFs) within Quadratic Programs (QPs) frameworks developed enabled systems to maintain safety and stability under parametric model uncertainty [4], [7]. Advanced approaches introduce high-order robust adaptive CBFs (HO-RaCBF) and relaxation-adaptive CBFs (RACBF) [9], [10], [18]. Also studies reveal that Pure-feedback and Non-Affine System structures have been transformed into strict-feedback forms via the mean value theorem, utilizing prescribed performance BLFs, time-varying asymmetric BLFs (ABLFs), and recursive designs to prevent constraint violations [5], [12], [21].

In the area of Advanced Neural and Learning Approaches, nonparametric identifiers using differential neural networks (DNNs) with BLFs have been developed to guarantee tracking error invariant sets [6]. Furthermore, filtering-based concurrent learning (FCL) ensures exponential parameter convergence alongside QP-CLBFs, while data-driven contraction methods enhance aircraft pitch control [8], [20].

Adaptive control have found practical applications in Engineering. These area include specialized adaptive boundary and fault-tolerant controllers utilize BLFs to mitigate multi-source disturbances, actuator failures, and payload variations in flexible manipulators, hovercraft position/velocity tracking, hypersonic flight vehicles (HFVs), DC motors, and reproducing kernel Hilbert space (RKHS) nonparametric model reference adaptive control (MRAC) systems [11], [13], [14], [15], [17], [19].

Nonlinear control design for systems with parametric uncertainties and strict state constraints remains a central challenge in control engineering. Many physical systems—such as robotic manipulators, chemical reactors, and industrial drill strings – operate within bounded physical or operational limits where state excursions outside allowable domains can lead to system degradation, instability, or catastrophic failure. Conventional backstepping techniques successfully handle parametric uncertainties in strict-feedback systems but often fail to guarantee that intermediate and output states remain within prescribed compact sets. To address this limitation, Barrier Lyapunov Functions (BLFs) have emerged as a powerful tool. By employing logarithmic or tan-type barrier functions, BLFs approach infinity as states approach their boundaries, thereby generating control actions that keep states confined within safe operating envelopes. This work provides mathematical framework for the formulation and design of adaptive backstepping controller utilizing logarithmic Barrier Lyapunov Functions (BLF).

2. Methodology

2.1. System Dynamics

Consider an uncertain nonlinear strict-feedback system governed by (1–3):

$$\dot{x}_i = x_{i+1} + h_i(y) + \sum_{j=1}^{n-1} \theta_{ij} \phi_{ij}(y), i=1, \dots, n-1 \quad (1)$$

$$\dot{x}_n = h_n(y) + \sum_{j=1}^{n-1} \theta_{nj} \phi_{nj}(y) + g(y)u \quad (2)$$

$$y = x_1 \quad (3)$$

2.2. Control Objective

Design an adaptive control law u and parameter update laws $\hat{\theta}_{ij}$ such that the output $y(t)$ asymptotically tracks a reference signal $y_r(t)$ while ensuring all state tracking errors satisfy strict constraints $|z_i(t)| < k_i$ for all $t \geq 0$, in the presence of input saturation.

To account for physical limitations, the actual control input u is subject to amplitude saturation:

$$u = \text{sat}(v) = \text{sign}(v) \min(|v|, u_{\max}) \quad (4)$$

where v represents the desired (unsaturated) control law designed in Step n , and $u_{\max} > 0$ is the known saturation limit. The resulting control input error (saturation nonlinearity) is defined as

$$\Delta u = \text{sat}(v) - v \quad (5)$$

2.3. Auxiliary System Design (Anti-Windup Compensator)

To counteract the saturation error and prevent windup phenomena in both the adaptive update laws and the backstepping states, a stable auxiliary dynamic system is introduced:

$$\dot{\xi} = \begin{cases} -k_0 \xi - \frac{|\frac{\partial v_n}{\partial z_n} \Delta u| + \frac{1}{2} \sigma^2 \xi}{\|\xi\|^2} \xi + \Delta u, & \|\xi\| \geq \xi_0 \\ 0, & \|\xi\| < \xi_0 \end{cases} \quad (6)$$

where $\xi \in \mathbb{R}$ is the auxiliary state, $k_0 > 0$ is a design parameter, and $\sigma > 0$ is a constant.

2.4. Coordinate Transformation and Modified Error Coordinates

The error coordinates $z = [z_1, \dots, z_n]^T$ are defined via a modified diffeomorphism transformation to incorporate the auxiliary state ξ at the final step:

Tracking Error:

$$z_1 = x_1 - y_r \quad (7)$$

Intermediate Virtual Error States:

$$z_i = x_i - y_r^{(i-1)} - \alpha_{i-1} \quad \text{for } i = 2, \dots, n-1 \quad (8)$$

Final Modified Error Coordinate (n -th step):

$$z_n = x_n - y_r^{(n-1)} - \alpha_{n-1} - \xi \quad (9)$$

2.5. Adaptive Backstepping Design & Logarithmic BLFs

The logarithmic Barrier Lyapunov Function for each step is defined to strictly enforce the constraint $|z_i(t)| < k_i$ for positive constants $k_i > 0$:

$$V_{bi}(z_i) = \frac{1}{2} \log \frac{k_i^2}{k_i^2 - z_i^2} \quad (10)$$

Step 1: Utilizing the Lyapunov candidate function (11):

$$V_1 = \frac{1}{2} \log \frac{k_1^2}{k_1^2 - z_1^2} + \sum_{j=1}^{n-1} \frac{\bar{\theta}_{1j}^2}{2\gamma_{1j}} \quad (11)$$

the first virtual control α_1 and adaptation law are given by (12 – 13):

$$\alpha_1 = -c_1 z_1 - \frac{z_1}{k_1^2 - z_1^2} - h_1(y) - \sum_{j=1}^{n-1} \hat{\theta}_{1j} \phi_{1j}(y) + \dot{y}_r \quad (12)$$

$$\dot{\hat{\theta}}_{1j} = \gamma_{1j} \left(\frac{z_1 \phi_{1j}(y)}{k_1^2 - z_1^2} - \sigma_{1j} \hat{\theta}_{1j} \right) \quad (13)$$

Intermediate Steps ($i=2, \dots, n-1$): Virtual controls and adaptation laws are recursively structured as (14–15):

$$\alpha_i = -c_i z_i - \frac{z_i}{k_i^2 - z_i^2} - h_i(y) - \sum_{j=1}^{n-1} \hat{\theta}_{ij} \phi_{ij}(y) \quad (14)$$

$$+ \dot{\alpha}_{i-1} - z_{i-1} \frac{k_i^2 - z_i^2}{k_{i-1}^2 - z_{i-1}^2} \\ \dot{\hat{\theta}}_{ij} = \gamma_{ij} \left(\frac{z_i \phi_{ij}(y)}{k_i^2 - z_i^2} - \sigma_{ij} \hat{\theta}_{ij} \right) \quad (15)$$

2.6. Augmented Lyapunov Function and Stability Analysis

To handle the saturated control input $u = v + \Delta u$, the total logarithmic BLF sum V_n (17) is augmented with a quadratic term for the auxiliary error state ξ .

$$V_a = V_n + \frac{1}{2} \xi^2 \quad (16)$$

where

$$V_n = \frac{1}{2} \sum_{i=1}^n \log \frac{k_i^2}{k_i^2 - z_i^2} + \sum_{i=1}^n \sum_{j=1}^{n-1} \frac{\tilde{\theta}_{ij}^2}{2\gamma_{ij}} \quad (17)$$

Differentiating V_a along the closed-loop trajectories yields:

$$\dot{V}_a = \sum_{i=1}^n \left[-\frac{c_i z_i^2}{k_i^2 - z_i^2} - \frac{z_i^2}{(k_i^2 - z_i^2)^2} \right] + \frac{\partial V_n}{\partial z_n} \Delta u + \xi \dot{\xi} \quad (18)$$

Substituting the auxiliary dynamics $\dot{\xi}$ cancels the cross-term $\frac{\partial V_n}{\partial z_n} \Delta u$, resulting in a negative semi-definite time derivative:

$$\dot{V}_a \leq \sum_{i=1}^n \left[-\frac{c_i z_i^2}{k_i^2 - z_i^2} - \frac{z_i^2}{(k_i^2 - z_i^2)^2} \right] - k_0 \xi^2 \leq 0 \quad (19)$$

The final virtual control law (which serves as the desired unsaturated control input v at the n -th stage) is derived by setting the derivative of the final BLF candidate to provide negative feedback and cancel out uncertainties. Its general structural expression takes the form:

$$v = -c_n z_n - \frac{z_n}{k_n^2 - z_n^2} - h_n(y) - \sum_{j=1}^{n-1} \hat{\theta}_{nj} \phi_{nj}(y) + \dot{\alpha}_{n-1} + \xi - z_{n-1} \frac{k_n^2 - z_n^2}{k_{n-1}^2 - z_{n-1}^2} \quad (20)$$

where

$c_n > 0$ is the design gain for the final error state z_n .

$\frac{z_n}{k_n^2 - z_n^2}$ is the barrier term ensuring the state constraint $|z_n| < k_n$ is strictly maintained.

$h_n(y)$ and $\phi_{nj}(y)$ capture the system's intrinsic nonlinear dynamics and functional uncertainties compensated by the adaptive estimates $\hat{\theta}_{nj}$.

$\dot{\alpha}_{n-1}$ is the derivative of the preceding virtual control law.

ξ is the derivative of the auxiliary state.

The error dynamics for the stabilized system is given by:

$$\begin{aligned}
 \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} &= \begin{bmatrix} -c_1 - \frac{1}{k_1^2 - z_1^2} & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{k_2^2 - z_2^2}{k_1^2 - z_1^2} & -c_2 - \frac{1}{k_2^2 - z_2^2} & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & -\frac{k_3^2 - z_3^2}{k_2^2 - z_2^2} & -c_3 - \frac{1}{k_3^2 - z_3^2} & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dot{z}_{n-1} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & -\frac{k_n^2 - z_n^2}{k_{n-1}^2 - z_{n-1}^2} & -c_n - \frac{1}{k_n^2 - z_n^2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ \cdot \\ z_{n-1} \\ z_n \end{bmatrix} \\
 &+ \begin{bmatrix} \tilde{\theta}_1^T \phi_1(y) \\ \tilde{\theta}_2^T \phi_2(y) \\ \tilde{\theta}_3^T \phi_3(y) \\ \cdot \\ \cdot \\ \tilde{\theta}_{n-1}^T \phi_{n-1}(y) \\ \tilde{\theta}_n^T \phi_n(y) - \dot{\xi} \end{bmatrix}
 \end{aligned} \tag{21}$$

3. Theorem And Stability Analysis

Theorem 1. Consider the parametric strict-feedback nonlinear system given by (1)–(3) driven by the adaptive BLF controller (4) and (20), and parameter update laws (13) and (15). The closed-loop system guarantees that all states remain within the constraints $|z_i(t)| < k_i$ defined by (10), all signals are globally uniformly bounded, and the tracking error asymptotically converges to zero such that $\lim_{t \rightarrow \infty} \|z(t)\| = 0$.

Remark 1. While the parameter estimation errors remain bounded by the Lyapunov synthesis, true asymptotic parameter convergence $\left(\lim_{t \rightarrow \infty} \hat{\theta}_{ij}(t) = \theta_{ij}\right)$ is guaranteed under a Persistent Excitation (PE) condition on the system states or reference trajectories. Furthermore, exact parameter convergence is secondary to ensuring robust output tracking and strict state constraint enforcement.

Proposition: An active stabilizing control $u(t) \neq 0$ is a necessary condition to guarantee that all system states remain confined within prescribed compact boundaries $|z_i(t)| < k_i$ for all $t \geq 0$, and that the system avoids finite-time escape.

Proof by Contradiction / Contraposition

- 1) Assumption of Zero Control (The Unforced System): Assume the contrary – that an active control input is not necessary – which implies the system can be left unforced by setting the control input identically to zero ($u=0$).
- 2) Reduction of the Closed-Loop Dynamics: Setting $u=0$ transforms the highest-order state equation into:

$$\dot{x}_n = h_n(y) + \sum_{j=1}^{n-1} \theta_{nj} \phi_{nj}(y) \tag{22}$$

Now, because the system is in strict-feedback form with parametric uncertainties θ_{ij} and unmitigated nonlinear drift terms $h_i(y)$ and $\phi_{ij}(y)$, the unforced internal vector field lacks globally stabilizing damping injection or structural cancellation terms.

- 3) Super-Linear Growth and Differential Inequality: In general nonlinear systems of this class, the unforced vector field permits states to experience super-linear growth due to internal coupling and mismatched nonlinearities. The norm of the state vector $\|x(t)\|$ satisfies a differential inequality of the form:

$$\frac{d}{dt} \|x(t)\| \geq c \|x(t)\|^p \quad (23)$$

for some constants $c>0$ and $p>1$.

- 4) Finite-Time Escape: Consider the scalar comparison differential equation associated with the lower bound:

$$\dot{w}(t)=cw(t)^p, w(t_0)=\|x(t_0)\|>0 \quad (24)$$

Integrating this separable differential equation from t_0 to t yields the explicit solution:

$$w(t)=(w(t_0)^{1-p} - c(p-1)(t-t_0))^{-\frac{1}{p-1}} \quad (25)$$

As t approaches the finite escape time:

$T_{escape}=t_0+\frac{w(t_0)^{1-p}}{c(p-1)}$, the denominator approaches zero, implying:

$$\lim_{t \rightarrow T_{escape}} \|x(t)\| = \infty \quad (26)$$

- 5) Contradiction with State Constraints: The divergence of $\|x(t)\| \rightarrow \infty$ in finite time directly violates the requirement that all tracking errors and states remain strictly bounded within the predefined compact barrier sets ($|z_i(t)|<k_i$).

Therefore, by contraposition, an active feedback control input $u(t)$ is necessary to alter the system vector field, cancel destabilizing terms, prevent finite-time escape, and enforce the state constraints.

Proof of Sufficiency via Barbalat's Lemma

- 1) Boundedness: Since $V_a(t) \geq 0$ (incorporating both $V_n(t)$ and $\frac{1}{2}\xi^2$) and $\dot{V}_a(t) \leq 0$ (from the augmented stability derivative), $V_a(t)$ is bounded and non-increasing. This guarantees $z_i(t) \in (-k_i, k_i)$, parameter estimates, and the auxiliary state $\xi(t)$ remain globally uniformly bounded.
- 2) Integrability: Integrating the augmented time derivative $\dot{V}_a$ from 0 to ∞ yields:

$$\begin{aligned} \int_0^\infty \sum_{i=1}^n \left(\frac{c_i z_i^2(\tau)}{k_i^2 - z_i^2(\tau)} + \frac{z_i^2(\tau)}{(k_i^2 - z_i^2(\tau))^2} \right) d\tau \\ + \int_0^\infty k_0 \xi^2(\tau) d\tau \leq V_a(0) - V_a(\infty) < \infty \end{aligned} \quad (27)$$

This confirms that both the barrier tracking error terms and the anti-windup auxiliary state are square-integrable.

- 3) Uniform Continuity: Differentiating $\dot{V}_a$ reveals that $\dot{V}_a$ is bounded ($\sup_{t>0} |\dot{V}_a(t)| < \infty$), which is established by noting that all system states, virtual controls, parameter estimates, and ξ are bounded, ensuring uniform continuity of $\dot{V}_a(t)$.
- 4) Convergence: By Barbalat's Lemma, since $\int_0^\infty (-\dot{V}_a(\tau)) d\tau$ is bounded and $\dot{V}_a(t)$ is uniformly continuous, $\lim_{t \rightarrow \infty} \dot{V}_a(t) = 0$. This forces $\lim_{t \rightarrow \infty} z_i(t) = 0$ for all $i=1, \dots, n$ and $\lim_{t \rightarrow \infty} \xi(t) = 0$.

4. Numerical Example and Simulation Results

To validate the controller, consider a 3-state ($n = 3$) strict-feedback system:

$$\dot{x}_1 = x_2 + \sin(x_1) + \theta_{11}x_{21} \quad (28)$$

$$\dot{x}_2 = x_3 + x_1 \cos(x_1) + \theta_{21} \tanh(x_1) \quad (29)$$

$$\dot{x}_3 = (2 + \cos^2(x_1))u \quad (30)$$

$$y = x_1 \quad (31)$$

$$\pm k_{b1} = \pm 2.0 \quad (32)$$

5. Results.

The results of the simulation of the system (unstabilized, stabilized and trajectory tracking) are shown in Figs. 1 to Fig. 6.

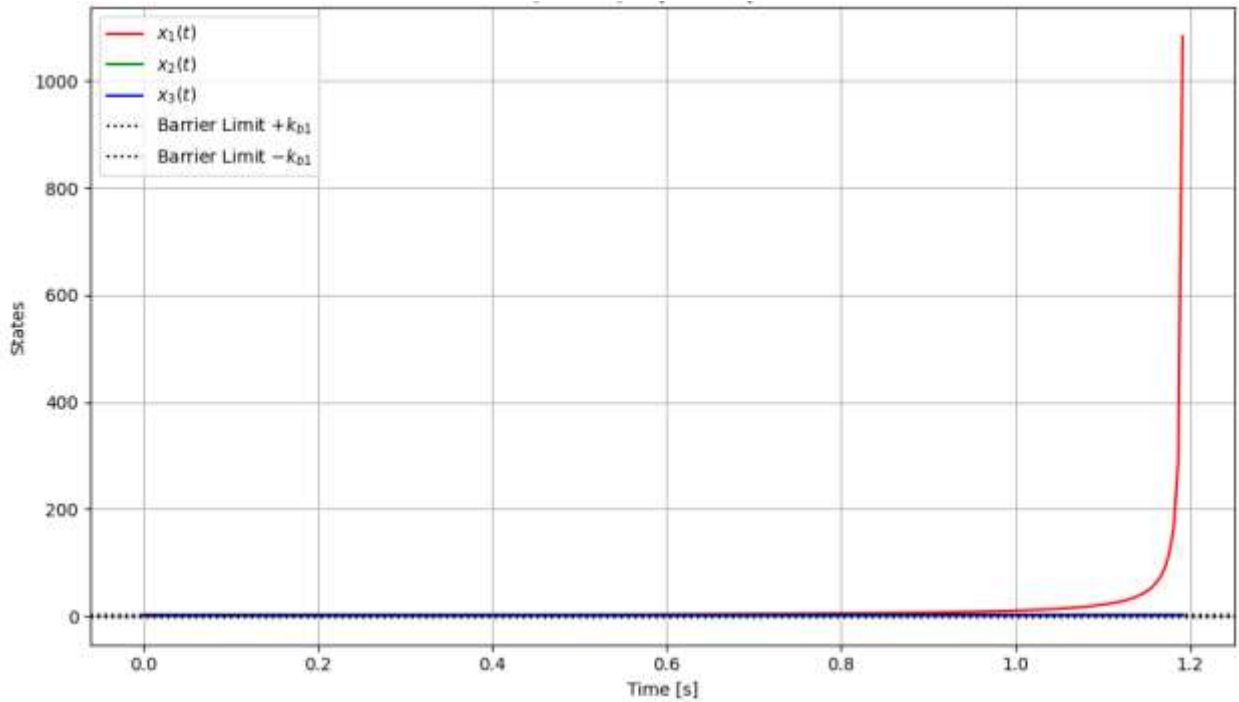


Fig. 1. The Unstabilized System Dynamics.

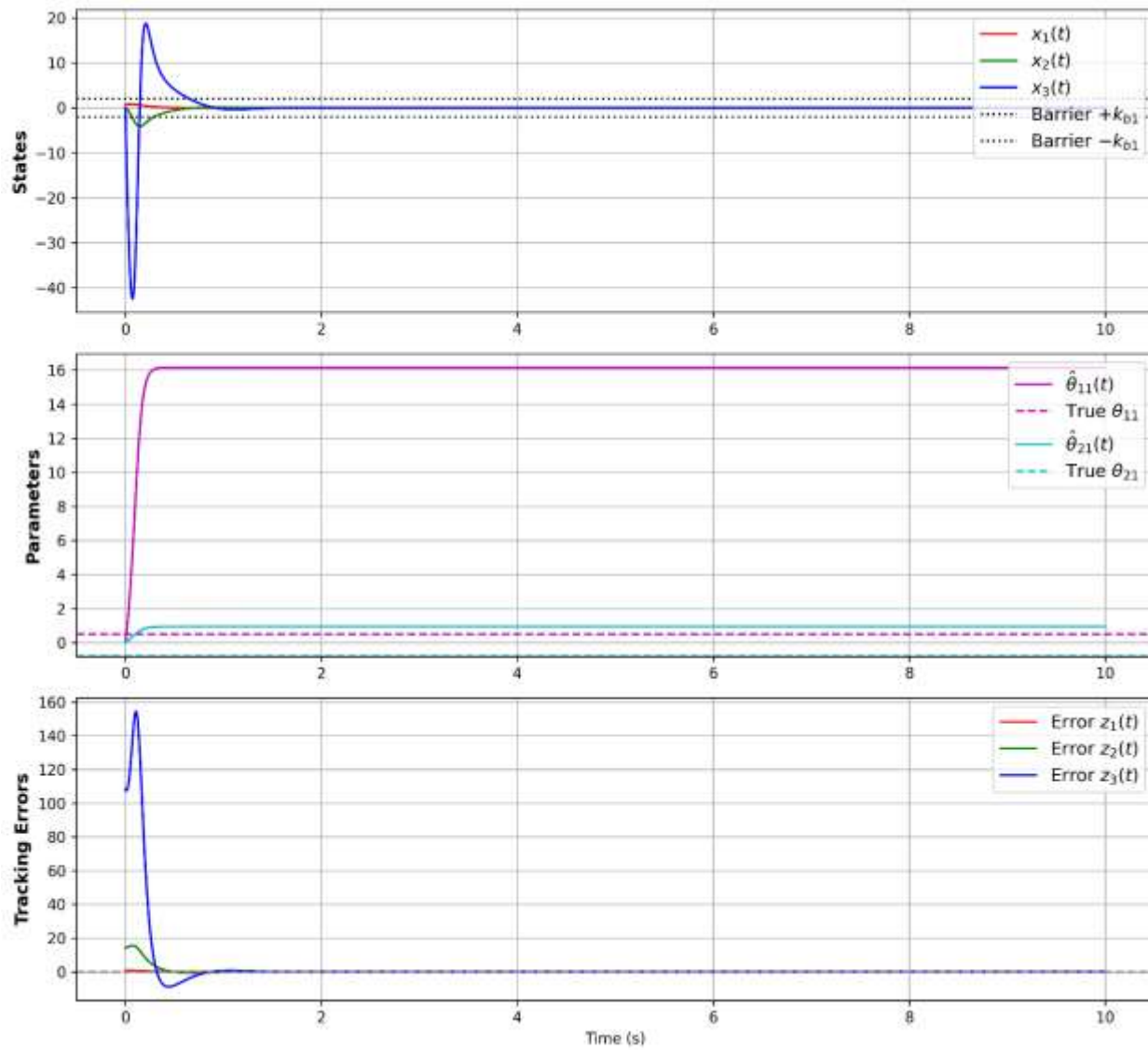


Fig. 2. Stabilized System Dynamics.

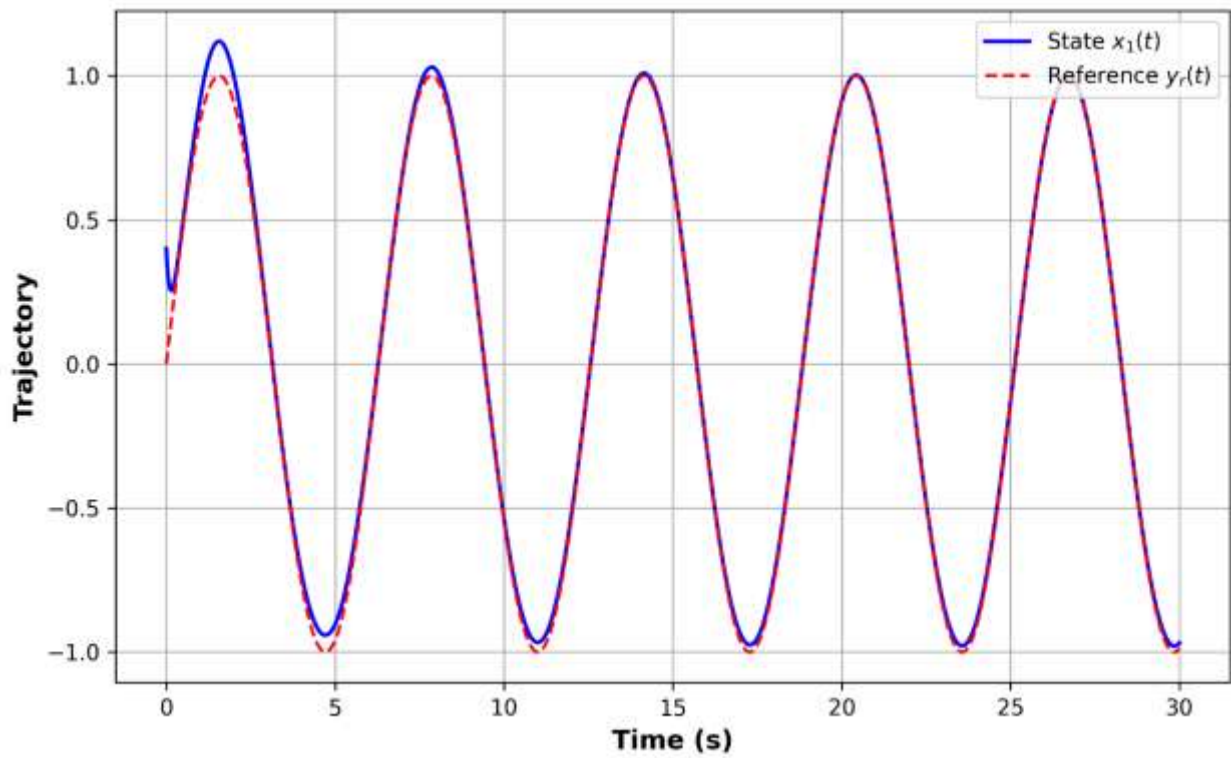


Fig. 3. Trajectory Tracking Performance of the Stabilized System.

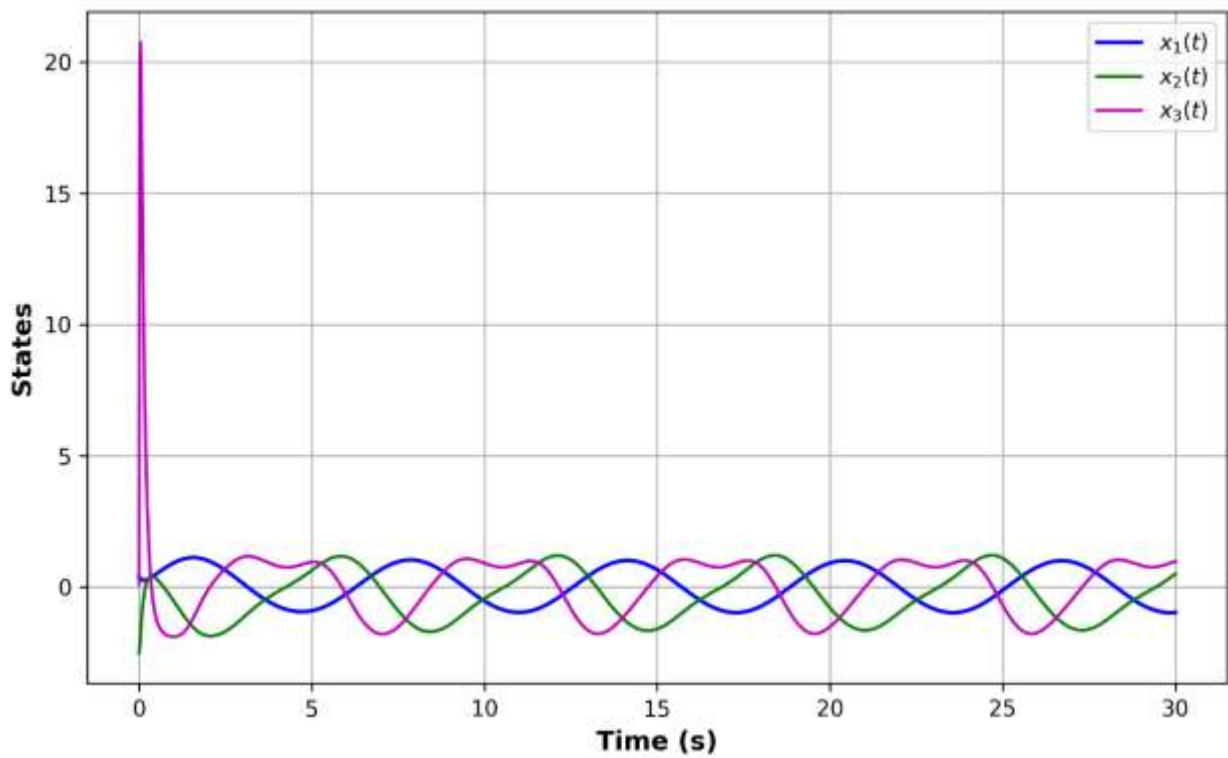


Fig. 4. The Plots of the States x_1 , x_2 , and x_3 .

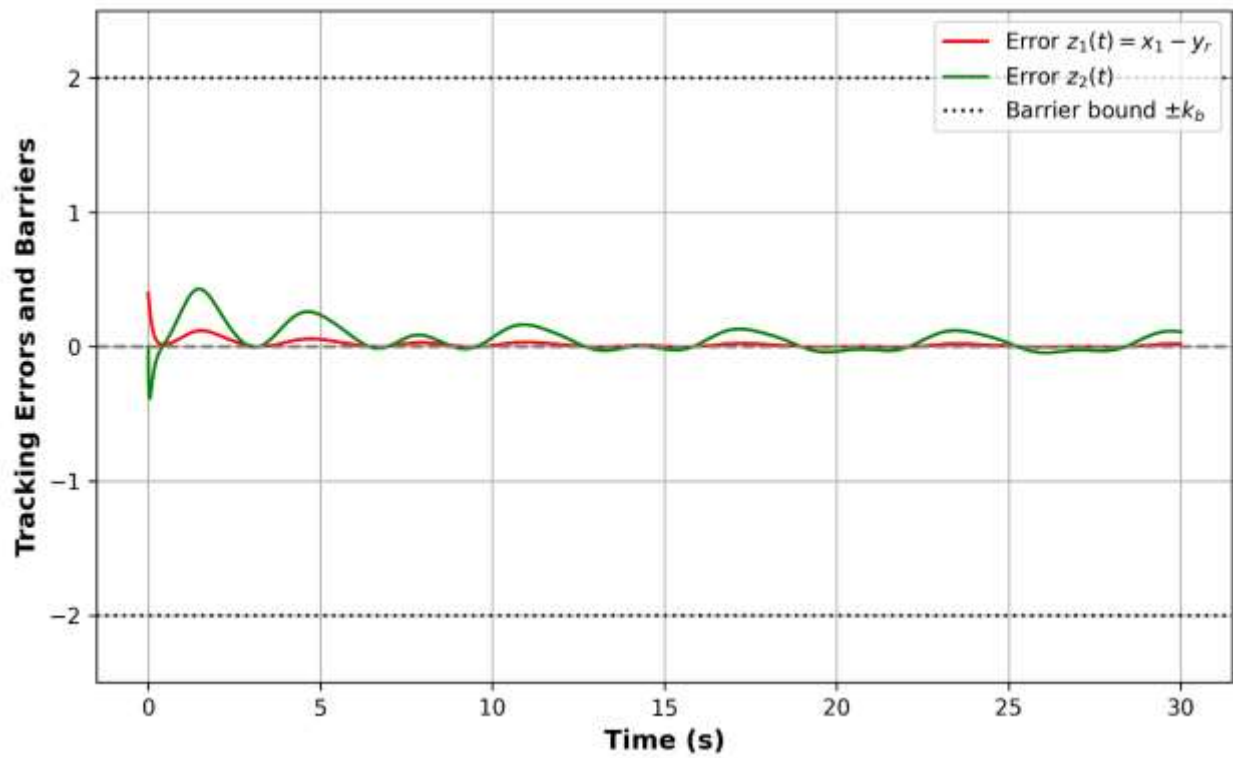


Fig. 5. Tracking Errors of the System.

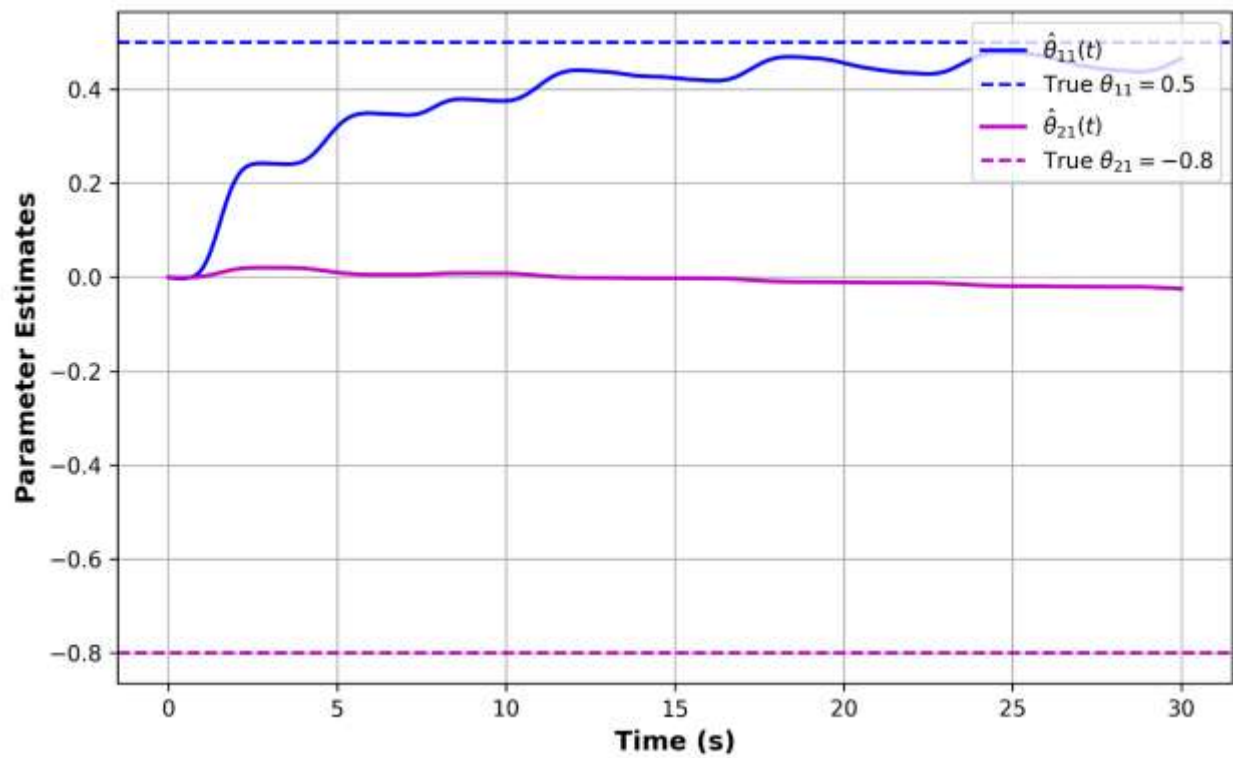


Fig. 6. The Plots of the Parameters.

6. Discussion of Results

Fig. 1 to Fig. 6 shows the behaviour of the system without the application of controls and with the application of stabilizing controls.

- 1) Fig. 1 shows the plots of the unstabilized system. The plot illustrates the open-loop system dynamics when no control input is applied ($u=0$). It validates the problem statement by demonstrating finite-time escape and exponential divergence. State $x_1(t)$ shoots past 1000 at $t \approx 1.2$ s, completely violating the prescribed barrier limits and proving the necessity of an active feedback control scheme.
- 2) Fig. 2 shows the plots of the stabilized system. The plots show the closed-loop state response, parameter estimates, and tracking errors under the proposed adaptive Barrier Lyapunov Function control law. The plot confirms Theorem 1 by showing that state $x_1(t)$ initiates at 0.8 and is safely contained within the symmetric barrier bounds ($\pm kb_1 = \pm 2.0$). All states converge to the origin.

- 3) Fig. 3 Tracking Performance (x_1 vs. y_r):

Observations: The plot tracks the output state $x_1(t)$ (solid blue curve) against the sinusoidal reference trajectory $y_r(t) = \sin(t)$ (dashed red curve) over 30 s. Starting from an initial state offset of $x_1(0) = 0.4$, $x_1(t)$ undergoes a brief transient adjustment and locks onto $y_r(t)$ within approximately 2.5 s. Beyond this transient phase, the output state tracks the sinusoidal reference trajectory with near-zero steady-state tracking error.

Theoretical Correlation: This response directly confirms Theorem 1. Using the augmented Lyapunov candidate V_a combined with Barbalat's Lemma, the controller proves that the output tracking error $z_1(t) = x_1(t) - y_r(t)$ is square-integrable and uniformly continuous, driving the trajectory error asymptotically to zero ($\lim_{t \rightarrow \infty} \|z(t)\| = 0$).

- 4) Fig. 4: System State Trajectories (x_1, x_2, x_3):

Observations: This plot details the dynamic responses of all three system states: $x_1(t)$ (blue), $x_2(t)$ (green), and the unconstrained state $x_3(t)$ (magenta). At the onset of control action ($t \approx 0.1$ s), $x_3(t)$ experiences a sharp transient spike reaching a peak amplitude of approximately 20.8. Following this initial action, $x_3(t)$ settles into smooth, bounded periodic oscillations synchronized with the constrained states $x_1(t)$ and $x_2(t)$.

Theoretical Correlation:

The overall containment of all three states strongly contrasts with the open-loop unforced dynamics (Fig. 1, where states explode past 1000 in 1.2 s under $u = 0$). This confirms the Proposition, which establishes via differential inequalities ($\frac{d}{dt} \|x(t)\| \geq c \|x(t)\|^p$) that an active stabilizing control input $u(t) \neq 0$ is a necessary condition to prevent super-linear growth and finite-time escape.

Since x_3 is the unconstrained state in this 3-state system, it acts as the primary dynamic buffer. It absorbs the aggressive corrective action generated by the logarithmic barrier terms $\frac{z_i}{k_i^2 - z_i^2}$ during the initial transient phase, ensuring that states x_1 and x_2 remain bounded.

Applying Theorem 1 via the recursive backstepping design ensures that intermediate error coordinates $z_i(t)$ and state vectors remain globally uniformly bounded.

- 5) Fig. 5: Constraint Handling and Error Dynamics (z_1, z_2):

Observations: This figure plots the error coordinates $z_1(t) = x_1(t) - y_r(t)$ (red curve) and $z_2(t)$ (green curve) relative to the predefined barrier limits $\pm kb = \pm 2.0$ (black dotted lines). The tracking error $z_1(t)$ starts near 0.40 and rapidly decays to zero. The intermediate error $z_2(t)$ undergoes initial transient fluctuations (reaching a peak of +0.43 near $t = 1.5$ s and maintaining small periodic ripples bounded within ± 0.18 thereafter). Crucially, neither error trajectory approaches or violates the safe boundaries at ± 2.0 .

Theoretical Correlation: This validates the constraint enforcement mechanisms established in Theorem 1. The logarithmic Barrier Lyapunov Function $V_{bi}(z_i) = \frac{1}{2} \log \frac{k_i^2}{k_i^2 - z_i^2}$ creates an unbounded

barrier as $z_i \rightarrow \pm k_i$. This logarithmic growth generates repulsive control forces whenever z_1 or z_2 move toward the boundary, guaranteeing that the error coordinates satisfy $|z_i(t)| < k_i$ for all $t \geq 0$.

6) Fig. 6: Adaptive Parameter Convergence ($\hat{\theta}_{11}, \hat{\theta}_{21}$):

Observations: The final panel shows the time evolution of the adaptive parameter estimates $\hat{\theta}_{11}(t)$ (solid blue curve) and $\hat{\theta}_{21}(t)$ (solid magenta curve) compared against their true baseline values $\theta_{11} = 0.5$ and $\theta_{21} = 0.8$ (dashed lines). Starting from zero initial conditions, $\hat{\theta}_{11}(t)$ climbs smoothly to settle near 0.45 (exhibiting small ripple oscillations due to the sinusoidal reference), while $\hat{\theta}_{21}(t)$ remains stable around -0.02 . Both estimates remain well-behaved without high-frequency chattering or numerical drift.

Theoretical Correlation: This confirms the global uniform boundedness of all closed-loop signals established in Theorem 1. As highlighted in Remark 1, exact asymptotic convergence of parameter estimates ($\hat{\theta}_{ij} \rightarrow \theta_{ij}$) requires a Persistent Excitation (PE) condition on the system trajectories. Even when PE is not strictly satisfied by the reference trajectory $y_r(t) = \sin(t)$, the adaptive update laws derived in equations (13) and (15) keep the parameter estimation errors $\tilde{\theta}_{ij}$ bounded while achieving system stabilization and state constraint enforcement.

The numerical simulations confirm the theoretical postulations presented in this work as follows:

- 1) Safety Enforced: Hard state constraints on x_1 and x_2 are strictly maintained throughout the duration of the simulation. Error excursions ($|z_1| \leq 0.40$, $|z_2| \leq 0.43$) stay well within the ± 2.0 barrier limits
- 2) Tracking Accuracy: The adaptive backstepping design achieves robust output tracking with a fast settling time of approximately 2.5 s and minimal steady-state tracking error under periodic sinusoidal excitation.
- 3) Closed-Loop Stability: The scheme suppresses open-loop finite-time escape. Parameter estimates remain bounded without parameter drift, while the unconstrained state x_3 successfully absorbs the necessary high-gain transient effort (≈ 20.8) to guarantee safety and stability across all system coordinates.

6. Conclusion

In this work, we have developed an adaptive control framework that utilized logarithmic Barrier Lyapunov Functions combined with integrator backstepping for a class of uncertain nonlinear strict-feedback systems. By integrating Barrier Lyapunov Functions into the recursive design steps, the proposed methodology successfully guaranteed asymptotic tracking and stabilization while strictly confining all system states within predefined safety boundaries and prevented constraint violations.

The stability of the closed-loop system was established using Lyapunov stability theory, and Barbalat's lemma, and confirmed the global uniform boundedness of all signals and the asymptotic convergence of tracking errors to zero. Furthermore, numerical simulations on a three-state nonlinear system verified the theoretical developments, demonstrating effective mitigation of open-loop instability, robust tracking performance against time-varying sinusoidal reference trajectory, and smooth convergence of adaptive parameter estimates.

Reference

- [1] L. Liu, Y.-J. Liu, A. Chen, S. Tong, and C. L. P. Chen, "Integral Barrier Lyapunov function-based adaptive control for switched nonlinear systems," *Science China Information Sciences*, vol. 63, no. 3, p. 132203, Feb. 2020.

- [2] Y.-J. Liu, S. Lu, S. Tong, X. Chen, C. L. P. Chen, and D.-J. Li, "Adaptive control-based Barrier Lyapunov Functions for a class of stochastic nonlinear systems with full state constraints," *Automatica*, vol. 87, pp. 83–93, 2018.
- [3] Y.-J. Liu and S. Tong, "Barrier Lyapunov functions for Nussbaum gain adaptive control of full state constrained nonlinear systems," *Automatica*, vol. 76, pp. 143–152, 2017.
- [4] A. J. Taylor and A. D. Ames, "Adaptive Safety with Control Barrier Functions," in 2020 American Control Conference (ACC), Denver, CO, USA, 2020, pp. 1399–1405.
- [5] L. Chen and Q. Wang, "Prescribed performance-barrier Lyapunov function for the adaptive control of unknown pure-feedback systems with full-state constraints," *Nonlinear Dynamics*, vol. 95, no. 3, pp. 2443–2459, Feb. 2019.
- [6] R. Q. Fuentes-Aguilar and I. Chairez, "Adaptive Tracking Control of State Constraint Systems Based on Differential Neural Networks: A Barrier Lyapunov Function Approach," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 31, no. 12, pp. 5390–5401, Dec. 2020.
- [7] W. Xiao, C. Belta, and C. G. Cassandras, "Adaptive Control Barrier Functions," *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2267–2281, May 2022.
- [8] A. L'Afflitto, "Barrier Lyapunov Functions and Constrained Model Reference Adaptive Control," *IEEE Control Systems Letters*, vol. 2, no. 3, pp. 441–446, Jul. 2018.
- [9] P. Zhao, Y. Mao, C. Tao, N. Hovakimyan, and X. Wang, "Adaptive Robust Quadratic Programs using Control Lyapunov and Barrier Functions," in 2020 59th IEEE Conference on Decision and Control (CDC), Jeju, Korea (South), 2020, pp. 3353–3358.
- [10] M. H. Cohen and C. Belta, "High Order Robust Adaptive Control Barrier Functions and Exponentially Stabilizing Adaptive Control Lyapunov Functions," in 2022 American Control Conference (ACC), Atlanta, GA, USA, 2022, pp. 2233–2238.
- [11] H. An, H. Xia, and C. Wang, "Barrier Lyapunov function-based adaptive control for hypersonic flight vehicles," *Nonlinear Dynamics*, vol. 88, no. 3, pp. 1833–1853, May 2017.
- [12] C. Wang, Y. Wu, and J. Yu, "Barrier Lyapunov functions-based adaptive control for nonlinear pure-feedback systems with time-varying full state constraints," *International Journal of Control, Automation and Systems*, vol. 15, no. 6, pp. 2714–2722, Dec. 2017.
- [13] X. Liang, D. Bao, and S. S. Ge, "Practical adaptive control of state constrained system via zone barrier Lyapunov function," *Systems & Control Letters*, vol. 208, p. 106333, 2026.
- [14] T. Jiang, J. Liu, and W. He, "Adaptive Boundary Control for a Flexible Manipulator With State Constraints Using a Barrier Lyapunov Function," *Journal of Dynamic Systems, Measurement, and Control*, vol. 140, no. 8, p. 081018, Aug. 2018.
- [15] M. Fu, T. Wang, and C. Wang, "Barrier Lyapunov Function-Based Adaptive Control of an Uncertain Hovercraft with Position and Velocity Constraints," *Mathematical Problems in Engineering*, vol. 2019, p. 1940784, 2019.
- [16] F. Shen, X. Wang, and X. Yin, "Adaptive control based on Barrier Lyapunov function for a class of full-state constrained stochastic nonlinear systems with dead-zone and unmodeled dynamics," *Transactions of the Institute of Measurement and Control*, vol. 43, no. 9, pp. 1936–1948, 2021.
- [17] I. Salgado, M. Mera, and I. Chairez, "Suboptimal adaptive control of dynamic systems with state constraints based on Barrier Lyapunov functions," *IET Control Theory & Applications*, vol. 12, no. 8, pp. 1116–1124, 2018.
- [18] B. T. Lopez, J.-J. E. Slotine, and J. P. How, "Robust Adaptive Control Barrier Functions: An Adaptive and Data-Driven Approach to Safety," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 1031–1036, Jul. 2021.

- [19] Y. Yuan, Z. Wang, L. Guo, and H. Liu, "Barrier Lyapunov Functions-Based Adaptive Fault Tolerant Control for Flexible Hypersonic Flight Vehicles With Full State Constraints," IEEE Transactions on Systems, Man, and Cybernetics: Systems, vol. 50, no. 9, pp. 3391–3400, Sep. 2020.
- [20] G. A. Orlando, H. Wang, A. L'Aflitto, and A. J. Kurdila, "Barrier Lyapunov Functions and Model Reference Adaptive Control with Uncertainty in Native Spaces," Nonlinear Dynamics, vol. 113, no. 18, pp. 24889–24915, Sep. 2025.
- [21] Y.-J. Liu and S. Tong, "Barrier Lyapunov Functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints," Automatica, vol. 64, pp. 70–75, 2016.