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APPLICATION OF ADOMIAN DECOMPOSITION METHODS IN SOLVING SOME SELECTED NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This paper considers the nonlinear partial differential equations. The Adomian Decomposition Method (ADM) was proposed as a method for solving a special kind of nonlinear partial differential equation with attention focused on some selected nonlinear partial differential equation. The nonlinear partial differential equation considered in this study includes advection equation, Sine-Gordon equation and Burger's equation, subject to given initial values. This method was subjected to test of convergence and it was observed that it converges rapidly.
Adomian decomposition method was applied in solving the three nonlinear partial equations highlighted above and the method demonstrates that the solution is obtained with a fast convergence, thereby, making the method mathematically tractable and less approximation error. The effect of the noise term was reduced and this makes the method attractive and convenient. There is no need to transform the nonlinear term to linear terms before solving the adomian polynomial. Hence, the method is effective and suitable for nonlinear partial differential equations with initial values
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Keywords**:** Adomian Decomposition Method, Advection equation, Sine-Gordon equation, Burgers' equation, Picard method, nonlinear partial differential equation

1. INTRODUCTION

Partial differential equation has a wide range of application in the field of medicine, sciences, engineering and commerce. Most mathematical model are usually expressed in the form of partial differential equation. Thus, the relevance of partial differential equation in physical phenomena becomes inevitable. In this paper, our focus is on trying to x-ray the solutions of non-linear partial differential equation using one of the major analytical approach in mathematics (Adomian Decomposition Method). This method has a wide solution technique which is used to solve Ordinary Differential equations (both linear and linear) and even partial differential equation without any doubt, this paper presents easy and convenient mathematical methods to solve nonlinear partial differential equation. Emphasis will also be focused on the theoretical framework which will act as a basis for the development of this method. Non-linear partial differential equation is considered because of the uniqueness and complexity of some of these non-linear differential equation. When the solutions to these nonlinear partial differential equations are obtained, it will bring further advancement in medicine, sciences (Physical and Biological), commerce and engineering. Thereby brings to bear the Adomian decomposition method which converges rapidly unlike most other numerical methods. In practical applications, we are face with more and more nonlinear problems [1]. Before now, researchers have proposed different methods to solve nonlinear problems. They include implicit hybrid methods [2], the residual power series method [3] and trigonometric basic functions [4]. In solving nonlinear problem, we transform them to linear problem called linearization before solving them. The nonlinear problem are solved easily and elegantly without linearizing the problem by using Adomian Decomposition [5].

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Some controversy in the literature on comparison between the picard method and the adomian decomposition method [6] examined that the ADM was equivalent to PM but only considered linear differential equation. The equivalence does not hold for non-linear differential equation. According to [7]. A partial differential equation (PDE) is an equation containing that contains the dependent variable (the unknown function) and its partial derivatives define a non-linear partial differential equation, we must know what linear partial differential equation is, then, the reverse is the nonlinear partial differential equation.

According to [7], a partial differential equation is linear if :

i. The power of the dependent variable and each partial derivative contained in the equation is one.
ii. The coefficient of the dependent variable and the coefficient of each partial derivative are constant or independent variables.

However, if any of these conditions is not satisfied, the equation is called nonlinear

2. METHODOLOGY

In this section, we use the functional equation to lay a theoretical framework for this work.

$$If\ U - Nu = f, \quad (2.1)$$

where N is a nonlinear operator a Hilbert space H into H, f is a given function (System input) in H, and u is an unknown function (System output) in H. [8], such a system is called nonlinear system. If it contains the nonlinear term $N(u)$. We are interested in finding the exact solution or approximate solution to (2.1). We adopt the Adomian decomposition Method (ADM) to find the approximate solution to equation (2.1) as follows:

$$Let\ U = \sum_{n=0}^{\infty} U_n \quad (2.2)$$

Hence, following the problem to determine every form U_n, however, there is a nonlinear term N in this equation, which brings about a great difficulty to complete our goal. Our aim is to decompose the nonlinear term N as:

$$Nu = \sum_{n=0}^{\infty} A_n \quad (2.3)$$

Where A_n are Adomian polynomial of $u_0, u_1, \dots u_n$, that is

$$A_n = A_n(u_0, u_1, \dots, u_n) \quad (2.4)$$

We set $\varphi = \sum_{n=0}^{\infty} \emptyset^n U_n$ and $N\varphi = \sum_{n=0}^{\infty} \emptyset^n A_n$ (2.5)

where $\emptyset$ is a parameter. It should be emphasized that $\emptyset = 1$, then $N\varphi = Nu$; the case $\emptyset = 1$. Hence, it is easy to see that

$$A_n = \frac{1}{n!}\frac{d^n}{d\emptyset^n} N\left(\sum_{n=0}^{\infty} \emptyset^n u_n\right)\Big|_{\lambda\neq 0} \quad (2.6)$$

Once A_n is obtained by (2.6) and substituting (2.2) and (2.3) into (2.1), we get:

$$\sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} A_n + f \quad (2.7)$$

We define the recurrent equation of the Adomian relation

$$u_0 = f,\ u_{n+1} = A_n(u_0, u_1, \dots, u_n) \quad (n\epsilon N) \quad (2.8)$$

From which u_n are solvable formally. If we can solve u_n up to $n \leq N$, then $u = \sum_{n\leq N} U_n$ is called the Nth approximate solution to (2.1)

If the non-linearity has the form $Nu = h(u)$, where h is a smooth function of u, then one can obtain the several Adomian polynomials. [9]

$$A_0 = h'(u_0); \quad (2.9)$$

$$A_1 = h'(u_0)u_1; \quad (2.10)$$

$$A_2 = h'(u_0)u_2 + h''(u_0)\frac{u_1^2}{2!}; \quad (2.11)$$

$$A_3 = h'(u_0)u_3 + h''(u_0)u_1u_2 + g^3(u_0)\frac{u_1^3}{3!} \quad (2.12)$$

$$A_4 = h'(u_0)u_4 + h''(u_0)\left(\frac{u_2^2}{2!} + u_1u_2\right) + g^3(u_0)\frac{u_4}{4!} \quad (2.13)$$

The Convergence Analysis of Adomian Decomposition Method

The proof of convergence of Adomian Decomposition Method was first developed by [8].

Consider the functional equation

$$y = y_0 + f(y);\ y\epsilon X \quad (2.14)$$

where X is a Banach space and $f(y): X \to X$ is analytic near y_0

$Y_n = y_0 + \sum_{k=1}^{n} y_k;$

$f_n(Y_n) = \sum_{k=0}^{n} A_k(y_0, y_1, \dots, y_n)$

The Adomian decomposition method is equivalent to determining a sequence $\{Y_n\}_{n\epsilon N}$

Now, $Y_0 = y_0$,

$Y_{n+1} = y_0 + f_n(Y_n),\ n \geq 0$

If there exists limits

$Y = \lim_{n\to\infty} Y_n, \qquad f = \lim_{n\to\infty} f_n,$

In the Banach space X, then Y solves the fixed point equation $Y = y_0 + f(Y)$ in X. The convergence of the Adomian Decomposition method is provided under the following conditions

$$||f(y)||_x \leq 1\ \forall\ y \in X \quad (2.15)$$

and

$$||f_n(Y_n) - f(Y)||_x \to 0 \quad \text{as } n \to \infty \quad (2.16)$$

Thus, equation (2.15) implies a constraint on the non-linear function $f(y)$ and equation (2.16) implies convergence of the series of Adomian polynomial to the locally analytic function $f(y)$.

The convergence of the decomposition series have been investigated by several authors. The theoretical treatment of the convergence of the decomposition method has been considered in [8] and [10]. The results of these convergence have been improved by [11] who proposed a new convergence proof of Adomian technique based on properties of convergence series. They obtained some results about the speed of convergence of this method providing us to solve linear and non-linear functional equations.

3. APPLICATION OF THE METHOD

The formula given in (2.6) for the first time was introduced by [12]. In this section, we are going to carry out the implementation of this method to three non-linear equations. The three non-linear equations includes

a) The Advection equation
b) The Sine-Gordon equation
c) The Burgers equation

In this application, we take advantage of the noise terms in [7] for the convergence of the solution. However, the noise terms are associated with inhomogeneous partial differential equation which is characterized by some of the equations outlined above. The noise term usually provide the solution after two consecutive iterations if they exist in components u_0 and u_1.

a) The Advection Equation

The Advection equation is given by $u_t + uu_x = x + xt^2$, $u(x,0) = 0,$ $t > 0$
where $u = u(x,t)$.
We now implement the Modified Adomian Decomposition Method to solve the equation

$$L_t u(x,t) = x + xt^2 - uu_x \quad (3.1a)$$

Applying the inverse operator, $L_t^{-\prime}$ to both sides;

$$L_t^{-\prime} L_t u(x,t) = L_t^{-\prime} x + L_t^{-\prime} xt^2 - L_t^{-\prime} uu_x$$

$$u(x,t) = L_t^{-\prime} x + L_t^{-\prime} xt^2 - L_t^{-\prime} uu_x$$

$$= xt + \frac{1}{3}xt^3 - L_t^{-\prime} uu_x \quad (3.2a)$$

Using the decomposition assumption for the linear term uu_x defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (3.3a)$$

$$uu_x = \sum_{n=0}^{\infty} A_n \quad (3.4a)$$

Substitute (3.3a) and (3.4a) in (3.2a)

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{1}{3}xt^3 - L_t^{-\prime}(A_n) \quad (3.5a)$$

The modified decomposition method admits the use of a modified recursive relation:

$$u_0(x,t) = xt$$

$$u_1(x,t) = \frac{1}{3}xt^3 - L_t^{-\prime}(A_0)$$

$$u_{k+2}(x,t) = -L_t^{-\prime}(A_{k+1}), \quad k \geq 0$$

Consequently, we obtain

$$u_0(x,t) = xt$$

$$u_1(x,t) = \frac{1}{3}xt^3 - L_t^{-\prime}(xt^2)$$

$$u_{k+1}(x,t) = 0, \quad k \geq 0 \quad (3.7a)$$

In view of (3.7a), the exact solution is given by $u(x,t) = xt$

b) The Sine-Gordon Equation

The Sine-Gordon Equation is given by

$$u_{tt} - u_{xx} = \sin u$$

$$u(x,0) = \frac{\pi}{2}, \quad u_t(x,0) = 1 \quad (3.1b)$$

To solve the equation (3.1b), we define the general Sine-Gordon equation by

$$u_{tt} - c^2 u_{xx} + \propto \sin u = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

where c and $\propto$ are constants (3.2b)

To solve this equation using the Adomian Decomposition Method, applying $L_t^{-\prime}$ to (3.2b), we get:

$$u(x,t) = f(x) + tg(x) + c^2 L_t^{-\prime}\big(u_{xx}(x,t)\big) - \propto L_t^{-\prime}\,(\sin u\,(x,t)) \quad (3.3b)$$

Taking the series on both sides,

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + L_t^{-\prime}[c^2(\sum_{n=0}^{\infty} u_n(x,t)) - \propto (\sum_{n=0}^{\infty} A_n)] \quad (3.4b)$$

This gives the recursive relation:

$$u_0(x,t) = f(x) + tg(x)$$

$$u_{k+1}(x,t) = c^2 L_t^{-\prime}\big(u_{xx}(x,t)\big) - \propto L_t^{-\prime}(A_n), \quad k \geq 0 \quad (3.5b)$$

Using the equation (3.5b) in (3.1b), we get:

$$u_0(x,t) = \frac{\pi}{2} + t$$

$$u_{k+1}(x,t) = L_t^{-\prime}\big(u_{kxx}(x,t)\big) + L_t^{-\prime}(A_k), \quad k \geq 0 \quad (3.6b)$$

Using Adomian polynomials for $\sin u$ as shown above gives:

$$u_0(x,t) = \frac{\pi}{2} + t;$$

$$u_1(x,t) = 1 - \cos t;$$

$u_2(x,t) = \sin t - \frac{3}{4}t - \frac{1}{8}\sin 2t;$ (3.7b)

Summing these iterates yields:

$u(x,t) = \frac{\pi}{2} + t + 1 - \cos t + \sin t - \frac{3}{4}t - \frac{1}{8}\sin 2t + \cdots$ (3.8b)

So that the series solution is:

$u(x,t) = \frac{\pi}{2} + t + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots,$

which is obtained using Taylor series expansion for trigonometric functions.

c) The Burgers' Equation

[13] considered one of the fundamental model equations in fluid mechanics. The equation demonstrated the coupling between diffusion and convection processes.

The standard form of Burgers' equation is given by

$u_t + uu_x = vu_{xx}, \quad t > 0$

where v is a constant that defines the Kinematic viscosity. If $v = 0$, the equation is called INVISCID BURGERS' EQUATION. This Inviscid Burgers' equation governs gas dynamics.

Non-linear Burgers' equation is described as one of the most simple nonlinear partial differential equation using the Adomian Decomposition method.

$u_t + uu_x = u_{xx}, \; u(x,0) = f(x)$ (3.1c)

We now solve a special case of equation (3.1c) as follows:

$u_t + uu_x = u_{xx}, \; u(x,0) = x, \quad t > 0$ (3.2c)

Taking the $L_t^{-'}$ on both sides of (3.2c), we have

$u(x,t) = x + L_t^{-'}(u_{xx}) - L_t^{-'}(u_{xx}) - L_t^{-'}(uu_x)$ (3.3c)

Using the Decomposition Series, we obtain

$\sum_{n=0}^{\infty} u(x,t) = x + L_t^{-'}(\sum_{n=0}^{\infty} u_n(x,t))_{xx}) - L_t^{-'}(\sum_{n=0}^{\infty} A_n)$ (3.4c)

This gives the recursive relation

$u_0(x,t) = x$

$u_{k+1}(x,t) = L_t^{-'}(u_{kxx}(x,t)) - L_t^{-'}(A_k), \; k \geq 0$ (3.5c)

Using the Adomian polynomial, we obtain

$u_0(x,t) = x;$

$u_1(x,t) = L_t^{-'}(u_{0xx}(x,t)) - L_t^{-'}(A_0) = -xt;$

$u_2(x,t) = L_t^{-'}(u_{1xx}(x,t)) - L_t^{-'}(A_1) = xt^2;$

$u_3(x,t) = L_t^{-'}(u_{2xx}(x,t)) - L_t^{-'}(A_2) = -xt^3;$ (3.6c)

Summing these iterates gives the series solution

$u(x,t) = x(1 - t + t^2 - t^3 + \cdots)$ (3.7c)

Consequently, the exact solution is given by

$u(x,t) = \frac{x}{1+t} \; ; \qquad |t| < 1$ (3.8c)

4. CONCLUSION

In this study, we considered the Adomian decomposition method in solving Advection equation, Sine-Gordon equation and Burgers' equation. The Adomian Decomposition was adopted in solving these non-linear partial differential equations. The method converges rapidly and very convenient in solving those equations. A convergence test was carried out on the method and it was discovered that the method is convergent compared to other methods for solving nonlinear ordinary or partial differential equations , there is no need for linearization of nonlinear term. This makes the method useful, easy and convenient. Hence, it is mathematically tractable and reduces approximation error. It can also be used to solve other nonlinear partial differential equations due to this property which gives it advantage over most methods. It can also be used to solve Boussinesq equation especially in their nonlinear form.

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