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SENSITIVITY INDICES OF A GENERIC INFECTIOUS DISEASE EPIDEMIC FOR DISEASE SPREAD AND ELIMINATION

S.T. Tonga, L.P. Kenda, D. Andeyangde and D.S. Joro

Department of Physics, Taraba State University, Jalingo-Nigeria

Abstract

This paper studies the sensitivity indices for natural death rate, rate of death due to disease, recovery rate, contact rate, and recruitment rate. The susceptible-infected-recovered (SIR) was used to develop the model equations, the disease-free equilibrium, endemic equilibrium was obtained using the model equation. The basic reproductive number was calculated using the next-generation matrix method and the eigenvalue equation of the Jacobian of the disease-free equilibrium point. The basic reproductive number was used to calculate the sensitivity indices for the natural death rate, the death rate due to disease, recovery rate, contact rate, and recruitment rate. It was discovered from the model assumption and the findings that the high contact rate and recruitment rate will spread further the disease otherwise the disease will be eliminated

Keywords: Basic reproduction number, sensitivity index, disease-free equilibrium, endemic equilibrium

Introduction

According to the World Health Organization (WHO), infectious diseases are responsible for a quarter to a third of all deaths worldwide. As of 2008, four of the top ten causes of death were due to infectious diseases; and in low-income countries, five of the top killers were due to infectious diseases [1]. Mathematical models are essential in the prediction of the future of disease [2,3]. According to [4] pandemic is an epidemic of an infectious disease that has spread across a large region, for instance, multiple continents or worldwide affecting a substantial number of individuals [5-6]. Disease epidemic has been a prevalent issue in the present community and many times this occurs due to an increase in population and contact rate with the infected individuals. Infectious diseases can be modeled using several methods [7-8]. It is important to know some of the parameters that play a major role in the spread and elimination of disease[9]. This article provides the sensitivity indices for the generic case of disease epidemic which can be adopted for any disease mode.

Model Formulation

The SIR model was adopted for this work with S as susceptible I infected and R recovered individuals

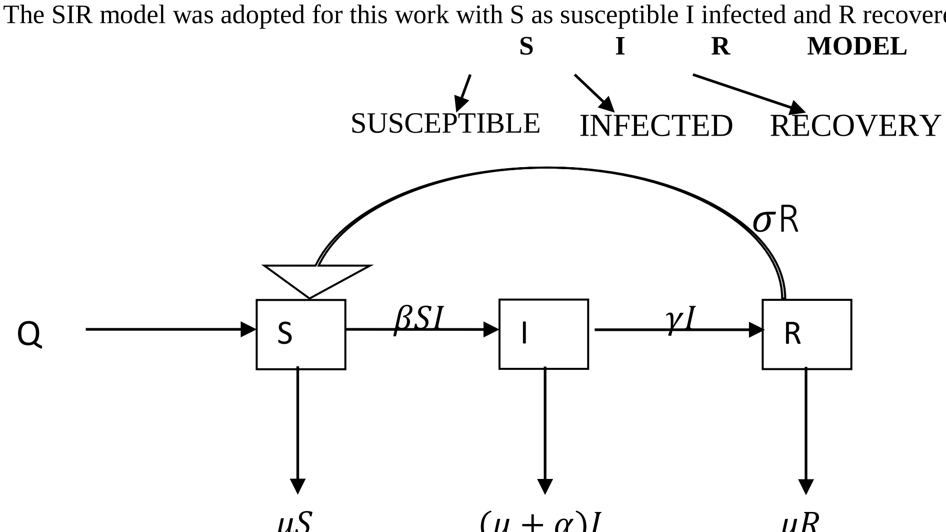

Figure 1: Model flow diagram

Corresponding Author**:** Tonga S.T., Email: tahiru.tonga@tsuniversity.edu.ng, Tel: +2348060507547

The model equations are

$$\frac{dS}{dt} = Q - \beta SI - \mu S + \sigma R \quad (1)$$

$$\frac{dI}{dt} = \beta SI - (\mu + \alpha + \gamma)I \quad (2)$$

$$\frac{dR}{dt} = \gamma I - (\mu + \sigma)R \quad (3)$$

Q = recruitment into susceptibility.
μ = natural death rate
α = death rate due to disease
β = contact rate
γ = rate at which the infected are treated to recover.
σ = rate at which recovered joined the susceptible.
βSI *= the contact rate of susceptible becoming infected*
μS *= the natural death rate of the susceptible*
σR *= rate of recovery joining the susceptible*
γI *= the rate at which the infected are treated to recover*
$(\mu + \sigma)R$ *= the rate at which the recovered died naturally and some joined the susceptible*
$(\mu + \alpha)I$ *= the rate at which the infected either died naturally or due to the disease*

4.2 Disease Free Equilibrium (DFE)

This is the point at which there are neither infected individuals nor recovered, as such, we set them to zero from the model equation. Asterisk represents the equilibrium point coordinate

The equilibrium point is (S, I*, R*), but I* = R* = 0*

Also at equilibrium

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

$$\frac{dS}{dt} = Q - \beta S^* \times 0 - \mu S^* + \sigma \times 0 = 0$$

$$\frac{dI}{dt} = \beta S^* \times 0 - (\mu + \gamma + \alpha) \times 0 = 0$$

$$\frac{dR}{dt} = \gamma \times 0 - (\mu + \sigma) \times 0 = 0$$

$$\frac{dS}{dt} = Q - \mu S^* = 0$$

$$Q - \mu S^* = 0$$

$$S^* = \frac{Q}{\mu} \quad (4)$$

Therefore the disease-free equilibrium point is $\left(\frac{Q}{\mu}, 0, 0\right)$

4.3 Endemic Equilibrium

Disease spread

The same condition is applied to this case also i.e

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

Using (1) to (3) again we have

$$\frac{dS}{dt} = Q - \beta S^* I^* - \mu S^* + \sigma R^* = 0$$

$$\frac{dI}{dt} = \beta S^* I^* - (\mu + \gamma + \alpha) I^* = 0$$

$$\frac{dR}{dt} = \gamma I^* - (\mu + \sigma) R^* = 0$$

$$\gamma I^* - (\mu + \sigma) R^* = 0$$

$$I^* = \frac{(\mu + \sigma) R^*}{\gamma} \quad (5)$$

$$\beta S^* I^* = (\mu + \gamma + \alpha) I^*$$

$$S^* = \frac{(\mu + \gamma + \alpha)}{\beta} \quad (6)$$

Putting (5) and (6) in (1)

$$Q - \beta\left(\frac{(\mu + \gamma + \alpha)}{\beta}\right)\left(\frac{(\mu + \sigma)}{\gamma}\right) R^* - \mu\left(\frac{(\mu + \gamma + \alpha)}{\beta}\right) + \sigma R^* = 0$$

$$R^*\left(\sigma - \frac{(\mu + \gamma + \alpha)}{\beta}\frac{(\mu + \sigma)}{\gamma}\right) = \mu\left(\frac{(\mu + \gamma + \alpha)}{\beta}\right) - Q$$

$$R^*\left(\sigma\gamma-\frac{(\mu+\gamma+\alpha)(\mu+\sigma)}{\gamma}\right)=\mu\left(\frac{(\mu+\gamma+\alpha)}{\beta}\right)-\beta Q$$

$$R^*=\left(\frac{\mu(\mu+\gamma+\alpha)-\beta Q}{\beta}\right)x\ \frac{\gamma}{[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]}$$

$$R^*=\frac{\gamma[\mu(\mu+\gamma+\alpha)-\beta Q]}{\beta[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]} \qquad (7)$$

$$I^*=\frac{(\mu+\sigma)}{\gamma}R^*$$

(7) into (5)

$$I^*=\frac{(\mu+\sigma)}{\gamma}\,x\left\{\frac{\gamma[\mu(\mu+\gamma+\alpha)-\beta Q]}{\beta[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]}\right\}$$

$$I^*=\frac{(\mu+\sigma)[\mu(\mu+\gamma+\alpha)-\beta Q]}{\beta[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]} \qquad (8)$$

Therefore, the endemic equilibrium point is

$$\left(\frac{(\mu+\gamma+\alpha)}{\beta},\frac{(\mu+\sigma)[\mu(\mu+\gamma+\alpha)-\beta Q]}{\beta[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]},\frac{\gamma[\mu(\mu+\gamma+\alpha)-\beta Q]}{\beta[\sigma\gamma-(\mu+\gamma+\alpha)(\mu+\sigma)]}\right)$$

4.4 Basic Reproductive Number

The inverse of the relative removal rate is called basic reproduction number or reproductive number, usually represented as R_o. The R_o is defined, from an epidemiologist perspective [10], as the average number of secondary 10 cases arising from an average primary case in an entirely susceptible population. The basic reproduction number is used to study the global impact that a disease can produce on a population, as a $R_o > 1$ would mean that the number of individuals infected will increase for the previous generation of infected individuals, and a $R_o < 1$ would mean the opposite, which is a decrement in that number, on the other hand, the sensitivity index can be used to determine the future of an infectious disease [11] as follows

Sensitivity index SI > 0 implies disease spread and SI < 0 implies disease elimination

We shall use the next generation matrix for the calculation of the basic reproduction number

$$R_O = fv^{-1} \qquad (9)$$

Where *f means disease entry and*
v means disease exit

$$f=\frac{\partial F}{\partial I} \qquad (10)$$

$$v=\frac{\partial V}{\partial I} \qquad (11)$$

$$F=\beta S^* I, \qquad (12)$$

$$V=(\mu+\gamma+\alpha)I \qquad (13)$$

$$\left.\begin{aligned}\frac{\partial F}{\partial I}&=\beta S^*\\ \frac{\partial V}{\partial I}&=(\mu+\gamma+\alpha)\end{aligned}\right\} \qquad (14)$$

Putting (4), (10), (11) and (14) in (9)

$$R_0=fv^{-1}=\frac{\partial F}{\partial I}x\frac{1}{\frac{\partial V}{\partial I}}=\frac{\beta Q}{\mu}x\frac{1}{(\mu+\gamma+\alpha)}$$

$$R_o=\frac{\beta Q}{\mu(\mu+\gamma+\alpha)} \qquad (15)$$

Another method for calculating the basic reproduction number is by finding the Jacobian of the model equation [12-13] at the disease-free mode.

The jacobian matrix is defined as the matrix of the first-order differential of a function, it is defined below as:

$$J(S,I,R)=\begin{pmatrix}\frac{\partial S'}{\partial S} & \frac{\partial S'}{\partial I} & \frac{\partial S'}{\partial R}\\ \frac{\partial I'}{\partial S} & \frac{\partial I'}{\partial I} & \frac{\partial I'}{\partial R}\\ \frac{\partial R'}{\partial S} & \frac{\partial R'}{\partial I} & \frac{\partial R'}{\partial R}\end{pmatrix} \qquad (16)$$

Applying (16) to (1) to (3) to our model equation we obtain

$$J(S,I,R)=\begin{pmatrix}-\beta I-\mu & -\beta S & \sigma\\ \beta I & \beta S-(\mu+\alpha+\gamma) & 0\\ 0 & \gamma & -(\mu+\sigma)\end{pmatrix} \quad (17)$$

The basic reproduction number is obtained at the disease-free equilibrium point i.e *S=I=0.*

Hence

$$J(S,0,0)=\begin{pmatrix}-\mu & -\beta S & \sigma\\ 0 & \beta S-(\mu+\alpha+\gamma) & 0\\ 0 & \gamma & -(\mu+\sigma)\end{pmatrix} \quad (18)$$

The eigenvalue of the jacobian was used to calculate the basic reproduction number.

This is done by taking the minimum eigenvalue and equating it to zero and taking the ratio of the right-hand side to the left-hand side. The ratio defines the basic reproduction which is seen below

$$|J(S,0,0)-\lambda I|=0 \quad (19)$$

$$\begin{vmatrix}-\mu-\lambda & -\beta S & \sigma\\ 0 & \left(\beta S-(\mu+\alpha+\gamma)\right)-\lambda & 0\\ 0 & \gamma & -(\mu+\sigma)-\lambda\end{vmatrix}=0 \quad (20)$$

The characteristic equation is

$$\lambda^3-(\mu+\alpha+\gamma+\sigma-\beta S)\lambda^2+(\alpha\sigma+\gamma\sigma-\mu^2-\beta S\sigma)\lambda+\mu(\mu^2+\mu\sigma+\alpha\sigma+\mu\gamma+\sigma\gamma+\alpha\mu-\beta S\mu-\beta S\sigma)=0$$

The respective values of the eigenvalues of the Jacobian determinant matrix are

$\lambda=-\mu,(\mu+\alpha+\gamma)-\beta S,(\mu+\sigma)$

Recall that

The dominant eigenvalue $(\mu+\alpha+\gamma)-\beta S$

$$(\mu+\alpha+\gamma)-\frac{\beta Q}{\mu}=0$$

But

$$S=\frac{Q}{\mu}$$

Hence

$$(\mu+\alpha+\gamma)=\frac{\beta Q}{\mu} \quad (21)$$

The ratio of the right-hand side to the left-hand side is constant R_O which is the basic reproduction number

$$R_O=\frac{\beta Q}{\mu(\mu+\alpha+\gamma)} \quad (22)$$

As the basic reproductive number

Sensitivity Index

The sensitivity index is generally given as

$$SI=\frac{\pi}{R_O}\frac{\partial R_O}{\partial \pi}$$

Where $\pi=\mu,\alpha,\gamma,Q \text{ and } \beta$

If the value of SI is greater than one, this implies disease spread

If the value of SI is lesser than one, this implies disease elimination

When $\pi=\mu$

$$\frac{\partial R_O}{\partial\mu}=\frac{-\beta Q(2\mu+\alpha+\gamma)}{(\mu^2+\mu\alpha+\mu\gamma)},\ SI=\frac{\mu}{R_O}\frac{-\beta Q(2\mu+\alpha+\gamma)}{(\mu^2+\mu\alpha+\mu\gamma)},\ SI=\frac{\mu}{R_O}\frac{\partial R_O}{\partial\mu}=\frac{-(2\mu+\alpha+\gamma)}{(\mu+\alpha+\gamma)}$$

When $\pi=\alpha$

$$\frac{\partial R_O}{\partial\alpha}=\frac{-\beta Q\mu}{(\mu^2+\mu\alpha+\mu\gamma)^2)},\ SI=\frac{\alpha}{R_O}\frac{-\beta Q\mu}{(\mu^2+\mu\alpha+\mu\gamma)},\ SI=\frac{\alpha}{R_O}\frac{\partial R_O}{\partial\alpha}=\frac{-\alpha}{(\mu+\alpha+\gamma)}$$

When $\pi=\gamma$

$$\frac{\partial R_O}{\partial\gamma}=\frac{-\beta Q\mu}{(\mu^2+\mu\alpha+\mu\gamma)^2)},\ SI=\frac{\gamma}{R_O}\frac{-\beta Q\mu}{(\mu^2+\mu\alpha+\mu\gamma)},\ SI=\frac{\gamma}{R_O}\frac{\partial R_O}{\partial\gamma}=\frac{-\gamma}{(\mu+\alpha+\gamma)}$$

When $\pi = \beta$

$\frac{\partial R_O}{\partial \beta} = \frac{Q}{(\mu^2+\mu\alpha+\mu\gamma)}, SI = \frac{\beta}{R_O}\frac{\partial R_O}{\partial \beta}, SI = \frac{\beta}{R_O}\frac{\partial R_O}{\partial \beta} = 1$

When $\pi = Q$

$\frac{\partial R_O}{\partial Q} = \frac{\beta}{(\mu^2+\mu\alpha+\mu\gamma)}, SI = \frac{Q}{R_O}\frac{\partial R_O}{\partial Q}, SI = \frac{Q}{R_O}\frac{\partial R_O}{\partial Q} = 1$

Table 1

Parameter	Sensitivity Index	Comment
μ	$\frac{-(2\mu+\alpha+\gamma)}{(\mu+\alpha+\gamma)}, < \mathbf{0}$	Disease elimination
α	$\frac{-\alpha}{(\mu+\alpha+\gamma)}, < \mathbf{0}$	Disease elimination
γ	$\frac{-\gamma}{(\mu^2+\mu\alpha+\mu\gamma)}, < \mathbf{0}$	Disease elimination
β	$1, > \mathbf{0}$	Disease spread
Q	$1, > \mathbf{0}$	Disease spread

Discussion

In table one above, we have shown that the sensitivity indices for natural death rate, the death rate due to disease, and the recovery rate are all negative which, according to our model assumption if the sensitivity index is less than zero implies disease elimination. This is to say when the population decrease through natural death rate or death due to disease, it reduces the risk of spreading the disease, and also high recovery from the disease will sure eliminate the disease. The sensitivity indices for contact rate and recruitment rate is unity which is greater than zero which by model assumption when the sensitivity index is greater than zero the disease is spread. we can therefore conclude that if the contact rate of infected individuals is high and the recruitment into being susceptible increases the diseases will spread further.

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