Journal of the Nigerian Association of Mathematical Physics
Volume 64, (April – September, 2022 Issue), **pp105 – 110**

A QUEUE MODEL TO ANALYZE THE PROBABILITY DISTRIBUTIONS OF VEHICLES' INTER-ARRIVAL AND SERVICE TIMES IN A TRAFFIC INTERSECTION

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Abstract

In this paper, we consider a queue model to analyze the probability distributions of vehicles inter arrival and service times in traffic intersection using Poisson, Gamma and Binomial distributions. The model consists of derivation of the mean and variance of the inter-arrival and service times of vehicles arriving and departing at road traffic intersection. From the model analysis, the queue system consists of n-vehicles with random arrival and departure at interval of time [t,T]. The probabilistic structure of the queuing model was described in terms of inter arrival and service time distributions. The model is synonymous with birth-and-death process, where no ambiguity arises in interpreting the corresponding inter-arrival and service times distribution assumptions provided. The model is applied to situations in which the input is a Poisson distribution and the service time for each vehicle is exponentially distributed.

***Keywords*: Queue, traffic, vehicles, distributions**

1. Introduction

Queuing theory is defined in [1] as the mathematical study of waiting lines in which queueing models are constructed so that queue lengths and waiting time can be predicted. As stated in [2], a queue is a waiting line made of customers requiring services from one or more facilities such as we experience in petrol stations, banks, toll gates, hospitals, super markets etc. Queues are formed when the demands for service exceeds its supply, and that waiting time depends on the number of customers in the queue system, the number of servers attending to the customers and the amount of service time for each customer [3].

Many authors have developed different queue models to address different situations involving queuing. For example, a queuing model on patient waiting in ante-natal care clinic to determine the number of doctors required so that the given percentage of pregnant women do not exceed a given waiting time and the number of expectant mothers in the queue do not exceed a given threshold is developed in [4], while a queuing model on toll gate with the aim of decongesting traffic on the high ways is developed in [5].

Long queues of vehicles are often found at various traffic light intersection in major cities in Nigeria. Such situations are caused by many factors which include the duration of traffic lights that do not match with the arrival of vehicles. In [6], traffic flows are divided into t two primary types: uninterrupted and interrupted traffic flows. Uninterrupted traffic flows are defined as all traffic flows regulated by vehicle to vehicle interaction and interactions between vehicles and the road way. Examples are vehicles travelling on a highway. Interrupted traffic flows are flows regulated by external means such as traffic light or traffic wardens.

A basic framework for modeling traffic flows with queuing theory is developed in [7] and was explained in detail and applied to several single stage queuing models in [8]. In [9] a queuing model where traffic is modelled on intersections with or without traffic lights, including the estimation of the maximum queue length is discussed. In [10], it was argued that roads have capacity for a finite number of cars.

Queuing models for predicting the stationary number of vehicles in a road link using generating functions is studied in [11] and was used to obtain the performance measures in a network of queues of varying degrees

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2. Model Assumptions and Mathematical Notations

The assumptions associated with the model formation are as follow:

a) The queue system consists of n-vehicles arriving per time t.

b) Vehicles arrival is randomly distributed.

c) A vehicle is assumed to join each line with equal probability regardless of the queue length.

d) Vehicles are attended to on first come, first serve basis.

e) It is also assumed that the lengths of the intervals between arrivals are independently and identically distributed.

Mathematical Notations

1. $\lambda =$ Average number of vehicles arriving per unit time (i.e arrival rate per unit time).
2. $\frac{1}{\lambda}$ =mean time between intervals
3. $h =$ The length of the interval $[T, T+h]$
   (where T and $T+h$ are intervals starting and ending points respectively)
4. $f(t)$ =density function for the time interval t between any two successive arrivals or departure.
5. $\mu =$ Average number of vehicles being served per unit time (i.e departure rate per unit time)
6. $\frac{1}{\mu}$ =mean time between departure

3. Model Description and Formulation

Random arrivals of vehicles into the queue system implies that the probability of an interval occurring in any small interval of time (T, $T+t$) depends only on the length of the interval h and not on the interval's starting point T or on the specific history of arrivals prior to T.

Probability Distributions of Inter-arrival Times

The assumption that the lengths of the intervals between arrivals are independently and identically distributed, describes a continuous density function in which the input (vehicles' arrivals) is termed 'a renewal process'.

Let $f(t) = (density\ function\ for\ the\ time\ \text{int}erval\ t\ between\ any\ two\ vehicles'\ successive\ arrival)$ Where $t > 0$, and also define

$\frac{1}{\lambda}$ =mean time between arrivals

So that $\lambda =$ arrival rate per unit of time.

We can determine λ from f(t) by taking the mathematical expectation of t, so that

$$\int_0^\infty tf(t)\,dt = \frac{1}{\lambda} \text{ (mean time between arrivals)} \tag{1}$$

The assumption of completely random arrival of vehicles corresponds to postulating

$$f(t) = \lambda e^{-\lambda t},\ t \geq o \text{ negative exponential distribution} \tag{2}$$

$$\left(where\ \ell = 2.71828, mean \frac{1}{\lambda}, variance = \frac{1}{\lambda^2}\right)$$

Then the probability that no arrival occurs in the interval (0, T) is the same as the probability that the first arrival occurs after T, hence

$$P[t \geq T] = \int_0^\infty \lambda \ell^{-\lambda} dt = \ell^{-\lambda T} \tag{3}$$

Thus, conditional probability that no arrival occurs in the interval $(0, T+h)$ given that no arrival occurs in the interval (0, T) is, by definition

$$\frac{P[t \geq T+h]}{P[t \geq T]} = \frac{\ell^{-\lambda(T+h)}}{\ell^{-\lambda T}} = P[t \geq h] \tag{4}$$

which depends only on h. According to equation (4), the probability of no arrival in the interval $(T, T+h)$ is the same regardless of whether there is no arrival in (0, T) or whether an arrival occurs at T and thereby "renews" the arrival process.

Suppose there are $n$vehicles' arrivals in the interval (0, T) then if the interval times are exponentially distributed, the n arrival times are independently and uniformly distributed over the interval (0, T).

A complementary insight into the assumption of exponential inter arrival times is gained by expressing $\ell^{-\lambda h}$ in its Taylor series expansion as:

$$p\begin{bmatrix} \text{no arrival in} \\ \text{any interval} \\ \text{of length h} \end{bmatrix} = \ell^{-\lambda h} = 1 - \lambda h + \frac{(-\lambda h)^2}{2!} + \frac{(-\lambda h)^3}{3!} + \tag{5}$$

For a very small, but positive, value of h, the term $1-\lambda h$ in (5) is relatively large as compared to the remaining terms in the summation. Therefore, this value can be used to approximate the probability in (5) when h is very small. We use the symbol ($\doteq$) to denote such an approximation. So we have, for *very* small h > 0

$$P\begin{bmatrix} no\ arrival\ in \\ any\ interval \\ of\ length\ h \end{bmatrix} \doteq 1-\lambda h \quad (h\ small) \qquad (6)$$

A verbally inexact, but nevertheless helpful, way to explain the mathematical manipulation below is to state that at most only one arrival occurs for a time interval $h \geq 0$ sufficiently small. Since the approximate probability of no arrival occurring in the interval of length h is given by (6), the corresponding approximate probability of one arrival occurring is

$$P\begin{bmatrix} single\ arrival \\ in\ any\ interval \\ of\ length\ h \end{bmatrix} \doteq \lambda h \qquad (h\ small)\ldots \qquad (7)$$

A more precise way of expressing the reasoning would be to display the exact probability of a single arrival, in a manner similar to (5), and then show that for very small h, the term λh is relatively large as compared to the remaining terms. We interpret the symbol ($\doteq$) as meaning that a quantity of relatively negligible magnitude is being ignored in the approximation. Given that the density function for inter arrival times \is exponential in equation (2), an immediate consequence is that the density function of the total arrival time y for any n consecutive arrivals is

$$g(y)=\frac{\lambda(\lambda y)^{n-1}\ell^{-\lambda y}}{(n-1)!}, y \geq 0 \quad (gamma\ distribution) \qquad (8)$$

Where n is a positive integer in equation (8) and y is the sum of n independent values drawn from the same exponential density in equation (2). Then

$$P\begin{bmatrix} total\ interval\ for \\ any\ n\ consecutive \\ arrivals \leq T \end{bmatrix} = \int_0^T g(y)\ dy = 1-\sum_{j=0}^{n-1}\frac{(\lambda T)^j\ell^{-\lambda T}}{j!} \qquad (9)$$

which can be verified by repeatedly applying integration by parts.

postulating that the probability distribution of the number vehicles' arrivals n in any interval length T is Poisson:

$$P\begin{bmatrix} n\ arrivals\ in \\ any\ interval \\ of\ length\ T \end{bmatrix} = \frac{(\lambda T)^n\ell^{-\lambda T}}{n!}\ for\ n=0,1,2,\ldots \qquad \text{(Poisson distribution)} \qquad (10)$$

With

$$E[n|T]=\lambda \mathrm{T} \text{ and Var } [\mathrm{n}|\mathrm{T}]=\lambda \mathrm{T} \text{ (Poisson interval of Length T)} \qquad (11)$$

From (9) and (10), it follows that

$$P\begin{bmatrix} \text{total interval for any n} \\ \text{consecutive arrivals} \leq T \end{bmatrix} = P\begin{bmatrix} \text{number of arrivals in} \\ \text{any interval T} \geq \text{ n} \end{bmatrix} \qquad (12)$$

In the following section, we present the probability distributions of service times.

Probability Distributions of Vehicles' Service Times

For a specified server (warden), assume that successive service times are independently and identically distributed, and described by a continuous density function. Let

$g(t)$ ≡density function for the length of time t to serve any vehicle, where $t \geq 0$(13)

and also let

$$mean\ service\ time = \int_0^\infty tg(t)\ dt \equiv \frac{1}{\mu} \qquad (14)$$

So that

$\mu =$ service rate per unit of time that the server is busy (15)

Frequently, the service time distribution is assumed to be exponential, hence

$$g(t)=\mu\ell^{-\mu t}, \quad t \geq 0 \qquad (16)$$

Accordingly, given assumption (4), if a vehicle is being served at Time t and we observe the system at Time $t+h$, then

P[service is not completed in interval of length h] $=\ell^{-\mu h}$ (17)

Consequently, for h > 0 very small,

P[service is not completed in interval of length h] $\doteq 1-\mu h$(h small) (18)

And

P[service is completed in interval of length h] $\doteq \mu h$(h small) (19)

Assume that there is a single server having an exponential time density in equation (16), and let

$P_n(T)$ =probability that n customers are in the system at Time T (20)

As in the preceding section, we calculate approximate probabilities by ignoring relatively small quantities. And in the same approximate vein, we say that at most only one departure occurs during a very small interval of time $h > 0$. Hence, where there are n vehicles at Time $T + h$, we consider only the possibilities that at Time T either there were n vehicles and none have departed, or there were $n + 1$ vehicles and one departed during the very small interval of length h. Consequently, for $1 \leq n < M$,

$$P_n(T+h) \doteq (1-\mu h)P_n(T) + (\mu h)P_{n+1}(T) \quad (21)$$

The first term on the right is the approximate probability that no service occurred in the interval of length h and that n vehicles were in the system at Time T, and similarly for second term on the right of (21). Rearranging terms yields

$$\frac{P_n(T+h)-P_n(T)}{h} \doteq -\mu P_n(T) + \mu P_{n+1}(T) \quad (22)$$

So that letting $h \to 0$,

$$\frac{dPn}{dT} = -\mu P_n(T) + \mu P_{n+1}(T) \text{ for } 1 \leq n < M \quad (23)$$

The reason equation (23) holds exactly, instead of approximately, is that all the terms of relatively small magnitude that were ignored in (22) actually disappear in the process of letting h approach 0 in the limit.

By a similar line of reasoning, you can determine that

$$\frac{dP_M}{dT} = -\mu P_M(T) \text{ for } n = M. \quad (24)$$

The unique solution to the system of linear differential equations (23) and (24) is

$$P_n(T) = \frac{(\mu T)^{M-n} e^{-\mu T}}{(M-n)!} \qquad for\ n = 1, 2, \dots, M \quad (25)$$

$$P_o(T) = 1 - \sum_{n=1}^{M} P_n(T) \text{ for n = 0} \quad (26)$$

The distribution in (25) and (26) is sometimes called a truncated Poisson.

If the *Mth* vehicle is the last to be served, then the total time y that the vehicle spends in the queue, including the vehicle's own service time, has the density given by the sum of M exponentially distributed variables.

$$h(y) = \frac{\mu(\mu y)^{M-1} e^{-\mu y}}{(M-1)!}, y \geq 0 \qquad \text{(gamma distribution)} \quad (27)$$

With

$$E[y] = \frac{M}{\mu} \quad Var[y] = \frac{M}{\mu^2} \qquad \text{(gamma)} \quad (28)$$

Consider a very small interval of time h > 0. Then because the servers are independent, you can apply binomial probability calculations, using the approximate expression in (18), to obtain, for a small time interval h

P[none of n vehicles departs] $\doteq (1-n\mu h)^n \doteq 1 - n\mu h$ (29)

P[one of n vehicles depart] $\doteq n\mu h$ (30)

Once again, the justification for (30) is that when interval h > 0 is very small, we can restrict attention to the events of no and one departure; the possibilities of more departures have relatively negligible probability. Consequently, when there are n vehicles in the queue system at Time $T + h$, we consider only the possibilities that at Time T there were n vehicles and none have departed, or there were n + 1 vehicles and one departed-giving, for $0 \leq n < M$,

$$P_n(T+h) \doteq (1-h\mu n)P_n(T) + (n+1)\mu h P_{n+1}(T) \quad (31)$$

Bringing $P_n(T)$ to the left hand side of (31), dividing by h, and letting $h \to 0$ yields

$$\frac{dP_n}{dT} = n\mu P_n(T) + (n+1)\mu P_n + 1(T) \text{ for } 0 \leq n < M \quad (32)$$

Similar reasoning gives

$$\frac{dPM}{dT} = -M\mu P_M(T) \text{ for } n = M \quad (33)$$

As can be verified by substitution, the complete solution to (32) and (33) is

$$P_n(T) = \binom{M}{n} (\ell^{-\mu T})^n (1 - \ell^{-\mu T})^{M-n} \qquad for\ n = 0, 1, 2, \dots, M \qquad (34)$$

With binomial distribution,

$$E[n|T] = M_e -^{\mu T} \qquad \text{Var}\,[n|T = M_e -^{\mu T}]\ (1-{}_e - {}_{\mu T}) \qquad (35)$$

4. Discussion

In equation (1) to (13), we considered the inter – arrival time distribution using the assumption that the lengths of the intervals between arrivals are independently and identically distributed, describes a continuous density function in which the input (vehicles' arrivals) is termed 'a renewal process'. This allows us to calculate approximate probabilities by ignoring relatively small quantities of time. In the same approximate vein, we say that at most only one departure occurs during a very small interval of time $h > 0$. Hence, where there are n vehicles at Time $T + h$, we consider only the possibilities that at Time T either there were n vehicles and none have departed, or there were $n + 1$ vehicles and one departed during the very small interval of length which accounted for the results for the probability distribution of service times in equations (14) to (22).

We notice that (23) holds exactly, instead of approximately, because all the terms of relatively small magnitude that were ignored in (22) actually disappear in the process of letting h approach 0 in the limit. Also, the justification for (30) and the equations that followed is that when interval h > 0 is very small, we can restrict attention to the events of no and one departure; the possibilities of more departures being relatively negligible.

5. Conclusion

In the queue theory literature, the three key words are input process, service distribution and the number of servers. M is an abbreviation for Markovian which is associated with random arrival or departure of customers in a queue system. ($M \equiv$ exponentially distributed interval or service time). The queue model discussed in this paper centred on derivation of mean and variance of probability distributions of inter arrivals and service times using Poisson distribution with a single server (MM/1). Traffic congestion is bound to occur when the flow rate arrival of vehicles is higher than the workspace capacity at a given time. In this paper, we have extended the queue models in [10] and [11] by deriving the mean and variance of probability distributions of inter-arrival and service times using Poisson and Gamma distribution with a simple server (MM/1).

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