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FIXED POINT COINCIDENCE THEOREMS FOR MAPPINGS SATISFYING A CONTRACTIVE CONDITION OF RATIONAL TYPE

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Abstract

In this research, we prove some coincidence point theorems for nonlinear contractive mappings with rational expressions in the context of metric spaces endowed with a partial order. Hence, this work serves as an improvement to the available results in the literature, and illustrations to support our claims were also presented.

Keywords**:** Partially ordered metric spaces, coincidence point, monotone f-nondecreasing, compatible and weakly compatible mappings.

1. Introduction

There are a lot of generalizations of the Banach contraction mapping principle in the literature. One of the most interesting of them is the result of Ran and Reurings [1] which studied the existence of fixed points for certain mappings in partially ordered metric spaces and applied their results to matrix equations. Their results were later extended by Nieto and Lopez [2] for non-decreasing mappings and obtained the solutions of certain partial differential equations with periodic boundary conditions.

Chatterji [3] considered various contractive conditions for self-mappings in metric space. Dass and Gupta [4] also investigated the rational type of contractions to obtain a unique fixed point in complete metric space. Very recently, Seshagiri and Kalyani [5] have explored some results on coupled fixed points for the mappings in partially ordered metric spaces.

In this manuscript, we establish some coincidence point for f-nondecreasing self mapping satisfying certain rational type contractions in the context of metric spaces endowed with partial order. We generalize and extend the results in some literatures. A few examples are given to support our results.

2. Preliminary Notes

We start with the following definitions and theorems that motivate our study:

Definition 2.1 [6] The triple $(X, d, \preccurlyeq)$ is called partially ordered metric spaces, if $(X, \preccurlyeq)$ is a partially ordered set and (X, d) is a metric space.

Definition 2.2 [6] If (X, d) is a complete metric space, then the triple $(X, d, \preccurlyeq)$ is called complete partially ordered metric spaces.

Definition 2.3 [7] Let $(X, \preccurlyeq)$ is called partially ordered set. A mapping $f: X \to X$ is said to be strictly increasing (strictly decreasing), if $f(x) < f(y)\big(f(x) > f(y)\big)$ for all $x, y \in X$ with $x < y$.

Definition 2.4 [7] A point $x \in A$, where A is a non-empty subset of a partially ordered set $(X, \preccurlyeq)$ is called a common fixed (coincidence) point of two self-mappings f and T, if $fx = Tx = x$ $(fx = Tx)$.

Definition 2.5 [7] The two self-mappings f and T defined over a subset A of a partially ordered metric space $(X, d, \preccurlyeq)$ are called commuting, if $fTx = Tfx$ for all $x, y \in A$.

Definition 2.6 [7] Two self-mappings f and T defined over $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} (Tfx_n, fTx_n) = 0$.

Definition 2.7 [7] Two self-mappings f and T defined over $A \subset X$ are said to be weakly compatible, if they commute only at their coincidence points (i.e., if $fx = Tx$ then $fTx = Tfx$).

Definition 2.8 [7] Let f and T be two self-mappings defined over a partially ordered set $(X, \preccurlyeq)$. A mapping T is called

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monotone f-nondecreasing, if $fx \leq fy \Rightarrow Tx \leq Ty$ for all $x, y \in X$.

Definition 2.9 [7] Let A be a non-empty subset of a partially ordered set $(X, \preccurlyeq)$. If every two elements of A are comparable, then it is called well ordered set.

Definition 2.10 [7] A partially ordered metric space $(X, d, \preccurlyeq)$ is called ordered complete if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following condition holds: either

i. if x_n is a non-increasing sequence in X such that $x_n \to x$ implies $x \preccurlyeq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$, or

ii. if x_n is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preccurlyeq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$.

Theorem 2.11 [8] Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that T and f are continuous self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping satisfying

$$d(Tx, Ty) \leq \alpha\left(\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}\right) + \beta\big(d(fx, fy)\big) \tag{2.1}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, and for some $\alpha, \beta \in [0, 1)$with $\alpha + \beta < 1$.

If there exists $x_0 \in X$ such that $f(x_0) \leq T(x_0)$ and T and f are compatible, then T and f have a coincidence point.

Theorem 2.12 [5] Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Suppose that $T: X \to X$ be a non-decreasing, continuous self mapping satisfying

$$d(Tx, Ty) \leq \begin{cases} \lambda d(x, y) + \eta[d(x, Ty) + d(y, Tx)] + \mu \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, & if\ A \neq 0 \\ 0 & , if\ A = 0 \end{cases} \tag{2.2}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, where $A = d(y, Tx) + d(x, Ty)$ and λ, η, μ are non negative real numbers with $\lambda + 2\eta + \mu < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a unique fixed point in X.

Theorem 2.13 [9] Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Suppose that $T, f: X \to X$ are continuous mappings and T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfying the following condition

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)[1 + d(fy, Ty)]}{1 + d(fx, fy)} + \beta[d(fx, Ty) + d(fy, Tx)] + \gamma d(fx, fy) \tag{2.3}$$

for all $x, y \in X$, for which fx and fy are comparable and for some $\alpha, \beta, \gamma, \delta, \varepsilon \in [0,1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are comparable, then T and f have coincidence point in X.

3. Main Results

Theorem 3.1 Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that $T, f: X \to X$ are continuous mappings and T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfying the following condition

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)[1 + d(fy, Ty)]}{1 + d(fx, fy)} + \beta \frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)} + \gamma d(fx, fy) \tag{3.1}$$

for all $x, y \in X$, for which fx and fy are comparable and for some $\alpha, \beta, \gamma \in [0,1)$ with $0 \leq \alpha + \beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $fx_0 < Tx_0$ and the mappings T and f are comparable, then T and f have coincidence point in X.

Proof. Let $x_0 \in X$ such that $fx_0 \leq Tx_0$. Since $T(X) \subseteq f(X)$, then we choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$, then there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. Recursively, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, for all $n \geq 0$.

From the hypotheses, we have $fx_0 = Tx_0 = fx_1$ and then the monotone property of f implies $Tx_0 \leq Tx_1$. Similarly, we have $Tx_1 \leq Tx_2$ as $fx_1 \leq fx_2$. Continuing the same process, we obtain

$Tx_0 \leq Tx_1 \leq Tx_2 \leq \cdots \leq Tx_n \leq Tx_{n+1} \leq \cdots$

Now, we have the following two cases:

Case 1: Suppose that $d(Tx_n, Tx_{n+1}) = 0$ for some n, then $Tx_n = Tx_{n+1}$. Thus, $Tx_{n+1} = Tx_n = fx_{n+1}$. Hence, x_{n+1} is a coincidence point of T and f.

Case 2: Suppose that $d(Tx_n, Tx_{n+1}) > 0$ for all $n \geq 0$. Then from (3.1), we have

$$d(Tx_{n+1}, Tx_n) \leq \alpha \frac{d(fx_{n+1}, Tx_{n+1})[1 + d(fx_n, Tx_n)]}{1 + d(fx_{n+1}, fx_n)} + \beta \frac{d(fx_{n+1}, Tx_{n+1})d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} + \gamma d(fx_{n+1}, fx_n)$$

which implies that

$$d(Tx_{n+1}, Tx_n) \leq \alpha d(Tx_n, Tx_{n+1}) + \beta d(Tx_n, Tx_{n+1}) + \gamma d(Tx_n, Tx_{n-1})$$

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma}{1 - \alpha - \beta}\right) d(Tx_{n-1}, Tx_n)$$

Inductively, we get

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma}{1 - \alpha - \beta}\right)^n d(Tx_1, Tx_0).$$

where $k = \frac{\gamma}{1-\alpha-\beta} < 1$
Now, we shall prove that $\{Tx_n\}$ is Cauchy sequence. For $m \geq n$, by triangular inequality we have
$d(Tx_m, Tx_n) \leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \cdots + d(Tx_{n+1}, Tx_n)$
$\leq (k^{m-1} + k^{m-2} + \cdots + k^n)d(Tx_1, Tx_0)$
$\leq \frac{k^n}{1-k} d(Tx_1, Tx_0),$
as $m, n \to +\infty, d(Tx_m, Tx_n) \to 0$, which shows that the sequence $\{Tx_n\}$ is Cauchy sequence in X. So, by the completeness of X, there exists a point $\lambda \in X$ such that $Tx_n \to \lambda$ as $n \to +\infty$.
By the continuity of T, we have
$T\lambda = T\left(\lim_{n\to+\infty} x_n\right)$
$= \lim_{n\to+\infty} Tx_n$
$= \lim_{n\to+\infty} x_{n+1}$
$= \lambda.$
Since, $Tx_n = fx_{n+1}$, then $fx_{n+1} \to \lambda$ as $n \to +\infty$. Thus, by compatibility of T and f, we get
$\lim_{n\to+\infty} (Tfx_n, fTx_n) = 0.$
Therefore, by triangular inequality, we get
$d(T\lambda, f\lambda) \leq d(T\lambda, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, f\lambda),$
Take the limit as $n \to +\infty$ and using the fact that T and f are continuous, we obtain that
$d(T\lambda, f\lambda) = 0.$
Hence, $T\lambda = f\lambda$. Therefore, λ is a coincidence point of T and f in X.

Theorem 3.2. In addition to the hypothesis of Theorem 3.1. If $f(X)$ is a complete subset of X, then T and f have coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and f are well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Suppose $f(X)$ is a complete subset of X.Following the proof of Theorem 3.1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence, also $\{fx_n\}$ is a Cauchy sequence in $(f(X), d)$, since $Tx_n = fx_{n+1}$ and $T(X) \subseteq f(X)$. But $f(X)$ is a complete, then there exists some $u \in f(X)$ such that
$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu.$
Notice that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing and then from the hypotheses, we get $Tx_n \leq fu$ which implies that $fx_n \leq fu$, for all $n \in \mathbb{N}$. Since T is monotone f-nondecreasing, then we have $Tx_n \leq Tu$ for all $n \in \mathbb{N}$.
Letting $n \to +\infty$, we get $fu \leq Tu$.
Suppose that $fu < Tu$. Define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 3.1, yields that the sequence$\{fu_n\}$ is a nondecreasing sequence and $\lim_{n\to+\infty} fu_n = \lim_{n\to+\infty} Tu_n = fv$ for some $v \in X$. Thu, from the hypothesis, we have $sup fu_n \leq fv$ and $sup Tu_n \leq fv$, for all $n \in \mathbb{N}$.
Therefore,
$fx_n \leq fu \leq fu_1 \leq fu_2 \leq \cdots \leq fu_n \leq \cdots \leq fv.$
Now, we also have the following two cases:
Case 1: Suppose, if there exists $n_0 \geq 1$ with $fx_{n_0} = fu_{n_0}$, then we have
$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu.$
Hence, u is a coincidence point of T and f in X.
Case 2: Suppose that, $fx_{n_0} \neq fu_{n_0}$ for all $n \in \mathbb{N}$. Then from (3.1), we have
$$d(fx_{n+1}, fu_{n+1}) = d(Tx_n, Tu_n) \leq \alpha \frac{d(fx_n, Tx_n)[1 + d(fu_n, Tu_n)]}{1 + d(fx_n, fu_n)} + \beta \frac{d(fx_n, Tx_n)d(fu_n, Tu_n)}{d(fx_n, fu_n)} + \gamma d(fx_n, fu_n)$$
Take the limit as $n \to +\infty$ to the above inequality, we get
$d(fu, fv) \leq \gamma d(fu, fv)$
$< d(fu, fv)$ since $\gamma < 1$.
Therefore, we have
$fu = fv = fu_1 = Tu.$
Hence, u is a coincidence point of T and f in X.
Suppose that T and f are weakly compatible and let w be the coincidence point. Then
$Tw = Tfz = fTz = fw$, since $w = Tz = fz$, for some $z \in X$.
From (3.1), we have

$$d(Tz,Tw) \le \alpha \frac{d(fz,Tz)[1+d(fw,Tw)]}{1+d(fz,fw)} + \beta \frac{d(fz,Tz)d(fw,Tw)}{d(fz,fw)} + \gamma d(fz,fw)$$
$$\le \gamma d(Tz,Tw),$$
as $\gamma < 1$, then, we have from $d(Tz,Tw) = 0$. Therefore, $Tz = Tw = fw = w$. Hence, w is a common fixed point of T and f in X.

Now, suppose that the set of common fixed point of T and f is well ordered. It is enough to prove that the common fixed point of T and f is unique. Let u and v be two common fixed point of T and f such that $u \neq v$. Then from (3.1), we have
$$d(Tu,Tv) \le \alpha \frac{d(fu,Tu)[1+d(fv,Tv)]}{1+d(fu,fv)} + \beta \frac{d(fu,Tu)d(fv,Tv)}{d(fu,fv)} + \gamma d(fu,fv)$$
$$\le \gamma d(u,v)$$
$$< d(u,v) \text{ since } \gamma < 1,$$
a contradiction and hence, $u = v$. Conversely, suppose T and f have only one common fixed point, then the set of common fixed points of T and f being a singleton is well ordered.

4. Applications

Example 4.1 Define a metric $d: X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $X = [0,1]$ with usual order $\le$.Let$T$ and f be two self mappings on X such that $Tx = \frac{x^2}{2}$ and $fx = \frac{2x^2}{1+x}$, then T and f have a coincidence point in X.

Proof. By definition of a metric $d, (X,d)$ is a complete metric space. Obviously, $(X,d,\le)$ is a complete partially ordered metric space with usual order. Let $x_0 = 0 \in X$, then $f(x_0) \le T(x_0)$ and also by definition, T and f are continuous, T is monotone f-nondecreasing and $T(X) \subseteq (X)$.

Now, for distinct x, y in X with $x < y$, we have
$$d(Tx,Ty) = \frac{1}{2}|x^2 - y^2| = \frac{1}{2}(x+y)|x-y| \le \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y|$$
$$\le \alpha \frac{2x^2|3-x|[(1+y)+y^2|3-y|]}{4(1+x)(1+y)+2|x-y|(x+y+xy)} + \beta \frac{x^2y^2|3-x||3-y|}{2|x-y|(x+y+xy)} + \gamma \frac{2(x+y+xy)}{(1+x)(1+y)}$$
$$\le \alpha \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{2(1+y)+y^2|3-y|}{2(1+y)}}{1+\frac{2|x-y|(x+y+xy)}{(1+x)(1+y)}} + \beta \frac{\frac{x^2|3-x|}{2(1+x)} \cdot \frac{y^2|3-y|}{2(1+y)}}{\frac{2|x-y|(x+y+xy)}{(1+x)(1+y)}} + \gamma \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y|$$
$$\le \alpha \frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \gamma d(fx,fy)$$
Then, the contraction condition in Theorem 3.1 holds by selecting proper values of $\alpha, \beta, \gamma \in [0,1)$ such that $0 \le \alpha + \beta + \gamma + < 1$. Therefore, T and f have a coincidence point 0 in X.

Example 4.2 Define a distance function $d: X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $X = [0,1]$ with usual order $\le$. Let T and f be two self mappings on X such that $Tx = x^3$ and $fx = x^4$, then T and f have two coincidence points $0, 1$ in X with $x_0 = \frac{1}{4}$.

5. Conclusion

This work has proven the existence of coincidence point for nonlinear contractive mappings with rational expressions in the context of metric space endowed with partial order. Some illustrations to support our findings were also given.

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