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COMPUTING CAUCHY INTEGRAL THEOREM OF A MATRIX FUNCTION VIA SIMILARITY TRANSFORMATION AND BEHAVIOR OF HYPERGEOMETRIC MATRIX DENSITY FUNCTION

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Abstract

It is shown that Cauchy integral matrix function is Lebesgue measurable on the contour integral based on the Little Wood's third formula for which the trapezoidal rule and Simpson composite method are applicable. As a follow up, it is demonstrated that Taylor series representation of Cauchy integral matrix function is commutable with similarity matrix transformation when Jordan canonical block along diagonal is taken into consideration.

The spectrum of the diagonalizable matrix A is computed using the Givens orthogonal matrix plane rotation. As an extension of ideas; a measure of effectiveness on the use of SVD in the computation process is emphasized and fully utilized which leads to demonstration with the exponential of a matrix function as an example. Further analytical reasoning on performance of Cauchy integral theorem for the matrix functional calculus leads to the method of Residue theorem. The density of a matrix function is calculated based on the hypergeometric series taking into consideration the behavior of gamma function.

Keywords: Cauchy integral matrix function, hypergeometric function, Bessel matrix polynomial, density of a matrix MSC 2020 (Mathematical Reviews and zbMath): primary 28A20, 28A25, 30E20, Secondary: 65F15, 65F60.

1.0 Introduction

The paper presents the Cauchy integral theorem of a matrix function $f(A)$, where, $A \in \mathbb{C}^{n\times n}$.

The Cauchy integral theorem for the matrix function in the Lebesgue measurable sense over S- dissections is presented using Little Wood's third formula in which trapezoidal rule becomes a useful tool. We pay a special attention to the application of Residue theorem in the analytic senses. The distribution of the spectrum of a diagonalizable matrix is emphasized based on the knowledge of Givens matrix orthogonal plane rotation and Singular Value Decomposition (SVD). Behavior in the analysis of Gamma functions and hypergeometric function as well as the Bessel matrix polynomials are carefully explained for purposes of computing the density of a matrix. Relevant to the analysis in this direction is the nature of the distribution spectrum $\sigma(A)$ of the matrix.

1.1 Literature Review/ Problem statement

In what follows, the presentation of the approximation theory to the Cauchy integral formula for the matrix function $f(A)$ is discussed drawing attention of the readers to the existence of [1]. We give information on the spectrum of the matrix A using the Givens orthogonal matrix plane rotations and Singular Value Decomposition (SVD). The accompanying inverse matrix in the calculation is computed by some numerical methods. Also the use of standard Residue theorems in the computation of the Cauchy integral matrix function problem is stated. We introduce in the computation the density of a matrix based on the hypergeometric and gamma functions provided the nature of the spectrum in the diagonalizable matrix A is known.

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We motivate our interest in the presentation of the paper using the Cauchy integral complex theorem and Taylor series expansion of a holomorphic matrix function [2].In particular, the Trapezoidal rule is employed for the complex Cauchy integral theorem based on the Lebesgue measure on a measurable space.

Let η be a closed contour in the complex plane enclosing all eigenvalues of matrix A. Assuming further that f be holomorphic on and inside the contour η. Then by the Cauchy integral formula matrix function $f(A)$ it is expressed in the form:

$$f(A)=\frac{1}{2\pi i}\oint_{\eta} f(z)\left(zI_N-A\right)^{-1}dz \qquad (z\in\mathbb{C}). \qquad (1.1)$$

The term I_N appearing in equation (1.1) is the identity matrix and where the circle $z=re^{i\theta}\ (0\le\theta\le 2\pi)$ is well defined, having center A and radius r such that $\|A-zI_N\|<r$ holds verbatim.

That $z=z_0+re^{i\theta}$, and z_0 is enclosed in the contour $\eta\in D\subset\mathbb{C}$ for which holds, $\partial z/\partial\theta=ire^{i\theta}$ is an interesting exercise.

To show that $f(A)$ defines a Cauchy integral, it is enough to show that

$$\oint_{\eta}\frac{f(z)}{zI-A}dz=\int_0^{2\pi} f\left(z_0+re^{i\theta}\right)\left(re^{i\theta}I-A\right)^{-1}ire^{i\theta}d\theta=\int_0^{2\pi} ire^{i\theta}\ f\left(z_0+re^{i\theta}\right)\left(re^{i\theta}I-A\right)^{-1}d\theta \qquad (1.2)$$

We then expand $(zI-A)^{-1}$ as

$$(zI-A)^{-1}=\frac{1}{z}\left(I-\frac{A}{z}\right)^{-1}=\frac{1}{z}\left(I+\frac{A}{z}+\frac{A^2}{z^2}+\frac{A^3}{z^3}+...\right) \qquad (1.3)$$

By substituting equation (1.3) into equation (1.2) we have that

$$f(A)=\int_0^{2\pi}\left\{f(z_0+re^{i\theta})\times\frac{1}{re^{i\theta}}\left(I+\frac{A}{re^{i\theta}}+\frac{A^2}{r^2e^{2i\theta}}+\frac{A^3}{r^3e^{3i\theta}}+....\right)\right\}ire^{i\theta}d\theta \qquad (1.4)$$

Using the fact that any contour not containing a pole is zero, then, we have that

$$f(A)=\int_0^{2\pi} f(re^{i\theta})\left(I+\frac{A}{re^{i\theta}}+\frac{A^2}{r^2e^{2i\theta}}+\frac{A^3}{r^3e^{3i\theta}}+....\right)id\theta$$

It is necessary that for convergence, the term $r>\|A-z_0I\|$ for the f being analytic on $D\subset\mathbb{C}$ is valid.

It must be noted that for a use of any small eigenvalues of the spectrum $\sigma(A)$,any noise on the component of A will cause a catastrophic consequences on $\frac{1}{re^{i\theta}}$ appearing in the series expansion which will amplify the noise to an unacceptable level in equation (1.4).Therefore we depress (take) the value of spectral radius of the matrix to be greater than maximum eigenvalue of A.

So for a large enough value of the radius in magnitude in the numerator part; the term $\lim_{r\to\infty}\left(I+\frac{A}{re^{i\theta}}+\frac{A^2}{r^2e^{2i\theta}}+\frac{A^3}{r^3e^{3i\theta}}+...\right)\Rightarrow I_N$ .Coupling together these facts, we then have that

$$\oint_{\eta}\frac{f(z)}{zI-A}dz=\int_0^{2\pi} f(z_0+re^{i\theta})\,z^{-1}(I-\frac{A}{z})^{-1}d\theta=\int_0^{2\pi} f(re^{i\theta})\frac{1}{re^{i\theta}}ire^{i\theta}Id\theta \qquad (1.5)$$

$$= f(A)i\theta\Big|_0^{2\pi}=2\pi i f(A)$$

Thus the Cauchy integral theorem for the matrix function given by

$f(A)=\frac{1}{2\pi i}\int_0^{2\pi} f(z)(Iz-A)^{-1}dz$ is well defined.

Immediately following, we define the Residue theorem for the accompanied Cauchy integral theorem as follows:

Theorem 1.1,[3]. Let $D\subset\mathbb{C}$ be a region bounded by a finite number of piecewise continuous curves. Assuming further that f be at the boundary ∂D which is inside D holomorphic up to a finite number of isolated singularities

$\{z_1, z_2, ...z_{n-1}, z_n\}$ that are all in D. Then

$$\oint_{\partial D} f(z)dz = 2\pi i \sum_{k=}^{n} \operatorname*{Res}_{z=z_k} f(z) \cdot$$

Fact 1:
Let $f(z_0)$ have at $z = z_0$ a pole of order m then the residue may be obtained in the form

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_{z0}} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \cdot \quad (1.7)$$

We notice computing $f(A)$ for the Cauchy integral theorem matrix using residue theorem involves the inversion of a matrix. This is possible for a matrix of small size but increasingly difficult for matrices of much higher orders since the matrix $(re^{i\theta} I - A)^{-1}$ is complex.
Therefore turning our attention again to equation (1.2) we create numerical integration out of the existing problem by writing that

$$f(A) = \frac{1}{2\pi i} \int_0^{2\pi} f(re^{i\theta})(re^{i\theta} I_N - A)^{-1} rie^{i\theta} \, d\theta = \int_0^{2\pi} g(\theta) d\theta \quad (1.6)$$

where, it is set that $g(\theta) = f(re^{i\theta})(re^{i\theta} I_N - A)^{-1} re^{i\theta} / 2\pi$ is holomorphic on the circle.
To do so, we construct an approximate value for $f(A)$ similar to a kind of Lebesgue measure over the measurable space [4] and the references therein, with dissection θ over S- slices such that $\theta_k = \frac{2\pi}{S} k$. The parameter k is a period which ranges over the circle { $re^{i\theta_k}$ | $k = 0,1,,2,..., S-1$}.
Definition 1.1. Let $\{x_1, x_2, ..., x_n\} \subset \mathbb{C}$ be a finite set of points and $\mathbb{C}$=$[a,b]$.
Let $a < x_1, x_2, ... < x_k < b \;\; \forall i < k$. Consider the set $P = \mathbb{C} \setminus \{x_1, x_2, ..., x_n\}$, the interval $\mathbb{C}$ without the points $x_1, x_2, .., x_n$. Let our measure be such that the measure of any interval with end points $a < b$ be defined as $b - a$ and $x \in [a,b]$. Then, measure of P is given as

$$\begin{aligned}
\mu(P) &= \mu([a, x_1) \cup (x_1, x_2) \cup ... \cup (x_{n-1}, x_n) \cup (x_n, b)) \\
&= \mu([a, x_1)) + \mu((x_1, x_2)) + ... + \mu((x_{n-1}, x_n)) + \mu((x_n, b]) \\
&= (x_1 - a) + (x_2 - x_1) + (x_3 - x_2) + ... + (b - x_n) \\
&= b - a \\
&= \mu(\mathbb{C})
\end{aligned}$$

We then define [5] the outer measure of any interval I on the complex number line with end points $a < b$ as $b - a$. The outer measure is denoted as $\mu^*(I)$ where, it is expressed that μ ($\mathbb{C}$) is the infimum of $\sum \ell(I_i)$ overall coverings $\mathbb{C} \subset \cup I_i$ by the countable unions of intervals. Then outer measure of any $\mu^*(P)$ of any set $P \subset \mathbb{C}$ is the glb $\mu^*(D) \,|P \subset D$ and D open in $\mathbb{C}$.
If we then denote inner measure of any set $P \subset \mathbb{C}$ by $\mu_*(P)$,it follows that
μ^* ($\mathbb{C}$)- μ^* ($\mathbb{C}$\P) is feasible where $\mathbb{C} \setminus P$ is the complement of P with respect to $\mathbb{C}$.
Using above procedures the outer measure μ^* exhibits sub-additivity for which the set $\{P_k \,|k = 1,2,3,...\}$ is a subset of $\mathbb{C}$, such that

$$\mu^*\left(\bigcup_{k=1}^{\infty} P_k\right) \leq \sum_{k=1}^{\infty} \mu^*(P_k) \cdot$$

This gives that $\mu^*(P) = \mu_*(P)$ for $P \subset \mathbb{C}$ is Lebesgue measurable set. In this case it is denoted as $\mu(P) = \mu^*(P) = \mu_*(P)$.

Whilst $P_i \in M$ for $i = 1,2,3,...,$ such that $P_1 \subset P_2 \subset ...,$ and $P = \bigcup_{i=1}^{\infty} P_i$ then it holds that $\mu(P_i) \to \mu(P)$ as $i \to \infty$ .Furthermore for $i = 1,2,...,$ such that $P_1 \supset P_2 \supset ....,$ and $\mu(P_1) < \infty$, $P = \bigcap_{i=1}^{\infty} P_i$, then $\mu(P_i) \to \mu(P)$ as $i \to \infty$ holds good.

Before proceeding further, we have to take note that if $\{f_n\}$ is a sequence of measurable functions converging to f , then f is also measurable. This means that there is a $\{f_n\}$ converging to f for which exists $\overline{\lim} f_n = \lim f_n = \lim_{n\to\infty} f_n = f$. This holds for $f = \overline{\lim} f_n = \lim f_n$. So f is measurable in the outer measure. The same analogy goes for inner measure, for this, we omit the proof here.

Theorem 1.1. [6]. A bounded measurable function f is Lebesgue integrable on a bounded measurable set $\mathbb{C}$ if and only if given $\varepsilon > 0$ there exist simple measurable functions $\underline{f} \le f \le \bar{f}$ such that $\int_{\mathbb{C}} \bar{f}\, dm - \int_{\mathbb{C}} \underline{f}\, dm < \varepsilon$.

Using Little Wood's principle (3rd principle) and ignoring a set P of small measure, then the uniform convergence is given in the form:

$\left| \int_{\mathbb{C}-P} f_n - f \right| < \int_{\mathbb{C}-P} |f_n - f| \le m(\mathbb{C} - P) \sup_{\mathbb{C}-P} |f_n - f| = 0$. It is understood that $\left| \int_P f_n \right|$ and $\left| \int_P f \right|$ are both less than $Mm(P)$ for M and m being outer and inner measures respectively.

By Fatou's Lemma [6] and since $\{f_n\}$ is a sequence of non-negative measurable function on $\mathbb{C}$ it follows that $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \{\text{glb } f_k(x) \mid k \ge n\}$ for every $x \in \mathbb{C}$. Thus $\int_{\mathbb{C}} f dm \le \lim_{n\to\infty} \int_{\mathbb{C}} f_n dm$ is certain.

We thus established that Cauchy integral matrix function over the contour integral $\frac{1}{2\pi i} \oint_{\eta} \frac{f(z)}{(zI_N - A)} dz$ is measurable in the Lebesgue sense provided that the matrix A is diagonalizable. We can now state as follows in the sense of [7]:

Let f and g be holomorphic functions in a connected region $D \subset \mathbb{C}$. If $f = g$ in the neighbourhood of some point a , then $f = g$ on D.

Having established the existence and uniform convergence of the Trapezoid rule applied on the Cauchy integral matrix function via the Lebesgue measurable, we in the vicinity of compactness give the Taylor Series expansion for complex function f in the form:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(i)}(z_0)(z - z_0)^k}{k!} \qquad (1.7)$$

If we replace $f(z)$ with $f(A)$ in equation this leads to the Cauchy integral matrix function expressed in Taylor series

$$f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)(A - z_0 I)^k}{k!} \qquad (1.4)$$

The error estimate can be obtained in the form

$\left\| f(A) - \sum_{k=0}^{N} \frac{f^{(k)}(z)(A - z_0 I)^k}{k!} \right\| \le C\gamma^N \to 0$ as $N \to \infty$. The quantity $\gamma > \frac{\| A - z_0 I \|}{r} < 1$ where $r > \| A - z_0 I \|$ for $f(z)$ is analytic on $D(z_0, r)$. Thus the Taylor series representation of matrix function [8] commutes with the similarity transformation $f(A) = Xf(A)X^{-1}$ where X a matrix of column vector. This will be discussed later on in section3 in the paper.

Thus the Cauchy integral matrix function becomes

$$\int_0^{2\pi} g(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (z(\theta) - \alpha) f(z(\theta))(z(\theta)I - A)^{-1} d\theta, \quad (z(\theta) = \alpha + re^{i\theta}). \qquad (1.5)$$

Differentiating the $g(\theta)$ in q times, then

$$h^q(\theta) = \frac{i^q}{2\pi}\sum_{j=0}^{q}(z(\theta)-\alpha)^{j+1}\sum_{i=0}^{j}c_{i,j}\, f^{(j-1)}(z(\theta)I - A)^{-(1+i)} \tag{1.6}$$

where $c_{i,jq}$ are some constants

In the real space the Trapezoidal rule is

$$\int_0^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{S-1} f(x_j) + f(x_S)\right] \quad , \tag{1.7}$$

$$h = \frac{b-a}{S}, x_j = a + jh,\ j = 1,2,..., S-1\cdot$$

This translates to equidistant trapezoidal rule :

$$T_S(f) = h_S\sum_{i=1}^{S-1} f(x_i) + \frac{h_S}{2}(f(x_0) + f(x_S)) \tag{1.8}$$

where $h_S = \frac{b-a}{S}, \quad x_i = a + ih_S\cdot$

Thus for $S = S_0, 2S_0, 4S_0$, then taking appropriate linear combination [9] we have that

$$T_S(f) = \frac{h_S}{2}(f(a) + f(b)) + 2\sum_{i=1}^{S-1} f(a + ih_S)\ , \tag{1.9}$$

$$T_{2S}(f) = \frac{h_S}{4}(f(a) + f(b)) + 2\sum_{i=1}^{S-1} f(a + ih_S) + 2\sum_{i=1}^{S} f\left(a + \left(i - \frac{1}{2}\right)h_S\right) \tag{1.10}$$

$$= \frac{1}{2}\left(T_S(f) + h_S\sum_{i=1}^{S} f\left(a + (i - \frac{1}{2}\right)h_S\right)$$

If the function possesses third order continuously differentiable $c^3[a,b]$ then it may be approximated by the Composite Simpson Rule:

$$S_S(f) = \frac{4T_{2S}(f) - T_S(f)}{3} = \frac{T_S(f) + 2R_S(f)}{3}$$

$$= \frac{h}{6}\left(f(a) + f(b) + 2\sum_{i=1}^{S-1} f(a + ih_S) + 4\sum_{i=1}^{S} f\left(a + \left(i - \frac{1}{2}\right)h_S\right)\right) \tag{1.11}$$

Since $f(a) = f(b)$ over the contour integral then equation (1.7) becomes

$$\int_0^b f(x)dx = \frac{b-a}{S}\sum_{j=0}^{S-1} f(x_j) \tag{1.12}$$

Turning our attention to our given problem under discussion in compact form, the Trapezoid rule applied on complex Cauchy integral matrix function $f(A)$ corresponding to equation (1.2) could be approximated in the form:

$$f(A) = \sum_{k=0}^{S-1}\int_{\theta_k}^{\theta_{k+1}} g(\theta)d\theta = \frac{1}{S}\sum_{k=0}^{S-1} f(re^{i\theta_k})re^{i\theta_k}\left(re^{i\theta_k}I_N - A\right)^{-1} = \sum_{k=0}^{S-1}(\theta_{k+1} - \theta_k)\frac{(g(\theta_{k+1}) + g(\theta_k))}{2} = f_S(A) \tag{1.13}$$

$$re^{i\theta_k}\left(re^{i\theta_k}I_N - A\right)^{-1} = \sum_{j=0}^{\infty}\left(\frac{A}{re^{i\theta_k}}\right)^j = \frac{1}{S}\sum_{k=0}^{S-1} f\left(re^{i\theta_k}\right)\sum_{j=0}^{\infty}\left(\frac{A}{re^{i\theta_k}}\right)^j \quad , \quad (\|A\| \le 1 < r) \tag{1.14}$$

γ, r and R are positive real numbers.

Besides, we also use the SVD factorization on the matrix $A = X^T DV$, where $D = diag(\lambda_1, \lambda_2, .., \lambda_n)$ to obtain

$$f(A) = \frac{1}{2\pi i}\int_0^{2\pi} f(z)X^{-1}(zI_N - \lambda_i)^{-1}Xdz = X^{-1}\frac{1}{2\pi i}\int_0^{2\pi} f(re^{i\theta})(zI - \lambda_i I)^{-1}X\, ire^{i\theta})d\theta$$

$$= X^{-1}\frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right)\left(re^{i\theta}I - \lambda_i I\right)^{-1}re^{i\theta}Xd\theta \tag{1.15}$$

Usually as advised in [10] we take Centre of the circle $z - \alpha$ to be $\alpha = \frac{1}{2}(\lambda_{\min} + \lambda_{\max}),\ \ r = \frac{1}{2}\lambda_{\max}\cdot$

The remaining part in the paper is arranged as follows. Section2 describes the hypergeometric function as it applies to the integral calculus. In section 3 we give the methodology required for the numerical computation of the Cauchy integral matrix function having in mind the point of focus on Trapezoid rule, the Residue theorem and the SingularValue Decomposition (SVD) of a diagonalizable matrix. We give error estimate arising from the Givens orthogonal matrix plane rotation. Section 4 in the paper gives the discussion aspect arising from results computed in section3. Section 5 gives the conclusion based on the strength of findings from the computed results.

2.0 Materials and method

We aim to use knowledge of eigenvalues of the diagonalizable matrix A to compute the Cauchy matrix function $f(A)$. We will use the hypergeometric function with a view of obtaining the density of the matrix. We give information on the gamma matrix function as well as the Di-gamma function. The Euler representation of gamma function was emphasized. To be lucid in our presentation, the use of SVD is a useful tool in the computation of the matrix function based on the knowledge of Jordan block matrix. Givens orthogonal matrix plane rotation is employed in calculating the eigenvalues via Sturm's sequence. The matrix exponential for the Cauchy integral theorem was calculated based on the knowledge of the Eigen space of the matrix.

2.1 The Hypergeometric Function And Density of A Matrix.

In lieu of making use of section 1 discussed earlier in the paper, matrix analysis has several uses in the sciences and engineering wherein, hypergeomtric and gamma functions play dominant role, see e.g,[11].The following preliminaries are adopted following[11,12],given that a,b,c are positive integers, the gamma and hypergeometric functions are described. Firstly we define the gamma function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, \qquad (\text{Re } z>0) \qquad (2.1)$$

where,

$t^{z-1}=e^{(z-1)\log t}$, and $\log t\in R$ so that $\Gamma(z+1)=z\Gamma z,\ \Gamma(1)=1,\ \Gamma(n+1)=n!$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \quad \text{with } \Gamma(\tfrac{1}{2})=\sqrt{\pi}. \qquad (2.2)$$

For $n\in N$, the first three rational (fractional) form of gamma functions are well known[13] in the form:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5...(2n-1)}{2^n}\sqrt{\pi}, \qquad (n=1,2,...) \qquad (2.3)$$

$$\Gamma\left(n+\frac{1}{3}\right)=\frac{1.4.7.(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right), \qquad (n=1,2,...), \qquad (2.4)$$

$$\Gamma\left(n+\frac{1}{4}\right)=\frac{1.5.9...(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right), \quad (n=1,2,...). \qquad (2.5)$$

Conversely, for negative integers their gamma functions [14] are defined in the form:

$$\Gamma(k)=\frac{\Gamma(k+n)}{k(k+1)(k+2)...(k+n-1)}, \text{ where }, -n<k<-n+1, \quad n\in N$$

The hypergeometric series with variable z is defined to be the equation

$$F(a,b,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n\, n!} \qquad (2.6)$$

The $(a)_0=1,(a)_1=a,\ (a)_2=a(a+n)$,

$$(a)_n=a(a+1)...(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)},\ (1)_n=n!,(b)_n=\frac{\Gamma(b+n)}{\Gamma(b)},(c)_n=\frac{\Gamma(c+n)}{\Gamma(c)}.$$

We give the zero balanced function for the hypergeometric function [15] assuming that $c=a+b$ with the asymptotic behavior for $F(a,b;c;x)$ as $x\to 1$ in the form

$$\begin{cases} F(a,b;c;1)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & ;if\ a+b<c \\ B(a,b)\Gamma(a,b;c;z)+\log(1-z)-R(a,b)+O(1-z)\log(1-z) & if\ a+b+c \\ F(a,b;c;z)=(1-z)^{(-a-b)}F(c-a,(-b,c;z))) & ,if\ a+b>c \end{cases} \qquad (2.7)$$

Where,

$R(a,b) = -\psi(a) - \psi(b) - 2\gamma$; $\psi\left(z = \frac{\Gamma^{/}(z)}{\Gamma(z)}\right)$

$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577..$

The symbol γ is the Euler-Mascheroni constant.

To this end, the quadratic Hypergeometric function follows from [15] in the form

$$F\left(\frac{1}{4},\frac{3}{4};1;\frac{8r(1+r)}{(1+3r)^2}\right) = \sqrt{1+3r}\,F\left(\frac{1}{4},\frac{3}{4};1;r^2\right) \quad , r \in (0,1) \tag{2.8}$$

The duplication formula inequality for the hypergeometric function in the Grotzsch ring function is then given by

$$\mu_{a,b}(r) = \frac{B(a,b)\Gamma\left(a,b;\frac{a+b+1}{2};1-r^2\right)}{F\left(a,b;\frac{a+b+1}{2};r^2\right)} \tag{2.9}$$

$$F\left(a+1,b+1,c+1,k^2\right) = \frac{c(c-1)}{abk^2}\left(F\left[a,b,c-1,k^2\right] - F\left[a,b,c,k^2\right]\right) \quad ; \tag{2.10}$$

and

$$F(a,b,c,k) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-kt)^a\,dt \tag{2.11}$$

Using above for both gamma and hypergeometric functions it follows that a diagonalizable matrix A has the gamma representation

$$\Gamma(A) = (A)_n = \Gamma(A+nI)\Gamma^{-1}(A)\ ,\ n \ge I, (A)_0 = I\ , \tag{2.12}$$

where the Euler's representation is defined in the form

$$\Gamma(A) = \lim_{n\to\infty}(n-1)!\left[(A)_n\right]^{-1} n^A \tag{2.13}$$

Also we have that

$$(1-z)^{-A} = \sum_{n=0}^{\infty}\frac{(A)_n}{n!}z^n\ , \quad |z| < 1 \tag{2.14}$$

Thus the gamma matrix function and digamma (inverse function) matrix function $\psi(A)$ are given by

$$\Gamma(A) = \int_0^{\infty} e^{-t}t^{A-I}dt\ , \quad t^{A-I} = \exp((A-I)\ln t) \tag{2.15}$$

$$\psi(A) = \frac{d}{dA}\ln\Gamma(A) = \Gamma^{-1}(A)\,\Gamma^{/}(A) \tag{2.16}$$

Besides for a single function, the Beta function in the sense of [16] is defined in the form

$$g(z) = \int_0^1 (1-s)^n s^{z-1}ds = n!\left[z(z+1)(z+2)...(z+n)\right]^{-1}\ ,\ \mathrm{Re}(z) > 0 \tag{2.17}$$

$$f(z) = \int_0^1 (1-\frac{s}{n})^n s^{z-1}ds = n!n^z\left[z(z-1)(z-2)...(z+n)\right]^{-1}\ ,\ \mathrm{Re}(z) > 0 \tag{2.18}$$

Since $f(z)$ and $g(z)$ are holomorphic then we define the gamma matrix function [16] that

$$g(A) = \int_0^1 (I-A)^s s^{z-1}ds = n!\left[A(A+I)(A+2I)...(A+nI\right]^{-1} \tag{2.19}$$

$$f(A) = \int_0^n \left(1-\frac{s}{n}\right)^n s^{A-I}ds = n!n^A\left[A(A+I)(A+2I)....(A+nI)\right]^{-1} \tag{2.20}$$

We then define the Bessel matrix function $J_A(z)$ which is given by the equation

$$J_A(z)=\left(\frac{z}{2}\right)A\Gamma^{-1}\frac{\left(A+\frac{1}{2}\right)}{\sqrt{2}}\int_{-1}^{1}\left(1-t^2\right)^{A-\frac{1}{2}}\cosh(zt)dt \tag{2.15}$$

The modification formula for the Bessel function $K_A(z)$, known as Macdonald function is

$$K_A(z)=\frac{1}{2}\left(\frac{z}{2}\right)^A\int_0^A \exp\left(-t-\frac{z^2}{4t}\right)t^{-(A+I)}\,dt \tag{2.21}$$

The task of computing $f(A)$ implies calculating the square root of a diagonalizable matrix [17]. To this extent, let A_1 and A_2 be independently distributed matrices of same dimension such that $A_{1,}, A_2 \sim p_n(\beta_i), i=1,2$.

Let $S=A_1,A_2$ and $R=(A_1+A_2)^{-\frac{1}{2}}$. The density of S in the sense of [18] for a square matrix A of order n with parameters (α,β) where $\alpha>\frac{1}{2}(n-1), \beta>\frac{1}{2}(n-1)$ is in the form

$$\left\{\frac{\Gamma_n\left(\beta_1+\frac{n+1}{2}\right)\Gamma_n\left(\beta_2+\frac{n+1}{2}\right)\det(S)^{\frac{n+1}{2}}\det\left(I_n+S\right)^{-(\beta_1+\beta_2+n+1)}}{\Gamma_n(\beta_1)\Gamma_n^2\left(\frac{n+1}{2}\right)\Gamma_n(\beta_2)}\right\}\times$$

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\sum_k\sum_\lambda\left(\beta_1+\frac{n+1}{2}\right)_k\left(\beta_2+\frac{n+1}{2}\right)_\lambda}{k!l!} \times\sum_{\phi\in k-\lambda}\left(\theta_\phi^{k,\lambda}\right)^2\frac{\Gamma\left(\left(\frac{n+1}{2},k\right)\right)\Gamma\left(\left(\frac{n+1}{2}\right),\lambda\right)c_\phi\left(I_n+S\right)^{-1}S}{\Gamma(n+1,\phi)}, S>O \tag{2.22}$$

Then, the density of R is

$$\frac{\Gamma_n\left(\beta_1+\left(\frac{n+1}{2}\right)\right)\Gamma_n\left(\beta_2+\left(\frac{n+1}{2}\right)\right)}{\Gamma_n(\beta_1)\Gamma_n(\beta_2)\Gamma_n^2\left(\frac{n+1}{2}\right)}\times\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_k\sum_\lambda\frac{\left(\beta_1+\left(\frac{n+1}{2}\right)\right)_k\left(\beta_2+\left(\frac{n+1}{2}\right)\right)_\lambda}{k!l!}$$
$$\times\sum_{\phi\in k,\lambda}\frac{\Gamma(n+1,\phi)\Gamma_n(\beta_1+\beta_2)}{\Gamma_n(\beta_1+\beta_2+n+1,\phi)}\theta_\phi^{k,\lambda}C_\phi^{k,\lambda}(R,I_n-R), \qquad 0<R<I_n \tag{2.23}$$

The $\phi_\theta^{k,\lambda}\phi_\theta^{k,\lambda}(R,I_n-R)$ are polynomials.

Difficulty in implementing methods of equations (2.23), (2.22) due to high computational work needs a simplified joint density of A_1 and A_2 in the form

$$\frac{\Gamma_n\left(\beta_1+\left(\frac{n+1}{2}\right)\right)\Gamma_n\left(\beta_2+\left(\frac{n+1}{2}\right)\right)}{\Gamma_n(\beta_1)\Gamma_n(\beta_2)\Gamma_n^2\left(\frac{n+1}{2}\right)}\times\det\left(I_n+A_1\right)^{-\beta_1-\left(\frac{n+1}{2}\right)}\det\left(I_n+A_2\right)^{-\beta_2-\left(\frac{n+1}{2}\right)} \tag{2.24}$$

where, $A_1, A_2>O$.

We compute $A_{1,}$ A_2 using the transformation

$$A_1=S^{\frac{1}{2}}(I_m-R)S^{\frac{1}{2}}, \quad A_2=S^{\frac{1}{2}}RS^{\frac{1}{2}}. \tag{2.25}$$

3.0 Numerical Examples.

Computing the Cauchy integral matrix function $f(A)$ involves computing the matrix eigenvalues of A and the residue of the integral function. As a result, the numerical computation of the matrix eigenvalues is provided by the Givens orthogonal matrix plane rotation [19,20] for a reasonable matrix size which reduces a symmetric matrix to a tridiagonal form from which comes in hand the fast higher order Newton's method to compute roots to the resulting polynomial.This is achievable by using the MATLAB Givens matrix plane rotation subroutines.

The number of finite rotation of matrix of size $n \times n$ is given by

$N = \frac{n(n-1)}{n}$. This is accompanied by the unitary matrices with $O\left(\frac{n^2}{\log n}\right)$ Givens factors to be optimally approximated.

Particularly, by the Kantorovich inequality [10], there are $0 < \lambda_1 \le \lambda_2 \dots \lambda_n$ the spectrum of the symmetric matrix A such that for any vector x for which holds

$$\frac{(x^T Ax)(x^T Ax)}{(x^T x)^2} \le \frac{1}{4}\left(K^{\frac{1}{2}} + K^{-\frac{1}{2}}\right)^2, \qquad (3.1)$$

where $\left(K = \frac{\lambda_n}{\lambda_1}\right)$ is the condition number of A

A prototype of Givens orthogonal matrix rotation [19],20] is

$$G(i,j,\theta) = \begin{bmatrix} 1 & 0 & 0\cdots & & \cdots & & 0 \\ 0 & 1 & 0\cdots & & \cdots & & 0 \\ \vdots & \ddots & \vdots & & \cdots & & \vdots \\ 0\cdots & & \cos(\theta)\cdots\sin(\theta)\cdots & & & & 0 \\ & & 1 & & & & \\ \vdots & & -\sin(\theta)\cdots\cos(\theta)\cdots & & & & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ 0\ \dots & & 0\dots & & 1\ \dots & & 0 \\ \vdots & & \vdots & & & \ddots & \vdots \\ 0\cdots & & 0\cdots & & 0\cdots & & 1 \end{bmatrix} \qquad (3.2)$$

The main purpose is to produce a matrix $T \in T(n)$ from the sequence of orthogonal plane matrix rotations $G_1, G_2, \dots, G_k$ such that

$\left\| T - \prod_k G_k \right\|$ is at minimum . (3.3)

We describe the perturbation error from the perturbation matrix $A = A + \delta I$, where δ is a parameter. Using $\prod_{k \in N} G_k$ as a product of Givens factors with angles of rotations θ_k and letting $\hat{G}_k$ be the corresponding perturbed factors with rotation angles $\theta_k + \delta_k$ for $0 < \delta_k < \delta$ .We then give the error [21,22] that

$$\left\| \prod_{k \in N} \hat{G}_k - \prod_{k \in N} G_k \right\|_F \le 2N\delta \ . \qquad (3.4)$$

Then

$$N_\varepsilon(T) = \min\left\{ \frac{N \left| \left\| T - G_j \dots G_N \right\|_F \right.}{\sqrt{n}} < \varepsilon \right\}, \qquad (3.5)$$

is the normalized approximation error [22,23,24] for a given ε assuming that $N_\varepsilon(T)$ is the number of Givens factors.

In addition, we may replace the matrix A by the Givens transformation matrix T_k in equation (1.1), since the eigenvalues of the matrix A are preserved under any similarity transformation.

Besides, assuming the QR - Factorization is considered. Let $A = QR$. Assuming D is the preconditioner to the matrix $\bar{A}$, and $\Delta A = \bar{A} - A$. We then define that

$$\gamma_P = K_P(A)\frac{\|\Delta A\|_p}{\|A\|_2}, \quad (p = 2, \infty). \tag{3.6}$$

We also verify [25] that A has full rank if $\gamma_2 < 1$, and its QR factorization satisfies the inequalities

$$\|\Delta Q\|_p \leq \frac{\sqrt{2}\ \gamma_F}{1-\gamma_2}, \qquad \frac{\|\Delta R\|_F}{\|R\|_2} \leq \frac{\sqrt{2}\ \gamma_F}{1-\gamma_2} \tag{3.7}$$

$$\frac{\|\Delta R\|_\infty}{\|R\|_2} \leq \left(\sqrt{6}+\sqrt{3}\right)\inf_{D\in D_n} \rho_D K_2\left(D^{-1}R\right)\frac{\|\Delta A\|_F}{\|A\|_2} \tag{3.8}$$

Where,

$$\rho_D = \left(1+\xi_D^2\right)^{\frac{1}{2}}, \xi_D = \max_{1\leq i\leq j\leq n}\frac{\delta_j}{\delta_i}, j = 1,2,...$$

EXAMPLE 1 . Consider the Matrix

$$A = \begin{bmatrix} 2 & -2 & 4 & 7 \\ -2 & 3 & 3 & 5 \\ 4 & 3 & 1 & 4 \\ 7 & 5 & 4 & 6 \end{bmatrix},$$

The aim is to reduce the matrix A to tridiagonal form using Givens matrix orthogonal plane rotations.

The first step is $A_0 \to A_1$. That is $T_1^{-1}A_0 = A_0^1 T_1 = T_1^{-1}AT = A_1, T_1 = T_{23}, T_1^{-1} = T_{23}^H$.

$\cos\theta_1 = \frac{-2}{r_1} = \frac{-2}{\sqrt{20}}$, $\sin\theta_1 = \frac{4}{r_1}$, where $r = \sqrt{(-2)^2 + 4^2} = 4.4721$.

We give only the respective phases of plane rotation matrices leading to the final tridiagonal matrices in Table 1.

TABLE1: PHASES OF ROTATION MATRIX VIA GIVENS PLANE MATRIX SIMILARITY TRANSFORMATION.

ITERATION k	PHASES OF ROTATION MATRIX A_k
1	$\begin{bmatrix} 2 & 4.472135955 & 0 & 7 \\ 4.472135955 & -1 & -1 & 1.341640786 \\ 0 & -1 & 5 & -6.260990337 \\ 7 & 40786 & -6.260990337 & 6 \end{bmatrix}$
2	$\begin{bmatrix} 2 & 8.306623863 & 0 & 0 \\ 8.306623863 & 5.188405797 & -5.814524542 & 2.611986652 \\ 0 & -5.814524542 & 5 & -2.528102914 \\ 0 & 2.611986652 & -2.528102915 & -0.1884057965 \end{bmatrix}$
3	$\begin{bmatrix} 2 & 8.306623863 & 0 & 0 \\ 8.306623863 & 5.188405797 & 6.408351146 & 0 \\ 0 & 6.408351146 & 6.083308727 & 0.2630538098 \\ 0 & 0 & 0.2630538098 & -1.220107193 \end{bmatrix}$

From the Table1 the original matrix A has been reduced to tridiagonal matrix as showed for k= 3.

The matrix $T_3 = \begin{bmatrix} 2 & 8.306623863 & 0 & 0 \\ 8.306623863 & 5.188405797 & 6.408351146 & 0 \\ 0 & 6.408351146 & 6.083308727 & 0.2630538098 \\ 0 & 0 & 0.2630538098 & -1.220107193 \end{bmatrix}$

Therefore we transform the matrix T_3 via Sturms' sequence to polynomial equation.
We can use a fast Newton solver to compute the zeros of the polynomial equation.
Thus
$\text{poly}(T_3) = \lambda^4 - 12\lambda^3 - 72\lambda^2 + 367\lambda + 528 = 0$.

The eigenvalues of T_3 are $(-6.3640, -1.2270, 4.5584, 15.0116)$.

While the eigenvalue of A are $(-6.3560, -1.2140, 4.5584, 15.0116)$ with polynomial
$\lambda^4 - 12.0516\lambda^3 - 72.2230\lambda^2 + 370.9761\lambda + 539.3870 = 0$.

We computed for the error in computing for $N_\varepsilon(T) = \min\left\{ N \,\middle|\, \frac{\|T - G_1.G_2.G_3\|}{\sqrt{N}} \right\} = \frac{14.4262}{\sqrt{4}} = 7.2131$,

which is quite reasonable to the degree of closeness of approximation to the Tridiagonal matrix T.

It is necessary to also compute for $e^A = \sum_{i=0}^{\infty} \frac{(A)^i}{i!} = I + A + \frac{(A)^2}{2!} + \frac{(A)^3}{3!} + \ldots$

Thus $e^A = \begin{bmatrix} 3.0369 & -1.9300 & 4.0676 & 7.0372 \\ -1.9300 & 4.0595 & 3.0957 & 5.0536 \\ 4.0676 & 3.0857 & 2.0589 & 4.0346 \\ 7.0372 & 5.0536 & 4.0345 & 7.0194 \end{bmatrix}$

Using SVD approach, we have $f(A) = Sf(A)S^{-1} = Sf(\Lambda)S^{-1}$

Hence $f(A) =$

$$\begin{bmatrix} -0.4619 & 0.6620 & 0.5896 & 0.0286 \\ -0.3238 & 0.4932 & -0.8074 & -0.0000 \\ -0.4064 & -0.3018 & -0.0214 & 0.8622 \\ -0.7188 & -0.4769 & -0.0030 & -0.5058 \end{bmatrix} \begin{bmatrix} e^{15.0116} & 0 & 0 & 0 \\ 0 & e^{6.3560} & 0 & 0 \\ 0 & 0 & e^{4.558} & 0 \\ 0 & 0 & 0 & e^{1.2140} \end{bmatrix} \begin{bmatrix} -0.4619 & 0.6620 & 0.5896 & 0.0286 \\ -0.3238 & 0.4932 & -0.8074 & -0.0000 \\ -0.4064 & -0.3018 & -0.0214 & 0.8622 \\ -0.7188 & -0.4769 & -0.0030 & -0.5058 \end{bmatrix}^{-1}$$

Hence, result computed with SVD for $f(A)$:

$$f(A) = e^A = (1.0e+006) \times \begin{bmatrix} 0.7058 & 0.4940 & 0.6206 & 1.0979 \\ 0.4947 & 0.4370 & 0.4351 & 0.7696 \\ 0.6206 & 0.4351 & 0.5462 & 0.9662 \\ 1.0977 & 0.7696 & 0.9661 & 1.7089 \end{bmatrix}.$$

Direct analytical result with MATLAB for e^A is

$$e^A = 1.0e+003 * \begin{bmatrix} 0.0074 & 0.0001 & 0.0546 & 1.0966 \\ 0.0001 & 0.0201 & 0.0201 & 0.1484 \\ 0.0546 & 0.0201 & 0.0027 & 0.0546 \\ 1.0966 & 0.1484 & 0.0546 & 0.4034 \end{bmatrix}.$$

Example 2. Consider again the matrix

$$A = \begin{bmatrix} 2 & -2 & 4 & 7 \\ -2 & 3 & 3 & 5 \\ 4 & 3 & 1 & 4 \\ 7 & 5 & 4 & 6 \end{bmatrix}, \quad f(z) = z^2.$$

Since $f(z)$ is analytic on the domain for which $z \in D \subset \mathbb{C}$, for D simply connected region containing the spectrum of

A .From $A = XJX^{-1} = X\, diag\left(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), ..., J_{n_k}(\lambda_i)\right)X^{-1}$

The Jordan canonical form of A is then expressed in the general form

$$f(A) = X\, diag\left(f(J_n(\lambda_1))\, f(J_n(\lambda_2))... f(J_n(\lambda_i))\right)X^{-1} \quad (3.9)$$

The analytic function f of a Jordan block

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f^{/}(\lambda_i) & f^{//}(\lambda_i) \ldots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\ & f(\lambda_i) & f^{/}(\lambda_i) \ldots & \\ & \ddots & \ddots & \vdots \\ & & & f^{/}(\lambda_i) \\ & & & f(\lambda_i) \end{bmatrix} \quad (3.10)$$

where it is true that $f(J_{ni}) = f(\lambda_i)I + \sum_{v=1}^{n-1} \frac{1}{v!} f^{(v)}(\lambda_i)N^v$.

Hence coupling all these we the write that

$$f(A) = Xf(J_n(\lambda_i))X^{-1} = \begin{bmatrix} 56.4926 & -92.1764 & -28.6572 & -2.4339 \\ 58.442 & -52.1285 & -18.5615 & -18.2651 \\ 41.9226 & -60.9774 & -54.0440 & 0.5403 \\ 72.5573 & -110.5404 & -93.7473 & -0.6897 \end{bmatrix}$$

In passing we remark that the pth root of a diagonalizable matrix can be computed as a byproduct of equation (3.9) free of additional cost.

We give a bound for the Cauchy integral matrix function. Firstly, consider that the first derivative of Cauchy integral theorem for matrix function $f(A)$ that is analytic within and on the boundary of the contour follows from same analogy of scalar function of complex variables. Hence we write that for a scalar complex function for Cauchy integral is in the form

$$f^{/}(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z)}{(z - z_0)^2}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(z_0 + re^{i\theta} - z_0)^2}\, dz \quad (3.11)$$

Using the fact that $z = z_0 + re^{i\theta}$; $z_0 \in z$, there holds that

$$f^{/}(A) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^2 e^{2i\theta}}\, ire^{i\theta} d\theta = \frac{1}{2\pi r}\int_0^{2\theta} f\left(z_0 + re^{i\theta}\right)e^{-i\theta} d\theta \cdot \quad (3.12)$$

Since the function $f(A)$ is an entire function, by Morera's theorem, $|f(A)| \le K < \infty$. Hence we have that $|f^{/}(A)| \le \frac{K}{r}$ for $r < \infty$. Thus $f^{/}(A) = 0, A \in \mathbb{C}^{n\times n}$. So it follows that $|f^{/}(A)| \le K$. A generalization to nth derivative of Cauchy's integral matrix function is then in the form

$$f^{n}(A) = \frac{n!}{2\pi i}\oint_{\eta} \frac{f(z_0 + re^{i\theta})}{(zI - A)^{n+1}} d\theta,\ n \in N^{+} \cdot \quad (3.13)$$

Therefore in view of equation (3.10) it holds that the expression $(zI - A)^{-1}$ is in the form

$$(zI - A)^{-1} = \begin{bmatrix} \frac{1}{(z-\lambda)} & \frac{1}{(z-\lambda)^2} & \cdots & \frac{1}{(z-\lambda)^n} \\ & \frac{1}{(z-\lambda)} \quad \frac{1}{(z-\lambda)^2} & \cdots & \vdots \\ & \ddots & & \vdots \\ & & \ddots & \frac{1}{(z-\lambda)^2} \\ & & & \frac{1}{(z-\lambda)} \end{bmatrix} \quad (3.14)$$

Then we write that $f(A)$ is given by

$$f(A)=\frac{1}{2\pi i}\oint f(z)(zI-A)^{-1}dz.$$

4.0 Discussion

The paper presented Cauchy integral theorem of matrix function wherein, the Lebesgue measure is discussed using Little Wood's Third formula [4]. We made reference to Trapezoidal rule and Composite Simpson's method [9]. It is showed that Cauchy integral matrix function is convergent with respect to the Eigen space of the matrix. The matrix A is symmetric and its eigenvalues are preserved under any similarity transformation. We related this with the Trapezoid rule where the contour is taken over the over Jordan arc.

Then, further using Taylor series, if $f(z)$ is analytic on the domain for which $z\in D\subset\mathbb{C}$, for D, it is commutable with the Jordan block matrix via the Singular Value Decomposition (SVD). The eigenvalues of the diagonalizable matrix is computed using the Givens orthogonal matrix plane rotation where the Sturmian sequence becomes a handy tool. The matrix exponential is then computed based on the aforementioned SVD. In particular the function of a matrix for the Cauchy integral is computed by this method.

Since $f(z)$ is analytic on the domain for which $z\in D\subset\mathbb{C}$, where, D is a simply connected region containing the spectrum of A and from $A=XJX^{-1}=X\,diag\left(J_{n_1}(\lambda_1),J_{n_2}(\lambda_2),...,J_{n_k}(\lambda_i)\right)X^{-1}$, it is computed the density of a matrix based on the use of hypergeometric series.

5.0 Conclusion

The paper presented numerical formulae for computing Cauchy integral matrix function. After some preliminary discussions we introduced the concept of Lebesgue measure and showed that Trapezoidal rule and Composite Simpson rule are Lebesgue measurable over the contour integral. In particular, it is established that Cauchy theorem for matrix function is convergent provided the spectral radius of the complex matrix $\|zI-A\|<<r$. This was proved in the series expansion for the inverse matrix $(zI-A)^{-1}$

Besides, we also computed the eigenvalues of the diagonalizable matrix A by the aid of Givens orthogonal matrix plane rotation wherein, the Sturm's sequence becomes a useful too. The error bound for the Givens rotation orthogonal matrices was computed. It is also given the method of testing for the rank of a matrix via QR decomposition method. We computed the exponential matrix using the Jordan canonical block. Then, as an extension, the use of Residue for the Cauchy matrix function explained and its limitation for a matrix of higher orders was stated. The use of hypergeometric series for the matrix function was given and stated its application in the testing for the density of a diagonalizable matrix. More of future works in this field of research may be found useful to the readership in this direction.

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