Transactions of the Nigerian Association of Mathematical Physics
***Volume 17, (October – December, 2021 Issue*), pp51 – 60**

SEMI-ANALYTIC METHODS FOR THE SOLUTION OF TWO EPIDEMIOLOGICAL MODELS

[1]B.T. Enoch, [2]T. T. Ashezua and [3]S.A. Somma

[1]Department of Mathematics, Federal University of Lafia, Lafia, Nasarawa State
[2]Department of Mathematics/Statistics/Computer Science, University of Agriculture, Makurdi, Nigeria
[3]Department of Mathematics, Federal University of Technology, Minna, Nigeria

Abstract

In this paper, we apply three semi-analytical methods, viz: the Differential Transform Method (DTM), Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM) to compute approximate solutions of a continuous mathematical model of Shigella diarrhea comprising of a non-constant population and a deterministic model on the impact of stress on the dynamics and treatment of Tuberculosis.

1. Introduction

Most epidemiological models are formulated using systems of linear and non-linear differential equations. After the stability analyses of the models are qualitatively obtained, the numerical solution of such models are in most cases conducted using the fourth-order Runge-Kutta method embedded in either MAPLE or MATLAB mathematical softwares. In a view to diversifying the approach, we employed the Differential Transform Method (DTM), the Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM) to solve the models by by Ojaswita *et al.* [1] and Namawejje *et al*. [2].

The concept of DTM was first proposed by Zhou [3] in 1986. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The method is very effective and powerful for solving various kinds of differential equations [4]. For examples, it was used to find the analytic approximate solution of the Volterra's model population growth of a species in a closed system [5], to solve a system of nonlinear differential equations, namely, the Coullet system [6], to find the solution of nonlinear PDEs [7], to solve Volterra integral and integro-differential equations with proportional delays [8] and so on.

The Homotopy Perturbation Method (HPM) was first proposed by He [9] and was developed and improved by He [10, 11, 12]. HPM is a coupling of the traditional perturbation method in topology, He [13]. The method provides an approximate analytical solution in a series form. HPM has been widely used by numerous researchers successfully for physical systems such as bifurcation, asymptotology, nonlinear wave equations, oscillations with discontinuities by He [14, 15, 16, 17]. It has also been applied to the reaction-diffusion equation [18] and the SIR infectious disease model [19].

The Variational Iteration Method (VIM), first envisioned by He [20] modifying the approach by Inokuti *et al.* [21], has been applied to many linear and nonlinear ODEs and PDEs. For example, He [22] employed VIM to give approximate solutions for some well – known nonlinear problems and in [23], he successfully applied VIM to autonomous systems of ODEs. Also, He [24] gave a solution for a seepage flow problem with fractional derivatives in porous media using VIM. Other researchers demonstrated further applications of VIM. For instance, Soliman [25] applied VIM to solve KdV – Burger's and Lax's seventh – order KdV equations. Momani and Abuasad [26] also used VIM to solve the Helmholtz equation and Odibat and Momani [27] solved nonlinear fractional differential equations via VIM; while Bildik [28] used VIM, DTM AND ADM to solve different types of nonlinear partial differential equations, and Abbasbandy [29] solved the quadratic Riccati differential equations by He's VIM with considering Adomian polynomials. Jufeng [30] applied VIM to solve two – point boundary value problems.

Corresponding Author**:** Enoch B.T., Email: blessingenoch774@yahoo.com, Tel: +2348105532965

The paper is organized as follows: In section 2, we give a brief description of VIM and in section 3, two mathematical models are solved using these methods. Some concluding remarks are given in section 4.

2.1 The Differential Transformation Method (DTM)

An arbitrary function $f(x)$ can be expanded in Taylor series about a point $x = 0$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0} \quad (1)$$

The differential transformation of $f(x)$ is defined as

$$F(x) = \frac{1}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0} \quad (2)$$

Then the inverse differential transform is

$$f(x) = \sum_{k=0}^{\infty} x^k F(k) \quad (3)$$

Table 1 - The Fundamental Mathematical Operations Performed by DTM [31]

Original Function	Transformed Function
$y(x) = g(x) \pm h(x)$	$Y(k) = G(k) \pm H(k)$
$y(x) = \alpha g(x)$	$Y(k) = \alpha G(k)$
$y(x) = \frac{dg(x)}{dx}$	$Y(k) = (k+1)G(k+1)$
$y(x) = \frac{d^2 g(x)}{dx^2}$	$Y(k) = (k+1)(k+2)G(k+2)$
$y(x) = \frac{d^m g(x)}{dx^m}$	$Y(k) = (k+1)(k+2)\dots(k+m)G(k+2)$
$y(x) = 1$	$Y(k) = \delta(k)$
$y(x) = x$	$Y(k) = \delta(k-1)$
$y(x) = x^m$	$Y(k) = \delta(k-m) = \{^{1,k=m}_{0,k\neq m}$
$y(x) = g(x)h(x)$	$Y(k) = \sum_{m=0}^{k} H(m)G(k-m)$
$y(x) = e^{(\lambda x)}$	$Y(k) = \frac{\lambda^k}{k!}$
$y(x) = (1+x)^m$	$Y(k) = \frac{m(m-1)\dots(m-k+1)}{k!}$

2.2 Homotopy Perturbation Method (HPM)

To illustrate the ideas of this method, the following nonlinear equation was considered, He [32].

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (4)$$

Subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \quad (5)$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Equation (4) can therefore be rewritten as follows:

$$L(u) + N(r) - f(r) = 0, \quad r \in \Omega \quad (6)$$

Using the homotopy technique, we construct a homotpy $v(r,p): \Omega \times [0,1] \to R$, which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \quad (7)$$

$$p \in [0,1], \qquad r \in \Omega \quad (8)$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(u) - f(r)] = 0 \quad (9)$$

Where

$L(u)$ is the part

$L(u) = L(v) - L(u_0) + pL(u_0)$

And $N(u)$ is the nonlinear part

$N(u) = pN(v)$

In eqn (7), $p \in [0.1]$ is an embedding parameter and u_0 is an initial approximation of (4) and (9), we will have

$H(v,0) = L(v) - L(u_0) = 0$ (10)
$H(v,1) = L(v) + N(v) - f(r) = 0$ (11)
The changing process of p from zero to unity is just of $H(v,r)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a small parameter and assume that the solution of eqn (7) and (9) can be written as power series in p:
$v = v_0 + pv_1 + p^2 v_2 + \cdots$ (12)
Setting $p = 1$ results in the approximate solution (4) as
$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$ (13)
The series (13) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following observations are made by He [11]:

- The derivatives of $N(v)$ with respect to v must be small because the parameter p may be large, i.e. $p \to 1$.
- The norm of $L^{-1}\frac{\partial N}{\partial V}$ must be smaller than one so that the series converges.

2.3 The Variational Iteration Method (VIM)
To illustrate the basic concept of the Variational Iteration Method, we consider the following general differential equation:
$Lu + Nu = g(x,t)$ (14)
where L and N are linear and non-linear operators respectively, and $g(x,t)$ is the source inhomogeneous term. He proposed the variational iteration method where a correction functional for (14) can be written as
$u_{i,n+1}(x,t) = u_{i,n}(x,t) + \int_{t_0}^{t} \lambda\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\}\mathrm{d}s,$ (15)
where $i = 1,2,\dots,m$, λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, n denotes the nth approximation and $\tilde{u}_{i,n}$ is considered as a restricted variation which means $\delta\tilde{u}_{i,n} = 0$. It is required first to determine the Lagrange multiplier λ [32] that will be identified optimally via integration by parts. The successive approximations $u_{i,n+1}(x,t), n \geq 0$, of the solution $u(x,t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. Having $\boldsymbol{\lambda}$ determined, then several approximations $u_j(x,t), j \geq 0$, can be determined. Consequently, the solution is given by
$u = \lim_{n\to\infty} u_n$

3. Application
3.1. Considering the Shigella diarrhea model by Ojaswita *et al.* [1],

$$\frac{dS}{dt} = \pi - \mu S - \beta SI + \alpha R$$
$$\frac{dI}{dt} = \beta SI - \mu I - \gamma I$$
$$\frac{dR}{dt} = \gamma I - \mu R - \alpha$$

Subject to the initial conditions $S(0) = 100,\ I(0) = 40,\ R(0) = 10$
where the parameters π is the population renewal rate, β is the infectivity rate, μ is the natural death rate, r is the recovery rate and α is immunity loss rate with their values given as $\pi = 10,\ \mu = 0.012$ [33], $\alpha = 0.25, r = 0.14$.
By applying the **DTM** to the model, we obtained the following recurrence relations:

$$S(k+1) = \frac{1}{k+1}\left[10 - 0.012S(k) - 0.0001\left(\sum_{m=o}^{k} S(m)I(k-m)\right) + 0.25R(k)\right]$$
$$I(k+1) = \frac{1}{k+1}\left[0.0001\left(\sum_{m=o}^{k} S(m)I(k-m)\right) - 0.152I(k)\right]$$
$$R(k+1) = \frac{1}{k+1}[0.14I(k) - 0.262R(k)]$$

Then the closed form of the solutions of order 8 are

$$s(t) = \sum_{k=0}^{8} S(k)t^k$$

$$s(t) = 100 + 10.9000t + 5.3137t^2 + 3.2399754t^3 + 2.493257432t^4 + 1.992024714t^5 + 1.661583302t^6 + 1.424919360t^7 + 1.247257316t^8$$

$$i(t) = \sum_{k=0}^{8} I(k)t^k$$

$$i(t) = 40 - 5.6800t + 0.42508t^2 - 0.01509925333t^3 + 0.003137287795t^4 + 0.001579329081t^5 + 0.001076797522t^6 + 0.0007809196739t^7 + 0.0005911186595t^8$$

$$r(t) = \sum_{k=0}^{8} R(k)t^k$$

$$r(t) = 10 + 2.980t - 0.78798t^2 + 0.08865398667t^3 - 0.006335309995t^4 + 0.0004198143020t^5 + 0.00001851912070t^6 + 0.00002084280621t^7 + 0.00001298349239t^8$$

Applying **HPM** to the model, we obtain the following:

$$(1-p)\frac{dS}{dt} + p\left[\frac{dS}{dt} + \mu S + \beta SI - \pi - \alpha R\right] = 0$$

$$(1-p)\frac{dI}{dt} + p\left[\frac{dI}{dt} + (\mu + \gamma)I - \beta SI\right] = 0$$

$$(1-p)\frac{dR}{dt} + p\left[\frac{dR}{dt} - \gamma I + (\mu + \alpha)R\right] = 0$$

Let $S = x_0 + px_1 + p^2x_2 + \cdots$

$I = y_0 + py_1 + p^2y_2 + \cdots$

$R = z_0 + pz_1 + p^2z_2 + \cdots$

Solving and substituting $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2$ into S, I and R above gives

$$S(t) = S_0 + p[\pi + \alpha R_0 - \mu S_0 - \beta S_0 I_0]t - p^2 - \{\beta S_0 I_0[\beta S_0 - (\mu + \gamma)] + (\pi + \alpha R_0 - \mu S_0 - \beta S_0 I_0)(\mu + \beta I_0)\}\frac{t^2}{2}$$

$$I(t) = I_0 + p[(\beta S_0 I_0 - (\mu + \gamma)I_0)t] + p^2\{\beta S_0 I_0[\beta S_0 - (\mu + \gamma)] + \beta I_0[\pi + \alpha R_0 - \mu S_0 - \beta S_0 I_0] - (\mu + \gamma)[\beta S_0 - (\mu + \gamma)]I_0\}\frac{t^2}{2}$$

$$R(t) = R_0 + p[(\gamma I_0 - (\mu + \alpha)R_0)t] + p^2\{\gamma I_0[\beta S_0 - (\mu + \alpha)] - (\mu + \alpha)[\alpha I_0 - (\mu + \alpha)R_0]\}\frac{t^2}{2}$$

Setting $p = 1$, the results follow.

Applying VIM to the same model, we construct the correction functional as follows

$$S^{n+1}(t) = S^n(t) + \int_0^t \lambda_1(\xi)\left[\frac{d}{d\xi}S^n(\xi) - \mu S^n(\xi) - \pi + \beta S^n(\xi)I^n(\xi) - \alpha R^n(\xi)\right]d\xi$$

$$I^{n+1}(t) = I^n(t) + \int_0^t \lambda_2(\xi)\left[\frac{d}{d\xi}I^n(\xi) - \beta S^n(\xi)I^n(\xi) + (\gamma + \mu)I^n(\xi)\right]d\xi$$

$$R^{n+1}(t) = R^n(t)\int_0^t \lambda_3(\xi)\left[\frac{d}{d\xi}R^n(\xi) + (\alpha + \mu)R^n(\xi) - \gamma I^n(\xi)\right]d\xi$$

Making the functional stationary, we obtain the following Lagrange multipliers

$\lambda_1(\xi) = -e^{\mu(\xi - t)}$, $\lambda_2(\xi) = -e^{(\mu+\gamma)(\xi-t)}$ and $\lambda_3(\xi) = -e^{(\mu+\alpha)(\xi-t)}$

Substituting these Lagrange multipliers into the correction functional to obtain

$$S^{n+1}(t) = S^n(t) + \int_0^t -e^{\mu(\xi-t)}\left[\frac{d}{d\xi}S^n(\xi) - \mu S^n(\xi) - \pi + \beta S^n(\xi)I^n(\xi) - \alpha R^n(\xi)\right]d\xi$$

$$I^{n+1}(t) = I^n(t) + \int_0^t -e^{(\mu+\gamma)(\xi-t)} \left[\frac{d}{d\xi} I^n(\xi) - \beta S^n(\xi) I^n(\xi) + (\gamma+\mu) I^n(\xi)\right] d\xi$$

$$R^{n+1}(t) = R^n(t) \int_0^t -e^{(\mu+\alpha)(\xi-t)} \left[\frac{d}{d\xi} R^n(\xi) + (\alpha+\mu) R^n(\xi) - \gamma I^n(\xi)\right] d\xi$$

The VIM, DTM and HPM algorithms are coded in the MAPLE software and the solutions are shown in Tables 1a and 1b. The graphs are also displayed in Fig. 1, 2, and 3.

3.2 Considering the model by Namawejje *et al*. [2];

$$\frac{dS}{dt} = \pi - (1-k)\alpha_1 \lambda S - k\alpha_2 \lambda S - \mu S$$
$$\frac{dL}{dt} = (1-k)\alpha_1 \lambda S - (\varphi\alpha_3\lambda + \gamma)L - \mu L$$
$$\frac{dI}{dt} = k\alpha_2 \lambda S + (\varphi\alpha_3\lambda + \gamma)L - (\nu + \mu + d)I + \alpha_4 \lambda \omega R$$
$$\frac{dR}{dt} = \nu I - \alpha_4 \lambda \omega R - \mu R$$

where $\lambda = \beta c \frac{I}{N}$

subject to the initial conditions $S(0) = 95000, I(0) = 25000, I(0) = 19500, R(0) = 13500$

where the parameters π is the recruitment rate of population, β is the probability of getting tuberculosis infection, μ is the per capita death rate due to the disease, γ is the progression rate of disease to active TB, φ is the level of re-infection, α_1 is the enhancement stress factor for slow progressors, α_2 is the enhancement stress factor for fast progressors, α_3 is the enhancement stress factor for re-infection (exogenous reactivation), α_4 is the enhancement stress factor for re-infection (endogenous reactivation), c is per capita contact rate, ν is the recovery rate, k is the proportion from S to I, ω is the rate of endogenous reactivation (from R to I) with their values given as $\pi = 2041, \alpha_1 = 1.6, \alpha_2 = 2, \alpha_3 = 1.3, \alpha_4 = 1, \beta = 0.35, c = 2, \mu = 0.02041, k = 0.05, \varphi = 0.06, \gamma = 0.05, \omega = 0.21, \nu = 0.2, d = 0.3$.

Applying the VIM to this model, we construct the correction functional as follows

$$S^{n+1}(t) = S^n(t) + \int_0^t \lambda_1(\xi) \left[\frac{d}{d\xi} S^n(\xi) - \pi + (1-k)\alpha_1 \beta c \frac{SI}{N} + k\alpha_2 \beta c \frac{SI}{N} + \mu S\right] d\xi$$

$$L^{n+1}(t) = L^n(t) + \int_0^t \lambda_2(\xi) \left[\frac{d}{d\xi} L^n(\xi) - (1-k)\alpha_1 \beta c \frac{SI}{N} + \varphi\alpha_3 \beta c \frac{LI}{N} + (\gamma+\mu)L\right] d\xi$$

$$I^{n+1}(t) = I^n(t) + \int_0^t \lambda_3(\xi) \left[\frac{d}{d\xi} I^n(\xi) - k\alpha_2 \beta c \frac{SI}{N} - \varphi\alpha_3 \beta c \frac{LI}{N} - \gamma L + (\nu+\mu+d)I - \alpha_4 \omega \beta c \frac{IR}{N}\right] d\xi$$

$$R^{n+1}(t) = R^n(t) + \int_0^t \lambda_4(\xi) \left[\frac{d}{d\xi} R^n(\xi) - \nu I + \alpha_4 \omega \beta c \frac{IR}{N} + \mu R\right] d\xi$$

Applying DTM to the model, we get

$$S(k+1) = \frac{1}{k+1}\left[\pi - (1-k)\alpha_1 \frac{\beta c}{N}\left(\sum_{m=o}^{k} S(m)I(k-m)\right) - k\alpha_2 \frac{\beta c}{N}\left(\sum_{m=o}^{k} S(m)I(k-m)\right) - \mu S(k)\right]$$

$$L(k+1) = \frac{1}{k+1}\left[(1-k)\alpha_1 \frac{\beta c}{N}\left(\sum_{m=o}^{k} S(m)I(k-m)\right) - \varphi\alpha_3 \frac{\beta c}{N}\left(\sum_{m=o}^{k} L(m)I(k-m)\right) - \gamma L(k) - \mu L(k)\right]$$

$$I(k+1) = \frac{1}{k+1}\left[k\alpha_2 \frac{\beta c}{N}\left(\sum_{m=o}^{k} S(m)I(k-m)\right) + \varphi\alpha_3 \frac{\beta c}{n}\left(\sum_{m=o}^{k} L(m)I(k-m)\right) + \gamma L(k) - (\nu+\mu+d)I(k) + \alpha_4 \omega \frac{\beta c}{N}\left(\sum_{m=o}^{k} R(m)I(k-m)\right)\right]$$

$$R(k+1) = \frac{1}{k+1}\left[\nu I(k) - \alpha_4 \omega \frac{\beta c}{N}\left(\sum_{m=o}^{k} R(m)I(k-m)\right) - \mu R(k)\right]$$

The VIM, DTM and the HPM algorithms are codded in MAPLE software and the solutions are shown in Tables 2a, 2b and 2c. The graphs are also displayed in Fig. 4, 5, 6 and 7.

Conclusion

Three semi-analytic methods, VIM, DTM and HPM have been used to solve two mathematical models and the results have shown that these methods are powerful mathematics tools for solving systems of Ordinary Differential Equations which appear in mathematical modelling of differential phenomena.

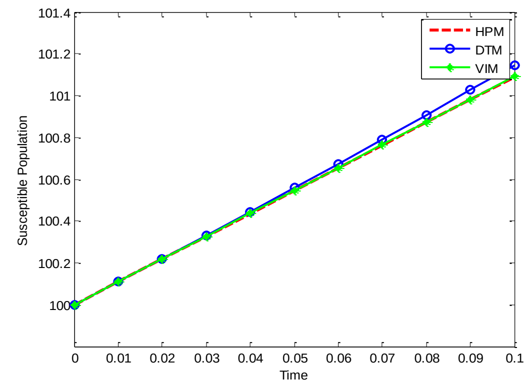

Fig. 1: Graph of the Solution Obtained using VIM, DTM and HPM for the Susceptible Population

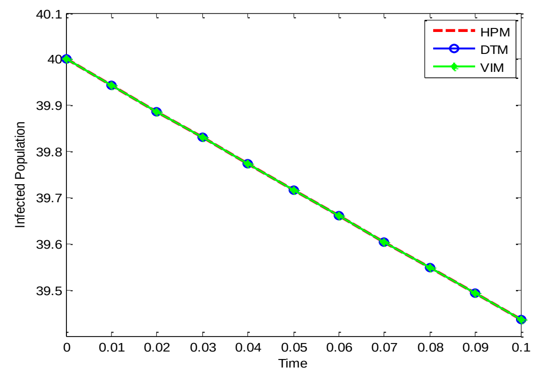

Fig. 2: Graph of the Solution Obtained using VIM, DTM and HPM for the Infected Population

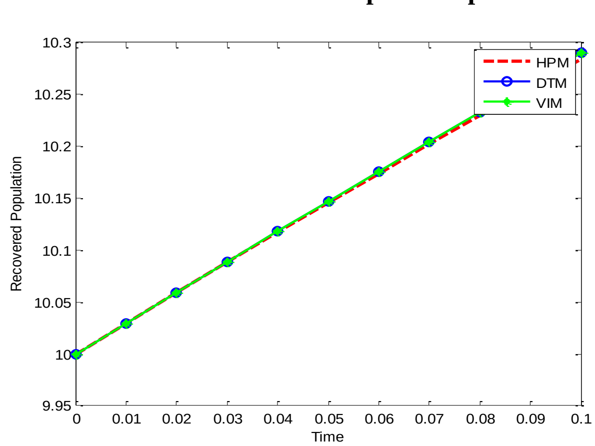

Fig. 3: Graph of the Solution Obtained using VIM, DTM and HPM for the Recovered Population

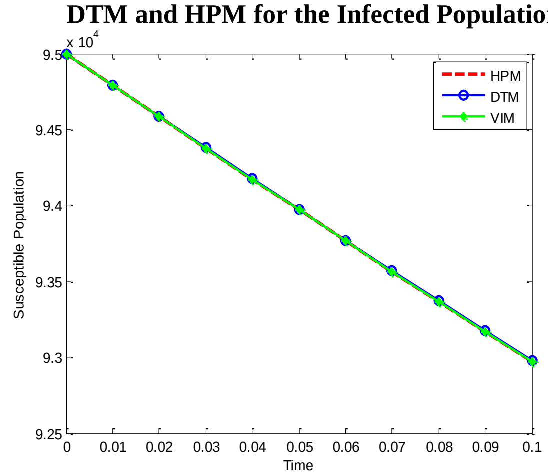

Fig. 4: Graph of the Solution Obtained using VIM, DTM and HPM for the Susce Population

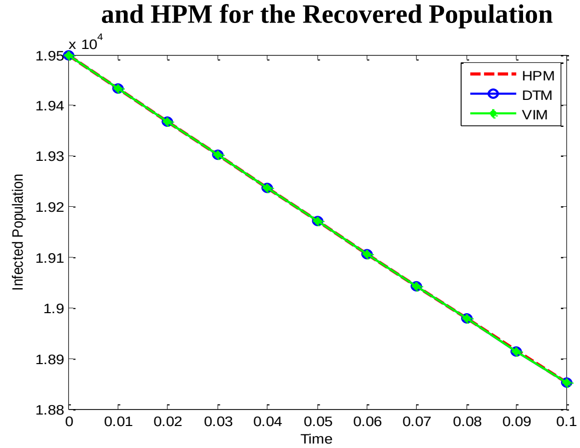

Fig. 5: Graph of the Solution Obtained using VIM, DTM and HPM for the Susce Population

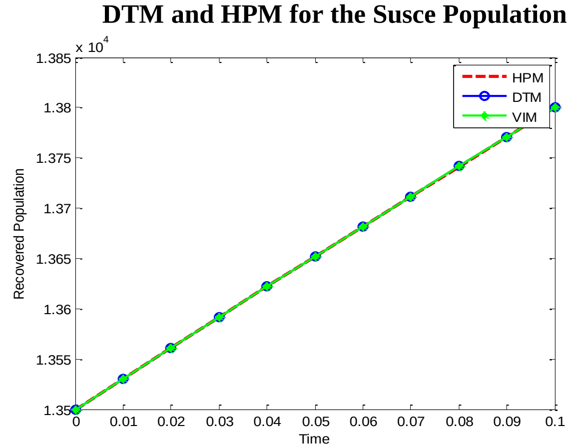

Fig. 6: Graph of the Solution Obtained using VIM, DTM and HPM for the Susce Population

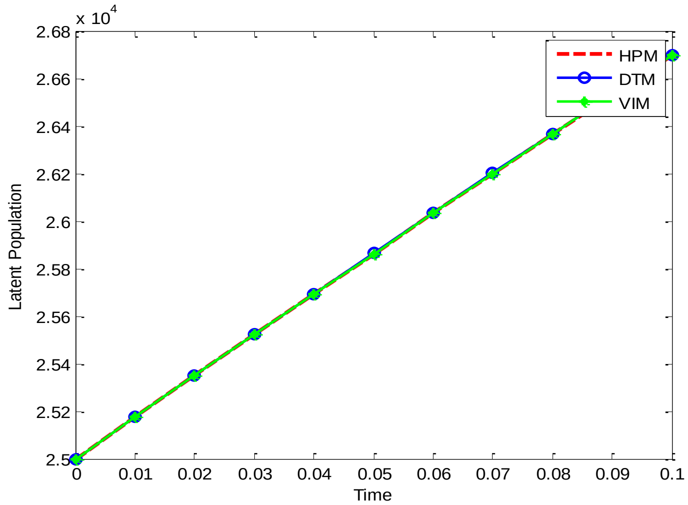

Fig. 7: Graph of the Solution Obtained using VIM, DTM and HPM for the Susce Population

Table 2a: Solution of the Shigella diarrhea model Using VIM and DTM

t	VIM			DTM		
	S(t)	I(t)	R(t)	S(t)	I(t)	R(t)
0	99.9999999	39.99999677	9.999999492	100	40	10
0.01	100.1090278	39.94324289	10.02972116	100.1095346	39.94324249	10.02972129
0.02	100.2181235	39.88656695	10.05928483	100.2201518	39.88656991	10.05928552
0.03	100.3272825	39.82998138	10.08869207	100.3318718	39.82998216	10.08869320
0.04	100.4365000	39.77347752	10.11794549	100.4447159	39.77347917	10.11794488
0.05	100.5457728	39.71705893	10.14704066	100.5587054	39.71706083	10.14704109
0.06	100.6551114	39.66072304	10.17598418	100.6738630	39.66072707	10.17598234
0.07	100.7645151	39.60447376	10.20476965	100.7902118	39.60447779	10.20476916
0.08	100.8739780	39.54830790	10.23340166	100.9077756	39.54831292	10.23340206
0.09	100.983491	39.49222588	10.26187979	101.0265793	39.49223236	10.26188157
0.1	101.0930687	39.43623225	10.29021066	101.1466480	39.43623603	10.29020822

Table 2b: Solution of the Shigella diarrhea model Using DTM

HPM		
S(t)	I(t)	R(t)
100	40	10
100.1089941	39.94324251	10.02966356
100.2179765	39.88657003	10.05905425
100.3269471	39.82998257	10.08817206
100.4359059	39.77348013	10.11701699
100.5448530	39.71706270	10.14558905
100.6537883	39.66073029	10.17388823
100.7627119	39.60448289	10.20191454
100.8716237	39.54832051	10.22966797
100.9805237	39.49224315	10.25714852
101.0894120	39.43625080	10.28435620

Table 3a: Solution of the Tuberculosis Model Using DTM

t	DTM			
	S(t)	I(t)	R(t)	L(t)
0	95000	19500	13500	25000
0.01	94791.65963	19433.50262	13530.64988	25175.08021
0.02	94584.74029	19367.37239	13561.16726	25348.92599
0.03	94379.23582	19301.60709	13591.55291	25521.54805
0.04	94175.14025	19236.20451	13621.80764	25692.95697
0.05	93972.44791	19171.16247	13651.93223	25863.16321
0.06	93771.15328	19106.47879	13681.92745	26032.17721
0.07	93571.25117	19042.15132	13711.79408	26200.00921
0.08	93372.73658	18978.17794	13741.53286	26366.66943
0.09	93175.60478	18914.55650	13771.14459	26532.16795
0.1	92979.85129	18851.28490	13800.63001	26696.51482

Table 3b: Solution of the Tuberculosis Model Using HPM

HPM			
S(t)	I(t)	R(t)	L(t)
95000	19500	13500	25000
94791.54439	19433.51028	13530.64975	25175.07840
94584.28355	19367.40452	13561.16619	25348.91161
94378.21748	19301.68272	13591.54932	25521.49963
94173.34618	19236.34489	13621.79915	25692.84245
93969.66966	19171.39101	13651.91567	25862.94007
93767.18791	19106.82109	13681.89888	26031.79251
93565.90093	19042.63514	13711.74879	26199.39975
93365.80873	18978.83314	13741.46538	26365.76179
93166.91130	18915.41511	13771.04868	26530.87864
92969.20864	18852.38104	13800.49866	26694.75030

Table 3c: Solution of the Tuberculosis Model Using VIM

t	VIM			
	S(t)	I(t)	R(t)	L(t)
0	95000	19500	13500	25000
0.01	94791.55697	19433.50261	13530.64988	25175.08014
0.02	94584.32723	19367.37235	13561.16726	25348.92543
0.03	94378.30081	19301.60696	13591.55291	25521.54614
0.04	94173.46787	19236.20420	13621.80764	25692.95242
0.05	93969.81868	19171.16188	13651.93223	25863.15431
0.06	93767.34359	19106.47778	13681.92744	26032.16178
0.07	93566.03307	19042.14971	13711.79406	26199.98464
0.08	93365.87765	18978.17551	13741.53285	26366.63266
0.09	93166.86801	18914.55305	13771.14457	26532.11546
0.1	92968.99484	18851.28015	13800.62997	26696.44263

REFERENCES

[1] Ojaswita, C., Tiny, M., Shedden, M. A. (2014). A Continuous Mathematical Model for Shigella Outbreaks. American Journal of Biomedical Engineering, 4(1): 10-16.

[2] Namawejje, H. (2011). Modelling the Effect of Stress on the Dynamics and Treatment of Tuberculosis, M.Sc. Dissertation, University of Dar se Salaam, Tanzania.

[3] Zhou, J. K. (1986). Differential Transformation and its Applications for Electrical Circuits. Huazhong University Press, Wuhan, China.

[4] Batiha, B. (2015). The Solution of the Prey and Predator Problem by Differential Transform Method. International Journal of Basic and Applied Sciences, 4 (1): 36-43.

[5] Paripour, M., karimi, L. and Abbasbandy, S. (2017). Differential Transform Method for Volterra's Population Growth Model. 24(1): 227-234.

[6] Mehmet, M., Ahmet, G. and Vedat, S. E. (2011). On Solving Coullet System by Differential Transform Method. *Cankaya University Journal of Science and Engineering*, 8(1): 111-121.

[7] Akinboro, F. S., Alao, S., Akinpelu, F. O. (2014). Numerical Solution of SIR Model using Differential Transform Method and Variational Iteration Method. General Mathematics Notes, 22(2): 82-92.

[8] Suayip, Y. and Nurbol I. (2014). Solving Systems of Volterra Integral and Integro-differential Equations with Proportional Delays by Differential Transform Method. doi:10.1155/2014/725648.

[9] He, J. H. (1999). Homotopy Perturbation Technique. Computer Methods in Applied Mechanics and Engineering, 178: 257-262.

[10] He J.H. (2006). New Interpretation of Homotopy Perturbation Method. International Journal of Modern Physics B, 20, 2561-2568.

[11] He, J. H. (2000). A Coupling Method of Homotopy Technique and Perturbation Technique for non-linear Problems. International Journal of Non-linear Mechanics, 35: 37-43

[12] He, J. H. (2006). Some Asymptotic Methods for Strongly Non-linear Equations. International Journal of Modern Physics B, 20: 1141-1199.

[13] He, J. H. (2000). New Perturbation Technique which is valid for Large Parameters. Journal of Sound and Vibration, 2229: 1257-1263.

[14] He, J. H. (2004). Asymptotology by Homotopy Perturbation Method. Appl. Math. Comput. 156 (3): 591-596.

[15] He, J. H. (2004). The Homotopy Perturbation Method for Nonlinear Oscillators with Discontinuities. Appl. Math. Comput. 151: 287-292.

[16] He, J. H. (2005). Application of Homotopy Perturbation Method to Nonlinear Wave Equations. Chaos Solitons Fractals, 26: 695-700.

[17] He, J. H. (2005). Homotopy Perturbation Method for Bifurcation on Nonlinear Problems. Int. Journal of Non-linear Sci. Numer. Simul. 8: 207-218.

[18] Ganji, D. D., Rajabi, A. (2006). Assessment of Homotopy -Perturbation and Perturbation Methods in Heat Radiation Equations. Int. Commun. Heat Mass Transfer, 33: 391-400.

[19] Abubakar, S., Akinwande, N. I., Jimoh, O. R., Oguntolu, F. A., Ogwumu, O. D. (2013). Approximate Solution of SIR Infectious Disease Model using Homotopy Perturbation Method. Pacific Journal of Science and Technology, 14 (2): 163-169.

[20] He, J. H. (1999). A New Approach to Nonlinear Partial Differential Equations. Commun. Nonlin. Sci. Numer. Simul, 2: 230-235.

[21] Inokuti, M., Sekine H., Mura T. (1978). General Use of Lagrange Multiplier in Nonlinear Mathematical Physics. In: Nemat – Nassed S, editor. Varaitional Method in Mechanics of Solids. Pergamon Press.

[22] He, J. H. (1999). Variational Iteration Method – a kind of nonlinear analytical technique: some examples. Int. J. Nonlin. Mech. 34: 699-708.

[23] He, J. H. (2000). Variational Iteration Method for Autonomous Ordinary Differential Systems. Applied Math. Comput, 114: 115-123.

[24] He, J. H. (1998). Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media. Comput. Meth. Appl. Mech. Engrg. 167: 57-68.

[25] Soliman, A. A. (2006). A Numerical Simulation and Explicit Solution of KdV-Burgers' and Lax's seventh-order KdV Equations. Chaos Solitons Fractals, 29(2): 294-302.

[26] Momani, S., Abuasad, S. (2006). Application of He's Variational Iteration Method to Helmholtz Equation . chaos Solitons Fractals, 27: 1119-1123.

[27] Odibat, Z. M., Momani, S. (2006). Application of Variational Iteration Method to Nonlinear Differential Equations of Fractional Order. Int. J. Nonlin. Sci. Numer. Simul., 7: 27-34.

[28] Bildik, N., Konuralp, A. (2006). The Use of Variational Iteration Method, Differential Transform Method and Adomian Decomposition Method for Solving Different Types of Nonlinear Partial Differential Equations. Int. J. Nolin. Sci. Numer. Simul. 7: 65-70.

[29] Abbasbandy, S. (2007). A New Application of He's Variational Iteration Method for Quadratic Riccati Differential Equation by Using Adomian Polynomials. J. Comput. Appl. Math. 207(1): 59-63.
[30] Jufeng, L. (2007). Variational Iteration Method for Solving two-point Boundary Value Problems. J. Comput. Appl. Math. 207(1): 92-95.
[31] Batiha, K. and Batiha, B. (2011). A New Algorithm for Solving Linear ODEs. World Applied Sciences Journal, 15 (12): 1774-1779.