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NUMERICAL SOLUTION TO SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION BY ITERATIVE DECOMPOSITION APPROXIMATION TECHNIQUE.

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Abstract

In this paper, we consider finding appropriate solution to nonlinear initial/boundary value problems. The numerical algorithm based on the iterative decomposition technique is applied to obtain analytic and approximate solution of such differential equation. No linearization nor perturbation is involved in obtaining the components of the power series solution that converges rapidly. Numerical examples are presented to elucidate the suitability, accuracy and efficiency of the new scheme

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Keywords: Differential equation, Iterative decomposition, Approximation, Accuracy.

1.0 Introduction

Second order nonlinear differential equations are often used to model real life phenomena in scientific and technological problems. Solutions to such models have been obtained by several authors [1-6] who used multi- step methods based on collocation and interpolation with different kinds of basis function , Tatari and Deghan [7] used Adomian method, Geng [8] used the approach of reproducing kernel. This paper focuses on differential equation of the form

$$y''(t) = f(t, y, y'),\ a \le t \le b,\ a = t_0 \prec t_1 \prec \ldots \prec t_{n-1} = b \quad (1)$$

$$y(t_0) = \alpha \quad (2)$$

$$y'(t_0) = \beta, \qquad \alpha, \beta\ \varepsilon\ \Re,\ h = \frac{b-a}{n},\ n = 1,2,3,\ldots \quad (3)$$

The existence and uniqueness of solution of equation (1) is ascertained according to Henrici [9].This class of problems is highly important for their application in biological sciences and control theory. Active research work is ongoing in this area with a number of numerical methods of solution developed [1-8], [10]. However, the decomposition methods are presently receiving more attention as efficient techniques for the solution of linear and nonlinear ordinary, partial, integral, integro-differential, deterministic or stochastic differential equations. [11-15]. These methods have been found to converge rapidly to the exact solution.

In this paper, a new class of the decomposition method is applied which offers further insight into convergence and, minimizes the already reduced volume of calculations introduced by the Adomian's method without jettisoning its accuracy and efficiency.

2. The Method

Consider a second order nonlinear initial value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) + N(y) = g(x) \quad (3)$$

$$y(x_0) = \alpha \quad (4)$$

$$y'(x_0) = \beta \quad (5)$$

where α and β are constants, $N(y)$ is a nonlinear term and $g(x)$ is the source term.

Equation (3) can be expressed in canonical form as

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$$Ly = g(x) - N(y) - p(x)y' - q(x)y \qquad (6)$$

in which the differential operator L is define as

$$L = \frac{d^2}{dx^2}(.) \qquad (7)$$

Thus, L is invertible, L^{-1} exist and is a two-fold definite integral from 0 to x.

That is, $L^{-1}(.) = \int_0^x \int_0^x (.)dsds = \int_0^x (x-t)N(y(t))dt$ (8)

Now, applying the inverse operator L^{-1} on equation (6) and using the initial conditions of equations (4-5) gives

$$y(x) = \alpha + \beta - L^{-1}g(x) - L^{-1}p(x)y' - \int_0^x (x-t)N(y(t))dt \qquad (9)$$

The iterative decomposition method assumes that the unknown function can be expressed in terms of an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (10)$$

So, that the component $y_n(x)$ can be determined iteratively. We also, observe and see that equation (9) is of the form [12]

$$y = N(y) + f \qquad (11)$$

where f is a constant and $N(y)$ is the nonlinear term

We split the nonlinear term as

$$\sum_{i=0}^{\infty} y_i = L^{-1}(y_0) + \sum_{j=0}^{\infty}\left\{L^{-1}\left(\sum_{i=0}^{\infty} y_i\right) - L^{-1}\left(\sum_{i=0}^{n-1} y_i\right)\right\} \qquad (12)$$

and substitute equations (9) and (12) into equation (11). We have

$$\sum_{n=0}^{\infty} y_n = f + L^{-1}(y_0) + \sum_{i=0}^{\infty}\left\{L^{-1}\left(\sum_{i=0}^{n} y_i\right) - L^{-1}\left(\sum_{i=0}^{n-1} y_i\right)\right\} \qquad (13)$$

which yields the recurrence scheme

$$\begin{cases} y_0 = f = \alpha + \beta + L^{-1}g(x) \\ y_1 = L^{-1}(y_0) \\ y_{n+1} = L^{-1}(y_0 + y_1 + ... + y_n) - L^{-1}(y_0 + y_1 + ... + y_{n-1}),\ n \geq 1 \end{cases} \qquad (14)$$

Finally, the n-th term approximation solution for equation (3) by the iterative decomposition technique is given by $y_n = \sum_{k=o}^{n-1} y_k$ and $y(x) = \lim_{n\to\infty} y_n$ as the exact solution in closed form or approximate solution.

3.0 Numerical Experiment

To give a clear exposition of the concept in this paper and further illustration of the above technique, we consider the following examples. In all cases considered, whenever the exact solutions are known, we defined the absolute error as

$$e_j = \max_{0\leq x\leq 1}|y(x) - y_n(x)|, \qquad j = 1,2...,N$$

Example 1.

Consider the problem of second order nonlinear homogeneous differential equation

$$y''(t) - t(y'(t))^2 = 0, \quad 0 \leq t \leq 1$$

$$y(0) = 1, \quad y'(0) = \frac{1}{2}, \quad h = \frac{1}{40}$$

The exact solution is

$$y(t)=\frac{1}{2}\ln\left(\frac{2+t}{2-t}\right)$$

Applying the iterative decomposition algorithm (14) to the above example, we get

$$y_0=1+\frac{1}{2}t$$

$$y_1=\frac{1}{24}t^3$$

$$y_2=\frac{1}{160}t^5+\frac{1}{2688}t^7$$

$$y_3=\frac{1}{1344}t^7+\frac{1}{6912}t^9+\frac{1}{67584}t^{11}+\frac{1}{958464}t^{13}+\frac{1}{30965760}t^{15}$$

Hence, the series solution is

$$y(t)\approx\sum_{k=0}^{3}y_k=1+\frac{1}{2}t+\frac{1}{24}t^3+\frac{1}{160}t^5+\frac{1}{896}t^7+\frac{1}{6912}t^9+\frac{1}{67584}t^{11}+\frac{1}{958464}t^{13}+\frac{1}{30965760}t^{15}+o(t^{16})$$

Table 1, the exact solution and corresponding absolute error for example 1

t	$y(t)-Exact$	$y(t)-computed$	$Error$
0.0025	1.001250001	1.001250001	0
0.0050	1.002500005	1.002500005	0
0.0075	1.003750018	1.003750018	0
0.0100	1.005000042	1.005000042	0
0.0125	1.006250081	1.006250081	0
0.0150	1.007500141	1.007500141	0
0.0175	1.008750223	1.008750223	0
0.0200	1.010000333	1.010000333	0
0.0225	1.011250475	1.011250475	0
0.0250	1.0125000651	1.0125000651	0

The comparison of the exact solution with the series solution is presented in
Table 1.The new scheme gives a better result than the results obtained by the 2-Step block method proposed by Adesanya et al [1] while our result coincides with MADM[15],but with fewer iterations.

Example 2

We consider the nonlinear oscillatory problem

$$u''(x)-u(x)+u^2(x)+(u'(x))^2-1=0, \qquad 0\le x\le 1$$

$$u(0)=2, \qquad u'(0)=0$$

The theoretical solution is $u(x)=1+\cos x$

Using the iterative scheme (14) in solving example two yields,

$$u_0=2+\frac{1}{2}x^2$$

$$u_1=-x^2+\frac{1}{24}x^4-\frac{1}{120}x^6$$

$$u_2=\frac{1}{144}x^6-\frac{23}{10080}x^8+\frac{67}{259200}x^{10}-\frac{23}{950400}x^{12}-\frac{1}{2620800}x^{14}$$

$$u(x) \approx 2 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{23}{10080}x^8 + \frac{67}{259200}x^{10} - \frac{23}{950400}x^{12} - \frac{1}{2620800}x^{14} + \ldots$$

The comparison of the exact solution with the approximate series using recursive algorithm [14] is shown in Table 2. The observed errors are quite negligible.

Table 2, the exact solution and corresponding absolute error for example 2

x	$u - Exact$	$u - computed$	Error
0.1	1.995004165	1.995004165	0
0.2	1.980066578	1.980066561	1.700E-09
0.3	1.955336489	1.955336037	4.520E-08
0.4	1.921060994	1.921056380	4.614E-07
0.5	1.877582562	1.877554254	2.831E-06
0.6	1.825335615	1.825209850	1.263E-05
0.7	1.764842187	1.764388610	4.536E-05
0.8	1.696706709	1.695310184	1.397E-05
0.9	1.621609968	1.617768156	3.842E-04
1.0	1.540302306	1.530591249	9.331E-04

Example 3

We seek for the solution of the three point second order nonlinear boundary value problem

$$y''(x) + \frac{3}{8}y(x) + \frac{2}{1089}(y'(x))^2 + 1 = 0 \ , \qquad 0 \le x \le 1$$

subject to the boundary conditions $y(0)$, $y(1/3) = y(1)$.

This problem in operator form becomes

$$y(x) = y(0) + xy'(x) - L^{-1}(1) - \frac{3}{8}L^{-1}y(x) - \frac{2}{1089}L^{-1}N(y'(x))^2$$

Applying the proposed method yields

$$y_0 = \alpha t - \frac{1}{2}t^2,$$

$$y_1 = \frac{-1}{16}\alpha t^3 + \frac{1}{64}t^4 - \frac{1}{6534}(\alpha - t)^4 ,$$

$$y_2 = \frac{-2030933}{10929447936}t^6 - \frac{10465225}{8331518616128}t^8 + \frac{2030933}{1821574658}\alpha t^5 + \frac{10465225}{10414397702016}\alpha t^7 - \frac{6237727}{2975542200576}\alpha^2 t^6$$
$$- \frac{1}{87120}\alpha^5 t + \frac{3139}{113848116}\alpha^4 t^2 - \frac{3139}{85}$$

So, $y(x) \approx y_0 + y_1 + y_2$ and imposing the boundary conditions $y(\frac{1}{3}) = y(1)$. Solving ,we get $\alpha = 0.7065$ which is substituted into $y(x)$ to obtain the approximate solution

$$y(x) \approx \frac{23790517}{100000000000000} + \frac{1766244949}{2500000000}x - \frac{5001511799}{10000000000}x^2 - \frac{437536707}{10000000000}x^3 + \frac{775718599}{50000000000}x^4$$
$$+ \frac{7876753571}{10000000000000}x^5 - \frac{2030933}{10929447936}x^6 - \frac{10165225}{83315181616128}x^8$$

Table 3 comparison of results from other methods and the new scheme

x	Successive iteration method[16]	Adomian [7]	Reproducing kernel method[8]	$y-computed$
o.1	0.0656	0.0656	0.0656	0.06560332653
0.2	0.1211	0.1209	0.1209	0.12093680840
0.3	0.1661	0.1658	0.1658	0.16583509450
0.4	0.2004	0.2001	0.2001	0.20013143050
0.5	0.2240	0.2236	0.2236	0.22360385080
0.6	0.2367	0.2363	0.2363	0.22363894040
0.7	0.2385	0.2382	0.2382	0.23815560570
0.8	02295	0.2291	0.2291	0.22907165420
0.9	0.2095	0.2091	0.2092	0.20913158120

It must be noted that the results from [7, 8, 16] were all approximated to four decimal places. This is in tandem with the proposed new scheme.

4.0 Conclusion

A numerical scheme of high accuracy has been proposed for the numerical solution of nonlinear second order initial/boundary value problems. In this paper, the series function representing the approximate solution proves to be a good estimate of the exact solution for the test examples .This suggests wider application of the method for more complicated problems of differential equations of higher order since the method is implemented with less stress computer coding which makes it cheaper and cost effective in implementation. The fact that nonlinear problems are solved without linearization, perturbation and utilization of special polynomial of any kind is an added advantage.

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