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THE EFFECTS OF WAVE BREAKING IN FAR FIELD OCEAN WATER FLOW AND THEIR GROUP VELOCITY

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Abstract

The paper presents the effects of wave breaking in the far field ocean water by wind generated from a given source with irregular heights and periods where the depth of water is considered deep and gravitational force important. We take into consideration Cariolis force and the Reynolds number from Laminar to turbulence typical of Rogue wave of the surface wind impact on Upper Ocean dynamic energy fluxes across boundary layers simultaneously interacting.

Dissipation of energy waves lost in the forms of white capping, depth induced wave breaking, bottom friction and, wave-wave interactions are analyzed. Associated equations for the wind growths for both linear and exponential forms are given. It is established that wave variance densities for both potential and kinetic energies with their wave energy modulations are dependent on the wave height and water depth. The cross correlation function for the stationary ergodic real valued process for the spectral density functions and their auto correlation spectral density functions for the wave horizontal velocities inform of Fourier transform for the wave velocity and acceleration transfer functions are discussed.

The regularity theory of energy minimizing harmonic maps into Riemannian manifolds in the sense of Schoelen and Uhlenbeck is introduced. The $2D$ Burgers equation for the Stokes waves is stated under special conditions. The eigenvalue bounds and backward stability for the flow of waves mechanism formed the peak of discussion for the pressure matrix based on pre-conditioned iterative solvers for the incompressible flow of the Navier –Stokes equation.

***Keywords*:** wave breaking, turbulence, spectral density functions, Navier –Stokes equations, eigenvalue problem

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1.0 INTRODUCTION

This paper aims at giving underlying understanding surface wind impact on upper ocean dynamics and the energy flux modulations across boundary layers. We give the mechanisms with which waves are created on and within the body of Ocean water and work done per unit area. For a single wave process the auto correlation and auto spectral density functions are respectively defined in the form of Fourier series embedding wave frequency and the acceleration transfer functions. The probabilistic nonlinear wave density function for the instantaneous force is expressed. The pressure matrix from the ergodic non stationary 2D Burgers' equation is presented where the spectral radius is important with their Riemannian manifolds.

The paper is motivated by effects of wind generated waves in the far field ocean water where depth of the water is considered deep and gravitational force important. The waves are generated from a source when wind blows across water surface for a long period of time with irregular heights and periods due to irregular nature of wind. The generation of the surface elevation of ocean water is a sum of harmonic waves obtained at different times and locations and is presumed statistically independent of their sources of origin.

We define deep water ocean waves as waves which are unaffected by the ocean bottom [1, 2] and cited references therein.

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‘’When air first flows in contact with the water, it creates a kind of ripples called capillary waves. The ocean surface [4] is then roughened and a mechanism of transfer of energy from the wind to the water is then made where gravitational attraction is expressed through still water’’.

Firstly, we give overviews of an ocean (e.g., Atlantic ocean). For instance, it has an average depth with its seas of 10, 925 feet (3,330 meters) in Lagos, Nigeria, while it has a maximum depth of 27,493 feet (8,380 meters) in Pueto Rico trench, north of the Island of Pueto Rico. The area of the Atlantic Ocean without its dependent seas is approximately 31,830, 000 square miles (82,440,000 square kilometers). When square kilometers of the seas are added, it has a total area of 41,100,000 square miles (106,460,000 square kilometers).

Using above information therefore, factors that do make up components of the ocean wave are the length, height, period and speed. The size of the wind produced is determined by the wind speed, wind duration and fetch within limited period. The primary objectives in the paper are to know information on ocean water waves based on impulse, potential energy, kinetic energy , mean square of bed, velocity, radiation stress , wave power, momentum flux and pressure equations.

1.1 PRELIMINARIES /LITERATURE REVIEWS

The relevant equations for the wind profiles are hereby presented. The equations for the wind speed, wind growths, steepness in wave front, wind skin friction velocity, wind drag coefficient and Pierson-Moskowitz spectrum are discussed.

As an illustration for this purpose when the wind blows, some energy is thereby transmitted to the water through the atmospheric air. However, during this period of energy transmission from the air to the water, some dissipation of energy takes place. The following ways [7,8,9, 10,12] are the means by which energy is lost via:

(i) generation by wind;

(ii) dissipation by white capping; depth-induced wave breaking; by bottom friction; and wave- wave interactions in both deep and shallow water.

The one point function for the sea surface elevation is described by the equation

$$\eta(t)=\sum_i a_i \cos(\delta_i t-\alpha_i) \qquad (1.1.1)$$

In equation (1.1.1) above, η is the surface elevation of the undisturbed surface of the ocean water, α is the amplitude of ith wave component, δ_i is the relative radian or circular frequency of the ith wave component in the presence of ambient current. The absolute radian frequency which is the Doppler shift, is defined in the form

$$\omega=\delta+\vec{k}.\vec{u} \qquad (1.1.2)$$

For instance, in equation (1.1.2), ω is the sum of relative frequency δ ,and the product of their ambient current velocity vectors $\vec{k}.\vec{u}$ and wave number.

We give equation for the wave speed in deep ocean water of depth h due to gravity in the form:

$$\omega^2=gk\tanh(kh) \qquad (1.1.3)$$

Let us take notice that Newton iteration method described in [3,18] can be used to refine an approximate value of ω^2 .

Wave breaking starts when the ratio of wave height over water depth is greater than a certain limit where dissipating energy acts rapidly.

The steepness in wave front [7] which controls the white capping expressed in the form:

$$S_{ds,w}(\delta,\theta)=-\gamma\bar{\delta}\frac{k}{\bar{k}}E(\delta,\theta) \quad , \qquad (1.1.4)$$

where, γ is the steepness dependent coefficient, κ is the wave number, θ is the wind direction, $\bar{\delta}$ and $\bar{k}$ are respectively the mean frequency and mean wave number.

Categorization of wave growth [12] by wind consisting of linear growth and exponential growth is

$$S_{in}=A+BE(\sigma,\theta) \qquad (1.1.5)$$

with the wind spread measured at $10m$ elevation U_{10} above sea level is highlighted. We then express [10,12] the wind skin friction velocity U_* in the form:

$$U_*^2=C_D U_{10}^2 \quad , \qquad (1.1.6)$$

where, C_D is the drag coefficient and can be obtained [12] in the form:

$$C_D(U_{10}) = \begin{cases} 1.2875\times 10^{-3}, & for \quad U_{10} < 7.5m/s \\ (0.8 + 0.065s/m\times U_{10})\times 10^{-3}, & for \quad U_{10} \geq 7.5m/s \end{cases} \quad (1.1.7)$$

Thus on a given day the drag wind coefficient of the ocean surface is sea state dependent. Hence wind stress measurements collected from ocean which may be used for investigation of dependence of drag coefficient and dynamic roughness of the ocean surface at different times and places in the ocean is a worth undertaking as a major task for enhancing oceanic activities on a large scale.

The practical purpose is to calculate the C_D and wind stress at each interval of time, say 12-hourly and then form monthly mean records of C_D and wind stress for each year. Proper investigation of studies may be carried out for over a period of twenty years or more, say from 2001 to 2021.

With this, one is able to compute the ratio of wind skin friction velocity U_* to that of phase speed c_{ρ_h} with high certainty of confidence.

The factor A in the linear growth expressed in equation (1.1.5) is given in the form

$$A = \frac{1.2\times 10^{-3}}{2\pi g^2}\left(U_* \max[0, \cos(\theta - \theta_w)]\right)^4 H, \quad (1.1.8)$$

where, ($H = \exp\left(-\left(\frac{\sigma}{\sigma^*_{pm}}\right)^{-4}\right), \sigma^*_{pm} = \frac{0.13g}{28U_*}2\pi$) (1.1.9)

In equation (1.1.8) the term θ_w is the wind direction, H is the filter and σ^*_{pm} is the peak frequency for the fully developed sea state [12].

The equation for the wind exponential growth [12] is expressed as:

$$\hat{B} = \beta\frac{\rho_a}{\rho_w}\left(\frac{U_*}{c_{\rho_h}}\right)^2 \max[0, \ \cos(\theta - \theta_w)]^2, \quad (1.1.10)$$

where, β is the Miles constant [5,6]. The Miles constant is calculated in the form:

$$\beta = \frac{1.2}{\kappa^2}\nu \ln^4(\nu), \qquad \nu \leq 1$$
$$\nu = \frac{gz_e}{c^2_{\rho_h}}e^r, \quad r = \frac{\kappa c}{|\, U_* \cos(\theta - \theta_w)\,|} \quad (1.1.11)$$

In equation (1.1.11) the term $\kappa = 0.41$ is the von Karman constant and z_e being the effective surface roughness.

Summing up, the wind profile could be described as given below

$$U(z) = \frac{U_*}{\kappa}\ln\left(\frac{z + z_e - z_0}{z_e}\right) \quad (1.1.12)$$

The total surface of wind [5,6 , 7] stress relative to the density of the air ρ_a is given by

$$\vec{\tau} = \rho_a |\vec{U}|\vec{U} \quad (1.1.13)$$

Wherefrom, the term z_e appearing in equation is**:**

$$z_e = \frac{z_0}{\sqrt{1 - \frac{\vec{r}_w}{|r|}}}, \qquad \text{(where, } z_0 = \vec{\alpha}\frac{U_*^2}{g}\text{)} \quad (1.1.14)$$

In equation (1.1.14), the value of α is taken as 0.01. Note that the value of the steepness γ appearing in equation (1.1.4) is obtained in the form

$$\gamma = C_{ds}\left((1-\delta)+\delta\frac{k}{\vec{k}}\right)\left(\frac{\bar{S}}{S_{pm}}\right)^{p} \qquad (1.1.15)$$

The parameters C_{ds}, δ and p are [5] the tunable coefficients and δ may be taken as 0.

The term $\bar{S}$ is the overall steepness of the wave whilst $\bar{S}_{pm}$ being the value of $\bar{S}$ for the Pierson-Moskowitz spectrum

$$\bar{S}_{pm} = \sqrt{3.02\times 10^{-3}} = \bar{k}\sqrt{E_{tol}} \qquad (1.1.13)$$

The rest parts in the papers are arranged as follows. Section2 gives materials and methods adopted .The values for mean frequency $\bar{\delta}$, mean wave number $\bar{k}$ and total wave energy E_{tol} are defined using relevant materials in the existing literatures. Based on the foregoing, the aim is to provide the numerical formulae for the mathematical models on the spectral short crested wind generated waves in the coastal regions and their group velocity. The Euler-Stokes equations as well as the accompanying auto-correlation and auto spectral variance density are discussed. It is established that the Cross correlation function for the stationary ergodic real valued processes $X(t)$ and $Y(t)$ in the directions of the wave profiles is the main statistical tool for analysis of wave variance density. We adopt the regularity theory for the energy minimizing harmonic maps into Riemannian manifolds. The strong differentiability of the function F implies the continuity of the Jacobian matrices with respect to the geodesic ball. We motivate this section principally with the work of [11. In section 3, the backward stability for linear system of equation arising from the discretization of Euler-Burgers equation in 2D is presented and the eigenvalue bounds are given utilizing the Weyl's theory in the sense of [2]. We give equations for encompassing square root of a diagonalizable matrix which may be useful in control theory in manner similar to [13,14].It is suggested that a modified fast linear solver such as the SOR method (Succesive Overelaxation method) could be used for the resulting system of equations.

2.0 MATERIALS AND METHODS

The values for mean frequency $\bar{\delta}$, mean wave number $\bar{k}$, and the total wave energy E_{tol} are defined by the equations

$$\bar{\delta} = \left(E_{tol}^{-1}\int_0^{2\pi}\int_0^{\infty}\frac{1}{\delta}E(\delta,\theta)d\delta d\theta\right)^{-1} \qquad (2.1.1)$$

$$\bar{k} = \left(E_{tol}^{-1}\int_0^{2\pi}\int_0^{\infty}\frac{1}{\sqrt{k}}E(\delta,\theta)d\delta d\theta\right)^{-1} \qquad (2.1.2)$$

$$\bar{E}_{tol} = \frac{1}{2}\rho_w g a^2 \quad (\text{ where, } a=\eta\ ) \qquad (2.1.3)$$

where the total wave energy for the dynamical wave pattern is

$$E_{tol} = \int_0^{2\pi}\int_0^{\infty}E(\delta,\theta)d\delta\, d\theta \qquad (2.1.4)$$

The equation expressing saturation point for the white capping is

$$S_{ds,w}(\delta,\theta) = -C_{ds}\left(\frac{B(k)}{B_k}\right)^{\frac{p}{2}}\tan(kh)^{\frac{2-p_0}{4}}\sqrt{gkE(\delta,\theta)} \qquad (2.1.5)$$

where,

$$B_k = \int_0^{2\pi}c_g k^3 E(\delta,\theta)d\theta \quad \text{(threshold saturation point)} \qquad (2.1.6)$$

$$p = \frac{p_0}{2}+\frac{p_0}{2}\tanh\left(10\left(\sqrt{\frac{B(k)}{B_0}}\right)-1\right). \qquad (2.1.7)$$

Waves start breaking when, $B(k) > B_k$.
The bottom friction equation for the ocean water [10] is expressed as:

$$S_{ds,b} = -C_b \frac{\delta^2}{g^2 \sinh^2(kh)} E(\delta,\theta) \quad (2.1.8)$$

The $C_b\ (= 0.067 m^2 s^{-2})$ is the bottom friction coefficient [8] independent of the bottom orbital motion

$$U_{rms}^2 = \int_0^{2\pi}\int_0^{\infty} \frac{\delta^2}{g^2 \sinh^2(kh)} E(\delta,\theta) d\delta d\theta \quad (2.1.9)$$

It follows from the fore goings that in a fully developed wave conditions in shallow water, the logarithm wind profile is the equation

$$\frac{1}{\sqrt{C_z}} = \frac{1}{k}\ln\frac{z}{z_0} \quad (2.1.10)$$

Where for instance, C_z is the wind drag coefficient in the reference point of wind speed at the elevation z for $k = 0.4$, the Karman constant. Strong wind forcing takes place when $\frac{U_*}{c_{\rho w}} > 0.1$.

2.2 THE MANIFOLDS FOR HARMONIC MAP

To begin with, we adopt the regularity theory for the energy minimizing harmonic maps into Riemannian manifolds. The strong differentiability of the function F implies the continuity of the Jacobian matrices with respect to the geodesic ball. We motivate this section as follows principally with the work of [11]:
Let $f: M^n \to N^m$ be a map between Riemannian manifolds of dimension n and m where M is compact and N is an open manifold. For $n=2$ the energy minimizing harmonic map is Holder continuous and smooth if M and N are smooth. Let $C^r(M,N)$ be the space of maps $f: M \to N$ which possesses continuous r- derivatives wherefrom, $C^r(M,N) \subseteq C^r(M,R^k)$ is a Banach space sub manifold [11].We also let $C^{r,\alpha}(M,N)$, ($\alpha \in (0,1]$) be the subset of $C^r(M,N)$ whose rth derivatives are Holder continuous with exponent α .The separable Hilbert space $L^2(M,R^k)$ will be the set of maps $f: M \to R^k$ whose component function have first derivatives in L^2. Notably, by $L_{1,0}^2(M,R^k)$ we mean those L_1^2 maps which are zero on the boundary ∂M .
Further assumption that $L_1^2(M,N) = \{f \in L_1^2(M,R^k) : f(x) \in N, a.e \ \ x \in M\}$ given that dim $M = 1$, then $L_1^2(M,N)$ is a Hilbert submanifold on $L_1^2(M,R^k)$ which needs not be so for dim $M > 1$.
It us note that the set $L_1^2(M,N)$ inherits strong and weak star topologies from $L_1^2(M.R^k)$.
As a result, the energy functional is expressed as

$$E(f) = \int_M \langle du(x), du(x)\rangle dV = \int_M e(u) \quad (2.2.1)$$

Where for instance, the Lagrangian $e(u)$ is given in local coordinates in the form:

$$e(u) = \sum_{\alpha,\beta}\sum_i g^{\alpha\beta} \frac{\partial u^i \partial u^i}{\partial x^\beta \partial x^\alpha} (\det g_{\eta\delta})^{\frac{1}{2}} dx \quad (2.2.2)$$

and $g_{\alpha\beta}$ is the metric tensor of M .
The energy norm is then given by

$$\|u\|_{1,2}^2 = E(u) + \int_M \sum_i (u^i(x))^2 dV, \quad (2.2.3)$$

wherefrom, dV is the volume element of M .
With these expositions therefore, it follows that a harmonic map is a solution to the Euler –Lagrange equation for E in $L_1^2(M,N)$.

The geodesic ball $B_\beta(a)$ about $a \in M$ is then the open or closed sphere in the topology defined where u is Holder continuous. In this case therefore a blow up harmonic map at the point vector $x^{(0)}$ is of particular interest for a $u^{(0)}$ minimizing tangent map.

2.3 THE EULER-STOKES- BURGER EQUATIONS

As said earlier, higher steepness of wave implies ratio of wave height to wave length with higher water particle velocities. In line with this, we proceed by giving equations of wave's motion. The relevant equations in the existing Literature [17,18] are hereby reviewed:

Equation of Euler:

$$\left.\begin{aligned}\frac{Du}{Dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial t}\\ \frac{Dv}{Dt} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y}\\ \frac{Dw}{Dt} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \qquad (2.3.1)$$

where,

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}, \qquad (2.3.2)$$

In equation (2.3.1), the parameters appearing denoted as ρ = mass density of fluid, X, Y, Z = abstract body force per unit mass along x, y, z directions.

By adding the kinematic viscosity to each of equations in equation (2.3.1), the Navier Stokes equations are in the form:

$$\left.\begin{aligned}\frac{Du}{Dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial t} + \nu\nabla^2 u\\ \frac{Dv}{Dt} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v\\ \frac{Dw}{Dt} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w\end{aligned}\right\} \qquad (2.3.3)$$

Where,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (2.3.4)$$

denotes the Laplace equation. For irrotationality, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$.

Alternatively we rewrite the term $\frac{u\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$, so that,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) \qquad (2.3.5)$$

The expression for potential function ϕ is given in the form:

$$\phi = \frac{gh\cosh(k(d+z))}{2\omega\cosh(kd)}\sin(kx - \omega t) \qquad (2.3.6)$$

But it is known that $\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g\eta = 0$, since at the free surface the pressure is zero , hence we have

$$g\eta + \frac{\partial \phi}{\partial t} = 0,$$

giving

$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}\Big|_{\eta = z}$. Further, at the free surface, $z = 0, \eta = z$ and

$$\frac{\partial \eta}{\partial t} \approx \frac{\partial \phi}{\partial z}\Big|_{z=0=\eta} \quad (2.3.7)$$

Using this, it follows that

$$-\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial z^2} \Rightarrow \frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2.3.8)$$

This has the solution in the form

$$\phi(x, y, z, t) = X(x).Z(z).T(t) = XZT \quad (2.3.9)$$

From the Laplace equation there follows immediately

$$\frac{\partial^2 X}{\partial x^2}(ZT) + \frac{\partial^2 Z}{\partial z^2}(XT) = 0 \quad (2.3.10)$$

We divide through by (XZT) to have

$$\frac{\partial^2 X}{\partial x^2}\left(\frac{1}{X}\right) + \frac{\partial^2 Z}{\partial^2 z}\left(\frac{1}{Z}\right) = 0 \quad (2.3.11)$$

Setting as

$$\frac{\partial^2 X}{\partial x^2}\left(\frac{1}{X}\right) = -k^2, \quad (2.3.12)$$

and

$$\frac{\partial^2 Z}{\partial z^2}\left(\frac{1}{Z}\right) = k^2 \quad (2.3.13)$$

The equation for pressure below the sea surface is written in the form

$$P = -\rho g z - \rho \frac{\partial \phi}{dt} = \quad . \quad (2.3.14)$$

$$P = -\rho g z + \rho g \eta \frac{\cosh k(d+z))}{\cosh(kd)}$$

The work done in unit time or energy carried across unit width of a section is

$$W = v\delta P = \int_{-h}^{0} \delta P \frac{\partial \phi}{\partial x} dz = \rho \int_{-h}^{0} \left(\frac{\partial \phi}{\partial x}\right)^2 dz$$

$$= \frac{g^2 \rho a^2}{\omega} k \frac{\sin^2(kx-\omega t)}{\cosh^2 kh} \int_{-h}^{0} \cosh^2 k(z+h) dz = \frac{g^2 \rho a^2 k}{\omega} \frac{\sin^2(kx-\omega t)}{\cosh^2 kh}\left(\frac{\sin 2kh}{4\omega} - \frac{h}{2}\right)$$

Because $\omega^2 = gk \tanh kh$, we then have that

$W = \frac{1}{2}\frac{g\rho a^2 k}{\omega}(1 + 2kh\cos ech2kh)\sin^2(kx-\omega t)$. Since the average of $\sin^2 \theta$ over any period is $\frac{1}{2}$ then it holds that $\sin^2\left(\overline{kx-\omega t}\right) = \frac{1}{2}$.

The energy is transferred at a speed equal to that of group velocity.
Therefore, we write that

$$\bar{W} = \frac{1}{2}\frac{g\rho a^2 k}{\omega}(1 + 2kh\cos ech2kh) = \frac{1}{2} g a^2 \rho U.$$

We define the statistical mechanics [12] for the Cross correlation function in the stationary ergodic and real valued processes $X(t)$ and $Y(t)$ by the equation

$$R_{XY}(t) = E\{X(t)Y(t)\} = \frac{1}{T}\int_0^T X(t)Y(t+\tau)dt, \quad (\tau = 0,1,2,...) \quad (2.3.15)$$

where the Fourier wave transform for the Cross correlation is the equation

$$S_{XY}(F) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i2\pi f\tau} d\tau \quad (2.3.16)$$

In equation (2.3.16), the real part is the spectrum while the imaginary part is the quadrature spectrum.
For a single wave process the auto correlation and auto spectral density functions respectively are defined [12,15] in the form

$$R_{XX}(\tau) = E\{X(t)Y(t+\tau)\} = R_X(\tau) \quad (2.3.17)$$

$$S_{XX}(f)=\int_{-\infty}^{\infty}(\tau)e^{-i2\pi f\tau}d\tau \qquad (2.3.18)$$

Therefore from the forgoing analysis, the spectral density functions of the wave horizontal velocities as described by the equations below:

$$u(t)=2\pi f\,\frac{\cosh k(d+z)}{\sinh kd}\eta(t) \qquad (2.3.19)$$

$$u(t+\tau)=2\pi f\,\frac{\cosh k(d+z)}{\sinh kd}\eta(t+\tau) \qquad (2.3.20)$$

$$\overline{u(t)u(t+\tau)}=(2\pi f)^2\frac{\cosh^2 k(d+z)}{\sinh^2 kd}\overline{\eta(t)\eta(t+\tau)} \qquad (2.3.21)$$

Wherefrom, the Fourier transform for the wave velocity profile is

$$S_{uu}(f)=\left\{4\pi^2 f^2\frac{\cosh^2 k(d+z)}{\sinh^2 kd}\right\}S_{\eta\eta}(f) \qquad (2.3.22)$$

Where for instance, $(2\pi f)^2=gk\tanh kd$ and f is the wave frequency and the acceleration transfer function is

$$S_{aa}(f)=\left\{\left(4\pi^2 f^2\right)\frac{\cosh^2 k(d+z)}{\sinh^2 kd}\right\}S_{\eta\eta} \qquad (2.3.23)$$

The probabilistic nonlinear wave density function for the instantaneous force is expressed as:

$$P(f)=\frac{1}{2\pi K\delta_u\delta_a}\int_{-\infty}^{\infty}\exp-\left(\frac{1}{2}\left(\frac{u^2}{(\delta_u)^2}+\frac{a^2}{(\delta_a)^2}\right)\right)du\,. \qquad (2.3.24)$$

In equation (2.3.24) above, the following notations denote that:

$\delta_a =$ Standard deviation of particle acceleration,

$a=F-C|u|u$,

$K=\frac{1}{4}C_m\rho\pi d^2$

$C_m =$ Effective inertial coefficient $=\frac{4A(ka)}{\pi(ka)^2}$; $C=C_d\rho\frac{d}{2}$.

The Reynolds number describing state of turbulence can be calculated from most ocean water waves which are expressed in the form of $2D$ Burgers equations. In line with [16] a prototype $2D$ Burgers equation can be written in the form:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{v\partial u}{\partial z}=\frac{1}{\nu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)\\&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}=\frac{1}{\nu}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)\\&(x,z)\in\Omega=(a,b)\times(c,d),t\in(0,T)\end{aligned}\right\} \qquad (2.3.25)$$

With boundary conditions

$$\left.\begin{aligned}&u(a,z,t)=f_{u1}(z,t);\ u(b,z,t)=f_{u2}(z,t)\\&u(x,c,t)=f_{u3}(x,t);\ u(x,d,t)=f_{u4}(z,t)\\&v(a,z,t)=f_{v1}(z,t);\ v(b,z,t)=f_{v2}(z,t)\\&v(x,c,t)=f_{v3}(x,t);\ v(x,d,t)=f_{v4}(x,t)\end{aligned}\right\} \qquad (2.3.26)$$

and the initial conditions

$$u(x,z,t)|_{t=0}=\phi(x,z),\quad v(x,z,t)|_{t=0}=\phi(x,z);\quad (x,z)\in\Omega \qquad (2.3.27)$$

From equation (2.3.27) above, ν is the Reynolds number, whilst $u(x,z,t)$ and $v(x,z,t)$ are the respective velocity components, h is the wave height, d is the water depth, k is the wave number $=\frac{2\pi}{\lambda}$, λ is the wave length, z is the vertical coordinate at the point where ϕ is considered and x is the horizontal coordinate with T being the wave period, t is time instant.

3.0 The Eigenvalue Bounds and Backward Stability Solution

Discretization of equation (2.3.26) in the x and z direction for the discrete functions on an $n_x \times n_z$ grids in space domain $\Omega = [a,b]\times[c,d]$ leads to a system of linear equation in the form:

$$As = b \qquad (3.1.1)$$

where the matrix $A \in R^{m\times n}$; $b \in R^m$. We are required to solve for the unknown variable s as ocean water phase vector as influenced by the fluid density ratio of air density to water density. The backward perturbation tube [13] is defined

$$\overset{\min}{\Delta s} \ \|\Delta A\|_F \ s.t \ \ (A+\Delta A)s = b \qquad (3.1.2)$$

where for instance, $\|\Delta A\| = \left(trace\left(\Delta A^T \Delta A\right)\right)^{\frac{1}{2}}$. The least squares solution is given by

$$\overset{\min}{s} \ \| (b-As)s^+ \|_F = \overset{\min}{s} \frac{\|b-As\|_2}{\|s\|} \quad (\text{for } F=2 \text{ }). \qquad (3.1.3)$$

Equation (3.1.3) has a solution $\hat{s}$ if

$$A^T\left(b - A\hat{s}\right) = -\hat{s}\frac{\left\|b - A\hat{s}\right\|_2^2}{\|s\|_2^2} \quad , \qquad (3.1.4)$$

and

$$\frac{\left\|b - A\hat{s}\right\|_2}{\|s\|_2} < \delta_{\min}(A) \qquad (3.1.5)$$

Because the boundary layers of the Ocean b in equation (3.1.1) is fixed, we need only consider worthwhile the perturbation in the matrix A. Therefore, using this technique, we write that

$$\overset{\min}{\Delta A}\|\Delta A\|_F \ni C^+{}_A = \left\{\Delta A : (A+\Delta A)^T (b-(A+\Delta A)s) = -s\frac{\|b-(A++\Delta A)s\|_2^2}{\|s\|_2^2}\right. \qquad (3.1.6)$$

The C_A is defined to be

$$C_A = \left\{ \Delta A : \Delta A \in C^+{}_A \text{ and } \frac{\|b-(A+\Delta A)s\|_2}{\|s\|_2} < \delta_{\min}(A+\Delta A)\right\} \qquad (3.1.7)$$

The holder continuity for the stream function $\phi : D \to R^n$ with exponent $\alpha \in (0,1)$ and $D \subseteq C^{n\times n}(D)$ which satisfies the union of spectrum $\cup\delta(A) \subseteq D$ with exponent α as

$$[\phi]_{\alpha,D} \le n^{\frac{1-\alpha}{2}} [\phi]_{\alpha,D} \qquad (3.1.8)$$

In particular, $\|\phi(X_{k+1}) - \phi(X_k)\|_F \le [\phi]_{\alpha,D}\, n^{\frac{1-\phi}{2}} \|X_{k+1} - X_k\|_F^\alpha$ (3.1.9)

for any $X_{k,}\ X_{k+1} \in D, \quad \alpha \in (0,1]$.The X_k is the principal value of matrix A.

The Gaussian function $\phi : t \to \exp\left(-mt^2\right)$ for $m > 0$ is Lipchitz continuous on R^n with constant $[\phi]_{1,R} = \sqrt{2m}\exp\left(\frac{-1}{2}\right)$ and thus for which holds the principal part for A the inequality

$$\left\|\exp\left(-mX_{k+1}^2\right) - \exp\left(-mX_k^2\right)\right\| \le \sqrt{2m}\ e^{-\frac{1}{2}} \|X_{k+1} - X_k\|_F \qquad (3.1.10)$$

The concave function resulting for two- phase wave matrices for a wave catching up with the one at the front is in the form:

$$\big\||A| - |B|\big\| \le \frac{2}{\pi}\left(2 + \log[\frac{\|A\|+\|B\|}{\|A-B\|}\|A-B\|]\right) \qquad (3.1.11)$$

Using elementary calculus, the matrix $\log A$ is given in the form:

$$\ln A = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(A-I)^j}{j!} \quad , \qquad (3.1.12)$$

whenever the wave series converges and I being the identity matrix. The series converges if and only if $\|A-I\|<1$. Note also that $\|\ln(I+A)\| \le -\ln(1-\|A\|) \le \frac{\|A\|}{1-\|A\|}$. As a result, the condition number for $\ln A$ is defined as

$$K \ln A = \frac{K(A)}{\| \ln A \|} . \qquad (3.1.13)$$

3.2 **The Spectral Representation And Sub-harmonics For the wave Equations.**

Particularly, we are interested in the amplification factor of high frequency noise corresponding to low eigenvalues for the wave equation accompanying the flow of Ocean water. This, is so from the decomposition of $A = \sum \lambda_i u_i u_i^T \ (\lambda_1 \ge \lambda_2 \ge \lambda_2 \ge ... \ge \lambda_n)$ from the spectral theory of symmetric matric with $u_i \in R^n \ (\|u_i\| = 1)$. The condition number is equal to $\frac{\lambda_1}{\lambda_n} = \kappa(A)$. For the noisy data b^δ from the linear system $As = b$, we give the Euclidean norm in the error data in the form

$\|b^\delta - b\| \le \delta$. The spectral representation follows from

$$s^\delta - s = \sum_{i=1}^{n} \lambda_i^{-1} u_i u_i^T (s^\delta - s) . \qquad (3.2.1)$$

Orthogonality of the eigenvectors implies that $\|s^\delta - s\|^2 = \sum_{i=1}^{n} \lambda_i^{-2} |u_i^T (s^\delta - s)|^2 \le \lambda_i^{-2} \|s^\delta - s\|^2$. This means that $\|s^\delta - s\|^2 \le \kappa \|s^\delta - s\| \le \kappa\delta$. Therefore a summary of the error estimate from the solution to the data error is in the form

$$s^\delta - s = \delta u_1 . \qquad (3.2.2)$$

The principal part of the matrix is pth root, e.g., [13,14] and the cited references therein; for the wave matrix via Newton iteration is computed using the equations:

$$G_{n+1} = G_n \left(\frac{(p-1) + M_n}{p} \right), G_0 = I \qquad (3.2.3)$$

$$M_{n+1} = \left(\frac{(p-1)I + M_n}{p} \right)^{-p} M_n , \quad (M_0 = A, \ (n = 0,1,2,...)) . \qquad (3.2.4)$$

Therefore as $n \to \infty, \ G_n \to A^{\frac{1}{p}} \ when \ M_n \to I$.

From the above iteration, $\ln A$ can be recovered as $\ln A = 2 \ln G_n - \ln M_n$.

The eigenvalues of A are contained in the equation

$$G_{n+1} = \frac{1}{p} \left((p-1) G_n + A^{\frac{1}{p}} G_n^{-1} A^{\frac{1}{p}} G_n^{-1} .... A^{\frac{1}{p}} G_n^{-1} \right)$$

$$= \frac{1}{p} \left((p-1) G_n + \left(A^{\frac{1}{p}} G_n^{-1} \right)^{p-1} A^{\frac{1}{p}} \right) , \ p \ge 2 . \qquad (3.2.5)$$

We discuss the sub harmonics of the stream function ϕ. By Rado's theorem, [2] for example, there are $\delta_1, \delta_2, ..., \delta_n$ singular values of the matrix ordered in decreasing sequence for which

$$\log \delta_1(f(\lambda)) + \log \delta_2(f(\lambda)) + ... + \log \delta_k(f(\lambda)) , \qquad (3.2.6)$$

and

$$\log^+ \delta_1(f(\lambda)) + \log^+ \delta_2(f(\lambda)) + ... + \log^+ \delta_k(f(\lambda)) , \qquad (3.2.7)$$

are subharmonics on Ω for $1 \le k \le n$. Weyl's theorem then shows that

$$\prod_{i=1}^{k} |\lambda_i(A)| \le \prod_{i=1}^{k} |\delta_K(A)| , \qquad (1 \le k \le n) \qquad (3.2.8)$$

where,

$$\delta_k(z) = \sum_{i=1}^{k} \log |\lambda_i(f(z))| .$$

By Cauchy's formula it follows that

$$\delta_1 f(z_0) \le \frac{1}{2\pi r} \int_{|z-z_0|} \left(\sum_{i=1}^{k} \log| \, v_i(f(z)) \, ||dz| \right) \quad (3.2.9)$$

is subharmonic on Ω.

which holds for all $v_i \, f(z)$ elements of $sp \; f(z)$ in the union $B(\lambda_i(f(z_0)), s), \; i = 1,2,...,k$.

Writing $A = (A^T A)^{\frac{1}{2}} Q_0$, and with QR as the Schur decomposition of A, then it can be rewritten that $\lambda_i(Q_0^{-1} A) = \delta_i(A)$, wherefrom, $Q \prod_{I=1}^{k} | \, \lambda_i(QA) | \approx \prod_{i=1}^{k} \delta_i(A)$. By the theorem due to [2] we thus set that:

$Q_k^Q(z) \le \frac{1}{2\pi i} \int_0^{2\pi} Q_k^Q(z_0 + re^{i\theta}) dz$ for the Cauchy Riemann equation and $\bar{B}(z_0, r) \subset \Omega$ is the Geodesic ball so that $Q_k^Q(z) = \sum_{i=1}^{k} \log| \, \lambda_i(Qf(z)) \, |$ is measurable with a measure μ.

4.0 Numerical Experiments

Particularly, we start with the improved SOR iterative formula in the algorithm stated below**:**

1) Choose ε - order of accuracy, $x^{(0)} \in R^n, \omega = 1, \; q = 1, k = 0$

2) For $i = 1, 2, ... n$ do

$$x_i^{(k+1)} = (1-\omega) x_i^{(k)} + \frac{\omega}{a_{ii}} \left[\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1} + \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

3) Compute the error estimate q in the form

$$q = \max_{1 \le i \le n} \frac{\left| x_i^{(k+1)} - x_i^{(k)} \right|}{\left| x_i^{(k)} - x_i^{(k-1)} \right|}$$

If $q > 1$ go to step 5

4) Break off criterion if

$$\frac{\max_{1 \le i \le n} \left| x_i^{(k+1)} - x_i^{(k)} \right|}{\max_{1 \le i \le n} \left| x_i^{(k+1)} \right|} \le \varepsilon (1-q)$$

5) Set $k = k + 1$ and go to step 2

endif,

endif

4.2 The improved SOR Method

The improvement to the iterative method [13] comes from the fact that after some steps of successive iterates of Gauss-Siedel method, define the error iterates in the form $d^{(k)} = x_i^{(k+1)} - x_i^{(k)}$ for $k = 0,1,...$, then we have that

$$x^{(k+1} - x^{(k)} = e^{(k+1)} - e^{(k)} = (L_\omega - I) e^{(k)} = (L_\omega - I) L_\omega^k e^{(0)}. \quad (4.2.1)$$

$$= L_\omega^k (L_\omega - I) e^{(0)} = L_\omega^k d^{(0)}$$

We distribute the error estimate in the form $d^{(k)} = L_\omega^k d^{(0)}$ and as $k \to \infty$ we compute

$$q_k = \max_{1 \le i \le n} \frac{\left| x_i^{(k+1)} - x_i^{(k)} \right|}{\left| x_i^{(k)} - x_i^{(k-1)} \right|} \quad (4.2.2)$$

as an approximation for L_ω.

Thus the computed optimal relaxation parameter $\bar{\omega}$ is

$$\bar{\omega} = \frac{2}{1 + \sqrt{1 - \frac{(q_k + \omega - 1)^2}{\omega^2 q_k}}}. \quad (4.2.3)$$

Therefore, it follows that

$$\left\| e^{(k)} \right\|_\infty \le \frac{1}{1 - \rho(L_\omega)} \left\| d^{(k)} \right\|_\infty. \quad (4.2.4)$$

Thus the implication of this is that the wave [13] at the front will have a break off criterion when $\left\| d^{(k)} \right\|_\infty \le (1-q)\varepsilon$ for the absolute error. It is in the form $\frac{\left\| d^{(k)} \right\|_\infty}{\left\| x^{(k+1)} \right\|_\infty} \le (1-q)\varepsilon$ for the relative error.

5.0 Conclusion

The paper presented methods of turbulence often seen in the ocean water caused by wind generated waves from a given source which moves into the far field ocean water. Various sources of energy dissipation which occur during this process are given and equations of wind speeds and wind skin frictional force are described. We mentioned that wind growths consist of linear and exponential growths. The values for mean frequency $\bar{\delta}$, mean wave number $\bar{k}$,and the total wave energy E_{tol} are defined by their equations in synergy with some existing literatures. We defined the statistical mechanics for the Cross correlation function in the stationary ergodic and real valued processes $X(t)$ and $Y(t)$ by an equation. It was adopted the regularity theory [11] for the energy minimizing harmonic maps into Riemannian manifolds where the Jacobians are assumed continuous with strong derivatives. The 2-D Burgers equations were described with the Reynolyds number dominating from Laminar to turbulence of Ocean water waves. The problem was reduced to solving a large linear system of equations where the spectral radius of the matrix was considered very important for analysis. Weyl's theorem [2] for sub-harmonicity for the Eigen space discussed where Rado's theorem implied.

The principal part of the matrix with respect to the pth root [14] was presented. The improved SOR method for the linear systems solvers was used for analysis and illustration.

As a whole, it is recommended that what have been discussed in the paper may be relevant to oil workers of Nigeria National Petroleum Company, the naval force, Sea farer who must must be able to adapt to ocean, air, litoral and riverine environments after reading this paper. The research paper will be found appealing for Naval operations in consonance with National oceanic and atmospheric administration and National Science foundation where investment in science and technology on oceanic [17] activities are highly needed both for academic activities and military operations with predictive capability to tactical missions. It will help in the naval capability warefare operations with focused on the improvement of air, surface and undersea weapon performances. The studies will also enable advanced electrical systems, components and survivable, agile, mobile sustainable manned and unmanned surface and sub-surface sea plateforms and undersea weapons, [17].

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