Transactions of the Nigerian Association of Mathematical Physics
***Volume 17, (October – December, 2021 Issue*), pp1 – 10**

AMENABLE FOURIER ALGEBRA FROM A CONSTRUCTIBLE GROUP

Daniel I.E.,[1] Olaleru J.O.[2] and Ogunsola O.J.[3]

[1]Department of Computer and Mathematics, Spiritan University Nneochi Abia state.
[2]Department of Mathematics, University of Lagos.
[3] Department of Mathematics, Federal University of Agriculture Abeokuta.

Abstract

The subject of amenability of Fourier algebra has been a subject of research for decades. It is generally known that Fourier algebras are not generally amenable even though they are operator amenable. In this study we are able to construct an amenable group G_T. The group has some features including a contractible identity. Its group algebra $L^1(G_T)$ is amenable following Barry Johnson's result. The Fourier algebra is obtained $A(G_T)$ is obtained by the actions of Fourier Transforms on $L^1(G_T)$, with G_T as an underlying group. $A(G_T)$ inherited the norm from its group algebra $L^1(G_T)$. Further, the amenability of $A(G_T)$ is obtained and studied. Some groups are paradoxical and non-amenable. The problems of non-amenability posed by these groups have given rise to non-amenable Fourier algebras. This brings some limitations to the study of these algebras which have much of applications in quantum mechanics. Various aspects of amenability and operator amenability can be studied under this Fourier algebra.

Keywords: Amenability, constructible group, Fourier transform, Fourier Algebra.

1. Introduction and Preliminaries

Fourier algebra consists of continuous functions in the category of operator space. It is the predual of the Von Neumann algebra $vN(G)$ and is a copltely contractive Banach algebra. John Von Neumann in 1929, on his investigation of Banach-Tarsk paradox, observed that non-abelian free groups are paradoxical, see [1]. In 1949 Mahlon M. Day introduced the English translation: amenable. His work centered on the amenability of locally compact groups. Barry Edward Johnson, an English Mathematician, initiated and linked the amenability of group to that of Banach algebras. Eymard in [2] introduced Fourier algebra $A(G)$. Forrest [3] extended Johnson's result to a complete characterization of amenable $A(G)$. He established the $A(G)$ is amenable when G has an abelian subgroup of finite index.

Consult [4] and [5] for generalized and various notions of amenable and operator amenable Banach algebra. In this research, we constructed a locally compact amenable group and the resulting Fourier algebra.

Definition 1.1 [6] Let A be Banach algebra and X a Banach A bimodule, a derivative D is a continuous bounded linear map $D: A \longrightarrow X$ such that

$D(ab) = a \cdot D(b) + b \cdot D(a), \quad (a, b \in X).$

Deviation of this form

$\delta_x(a) = a \cdot x - x \cdot a \qquad a \in A, \;\; x \in X \qquad are\ inner.$

The dual space X^* of X is also a Banach A bimodule, with operation

$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle \qquad a \in A, \;\; x \in X, x^* \in X^*.$

Definition 1.2 [6] A Banach algebra A is amenable if every derivation from A to the dual Banach A bimodule is inner.

Correspondence Author: Daniel I.E., Email: ifeyinwadaniel@yahoo.com, Tel: +2348024600455, +2348037265755 (OJO)

Definition 1.3 [5] A chain is a bounded linear map from Cartesian product of A to X, its bimodule, denoted by $T \in C^1(A, X)$. Let e a Banach algebra and X a Banach A bimodule. An n-cochain is a bounded n linear map from n-fold Cartesian product A^n to X, denoted by $T \in C^n(A, X)$.
Definition 1.4 [5] A chain is exact if the image of $(n+1)^{th}$ is always equal to the kernel of the nth map. Hochschild cohomology measures the extend the co-chain is exact. The vanishing first order Hochschild cohomology implies exactness of a sequence of $\in C^n(A, X)$.
Definition 1.5 [1] The normed algebra $(A, \|\cdot\|)$ is called a Banach algebra if $(\|\cdot\|)$ is a complete norm. A Banach algebra is unital if A has an identity $e \in A$ such that $(\|e\|) = 1$.
A Banach algebra A is amenable if the 1-dimensional cohomology module $H^1(A, H^*)$ vanishes, that is $H^1(A, H^*) = \{0\}$.
Theorem 1.6 [7] Fourier algebra is an operator amenable completely contractive Banach algebra if $D: A(G) \longrightarrow X^*$ is inner, for the dual X^*, and X completely contractive $A(G)$ bimodule.
Definition 1.7 [7] Let S and T be topological spaces. If f and g are mappings of S into T, then f and g are homotopic ($f \simeq g$) if there exists a mapping $h: S \times I^1 \longrightarrow T$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in S$. The mapping h is called a homology between f and g.
Definition 1.8 [8] Functors are the mechanisms that create images by rotations, translations, stretching and bending. Functors have characteristic feature that they form images not only of sets but also of continuous maps. Topologically related spaces have algebraic related images.
Continuous maps between spaces are projected onto homomorphism between their algebraic images.
Definition 1.9 [5] Let G be a locally compact topological group and μ its left Haar measure, consider the space $L^1(G)$ consisting of measurable functions $f: G \longrightarrow \mathbb{C}$ such that $\|f\| = \int_G |f| d(\mu)$.

The convolution product of functions $f, g \in L^1(G)$ is given by $\langle f * g \rangle(x) = \int f(y) g(y^{-1}x) dy, \quad x, y \in L^1(G)$. This a well defined product in $L^1(G)$, i.e. $f * g \in L^1(G)$.
Definition 1.10 [9] The Fourier transform on $f \in L^1(G)$ is defined by $\mathcal{F}\{f(t)\} = \hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi st} f(t) dt.$ The collection of Fourier transform on $L^1(G)$ is the Fourier algebra with the convolution defined by $\mathcal{F}(g)(t) \cdot \mathcal{F}(f)(x) = \hat{g}(s) \cdot \hat{f}(x)$ as the multiplication.
Definition 1.11 [8]A topological set T is contractible if there exist a point $p \in T$ such that the identity map i on T is homotopic to a constant mapping c such that $c: T \longrightarrow p$. Example is a disk.
Theorem 1.12 [8] If T is a contractible set, then each mapping $f: S \longrightarrow T$ is homotopic to a constant map $f: S \longrightarrow \{p\}$.
$Proof$ Let $f: S \longrightarrow T$ *and let* i be the identity mapping on T. Since T is contractible, $i \simeq c: T \longrightarrow \{p\}$ and $cf: S \longrightarrow \{p\}$, hus there exists a homotopy $h: T \times I^1 \longrightarrow T$ with
$h(y, 0) = y$ *and* $h(y, 1) = p \quad \forall\, y \in T.$
Let
$\hbar(x, t) = h(f(x), t) \quad for\ \langle x, t \rangle \in S \times I^1,$
f and h are continuous, $\hbar$ is also continuous. Moreover
$\hbar(x, 0) = h(f(x), 0) = f(x)$
and
$\hbar(x, 1) = h(f(x), 1) = p$ *and* $f \simeq cf.$
Example 1.13 [8] The simple torus $S^1 \times S^1$ and the solid torus $S^1 \times D^1$ are non-contractible structures.
Theorem 1.14 [8] If A, a Banach algebra is a retract of a contractible set S, then A is contractible.
$Proof$ Let $r: S \longrightarrow A$ be a retraction and continuous. Since S is contractible, the identity mapping i on S is homotopic to the constant mapping $c: S \longrightarrow \{p\}$ for some $p \in S$. There exist a Banach algebra A such that for $h \in A, h: S \times I^1 \longrightarrow S,$
with $h(x, 0) = x$ and $h(x, 1) = p$ for all $x \in S$. Let $\hbar(x, t) = r[h(x, t)] \in A \times I^1$. Since r and h are continuous, $\hbar$ is continuous.
Moreover
$\hbar(x, 0) = r[h(x, 0)] = r(x) = x$
and $\hbar(x, 1) = r[h(x, 1)] = r(p)$ for each $x \in A$. Thus the identity mapping $r|A \simeq \hat{c}: A \longrightarrow r\{p\}$, and A is contractible.

The following examples have contractible features [8]. The wedge sum of circles, it is a free group, the free product of copies of Z. The shrinking wedge of circles is the union of circle C_n of radius $\frac{1}{n}$ and centre $(1/n, 0) for\ n = 1,2,3 \cdots$.
The group constructed in this study is denoted by G_T, its group algebra by $L^1(G_T)$ and Fourier algebra $A(G_T)$.

2 The Construction

The construction is on the $T\ set \subset R^3$. It is made up of infinite number of paths or strings rooted at $(0,0)$ on (x, y) plane of (x, y, z). It is required to group paths to form classes. The tools used in this construction are two funtors namely k, the class forming and l, the loop forming functors. The paths in the T set takes direction through $[0, 2\pi]$ on the (x, y) plane. The strings or the paths denoted by σ_i with $0 \le i \le 2\pi$. The reading of the value of i is from $y = 0$ through anti-clockwise direction.
The class forming functor k also can map the whole strings to a single class, which gives rise to a trivial and contractive group [8], k: $\sigma_{0 \le i \le 2\pi} \longrightarrow t_0$.
The functor k groups all the paths into n classes, each class contains homotopy paths. The functor k maps some paths in a defined interval into a class. It maps each string $\sigma_i,\ i \neq n\frac{\pi}{16}$ by rotation of $\frac{\pi}{2}$ to t_n. Intervals are defined over
$\frac{(2n-1)\pi}{16} < t_n < \frac{(2n+1)\pi}{16}$:
$\frac{-\pi}{16} < t_0 < \frac{\pi}{16}$, $\frac{\pi}{16} < t_1 < \frac{3\pi}{16}$, $\frac{3\pi}{16} < t_2 < \frac{5\pi}{16}$, $\frac{5\pi}{16} < t_3 < \frac{7\pi}{16}$, $\frac{7\pi}{16} < t_4 < \frac{9\pi}{16} \cdots$ $\frac{(2n-1)\pi}{16} < t_n < \frac{(2n+1)\pi}{16} \cdots \frac{(2n-1)\pi}{16} < t_{15} < \frac{(2n+1)\pi}{16}$ $n = 0, 1, 2 \cdots 15$, for sixteen classes, having the rotation of $\frac{\pi}{2}$ through $\sigma_{\frac{n\pi}{16}}$ $in\ R^3$.
The second functor l forms loops of classes. It stretches the ends of the paths in a particular class to the origin$(0,0,0)$ through the base class. The stretching and the continuous deformation take clockwise direction. With this operation, loops are formed relative to the end point held fixed. There exists a massive number of paths in a class and we desire a way to relate one path to another. This is done by continuously deforming one path into another. However, we will require that paths in a class share the same endpoints and that during this deformation, the endpoints remain fixed. A loop in a T set is a path σ such that $\sigma(0) = \sigma(1)$, the starting and ending points meet. The loops in a class are related by position, end points and angle of rotation. The functor l forms, not only loops. But singularity properties of the loops.
Remark 2.1 consider in the plane, the problem of integrating a function f of a complex variable around a closed curve C, e.g., the unit circle. For example:
$\int_c z dz = 0\ and \int_c \frac{dz}{z} \neq 0$. What is the difference? From the point of that C can be shrunk to point within the domain of analyticity of z (i.e., the whole plane), hence integrating around C is equivalent to integrating at a point, which gives zero. On the contrary C cannot be shrunk to a point within the domain of $\frac{1}{z}$.
Some of the closed paths in this construction have singular points:

$$l: \frac{-\pi}{16} < \sigma_i < \frac{\pi}{16} \longrightarrow t^0 = \quad t_0\ \ Disk, no\ singular\ point$$

$$l: \frac{\pi}{16} < \sigma_i < \frac{3\pi}{16} \longrightarrow t^{\frac{\pi}{8}} = t_1, \qquad there\ is\ a\ singular\ point$$

$$\vdots$$

$$l: \frac{(2n-1)\pi}{16} < \sigma_i < \frac{(2n+1)\pi}{16} \longrightarrow t^{\frac{\pi}{8}} = t_1, \qquad there\ is\ a\ singular\ poin$$

Now consider the classes of infinite loops σ_i not as string but paths of continuous bounded functions.
Remark 2.2 (i) By the path σ_i operated by the two functors, we therefore mean continuous bounded maps.
(ii) The functors k and l have the virtue of being functorial: homomorphisms and continuous maps: $h: \sigma_i \rightarrow t_n$.
(iii) This infinite composition of loops is certainly continuous at each point and it is continuous at t_0, since every neighbourhood of the base point t_0 in the set contains all but finitely many of the loops.
Definition 2.3 [8] A composition functor $h = kl$ and its operation defined by $h: T\ set \longrightarrow G_T$is a functorial of kl.
Remark 2.4 Given a continuous function $t^k \in G_T$ the inverse of t^k is denoted by $\hat{t}^k$ and defined by $t^{2\pi - k}$.

2.5 The group elements and their measures

The following therefore the group elements There are n elements of the group, taking n= 16:
t^0 is the identity element. It can be denoted by t_0. 0 is the angle of
rotation of the homotopy class, representing the measure of the class.
$t^{\frac{\pi}{8}}$ denoted by t_1, $\frac{\pi}{8}$ is the angle of rotation of the homotopy class and the
measure.
$t^{\frac{\pi}{4}}$ denoted by t_2, $\frac{\pi}{4}$ is the angle of rotation of the homotopy class and the measure.
$t^{\frac{3\pi}{8}}$ denoted by t_3, $\frac{3\pi}{8}$ is the angle of rotation of the homotopy class and the measure.

.

.

.

$t^{\frac{2n\pi}{16}}$ denoted by t_n, $\frac{2n\pi}{16}$ is the angle of rotation of the homotopy class and the measure.

.

.

.

$t^{\frac{15\pi}{8}}$ denoted by t_{15}, $\frac{15\pi}{8}$ is the angle of rotation of the homotopy class and the measure.

2.6 The sub-groups of G_T

The proper non trivial subgroups of the G_T include:

$G_{Tsub1} = \left\{t^{2\pi}, t^{\frac{\pi}{4}}, t^{\frac{\pi}{2}}, t^{\frac{3\pi}{4}}, t^{\pi}, t^{\frac{5\pi}{4}}, t^{\frac{3\pi}{2}}, t^{\frac{7\pi}{4}}\right\}$, $G_{Tsub2} = \left\{t^{2\pi}, t^{\frac{\pi}{2}}, t^{\pi}, t^{\frac{3\pi}{2}}\right\}$, $G_{Tsub3} = \{t^{2\pi}, t^{\pi}\}$.

2.7 The computation of the results

These computations are made up of the binary operation of the G_T group and all the elements that yield the results in the Cayley tables.

$t_0 \circ t_n = t^0 \circ t^{\frac{n\pi}{8}} = t^{\frac{n\pi}{8}+2\pi} = t^{\frac{n\pi}{8}}$.

$t_1 \circ t_1 = t^{\frac{\pi}{8}} \circ t^{\frac{\pi}{8}} = t^{\frac{\pi}{8}+\frac{\pi}{8}} = t^{\frac{\pi}{4}}$; $t_1 \circ t_2 = t^{\frac{\pi}{8}} \circ t^{\frac{\pi}{4}} = t^{\frac{\pi}{8}+\frac{\pi}{4}} = t^{\frac{3\pi}{8}}$;

$t_1 \circ t_3 = t^{\frac{\pi}{8}} \circ t^{\frac{3\pi}{8}} = t^{\frac{\pi}{8}+\frac{3\pi}{8}} = t^{\frac{\pi}{2}}$; $t_1 \circ t_4 = t^{\frac{\pi}{8}} \circ t^{\frac{\pi}{2}} = t^{\frac{\pi}{8}+\frac{\pi}{2}} = t^{\frac{5\pi}{8}}$;

$t_1 \circ t_5 = t^{\frac{\pi}{8}} \circ t^{\frac{5\pi}{8}} = t^{\frac{\pi}{8}+\frac{5\pi}{8}} = t^{\frac{3\pi}{4}}$; $t_1 \circ t_6 = t^{\frac{\pi}{8}} \circ t^{\frac{2\pi}{3}} = t^{\frac{\pi}{8}+\frac{2\pi}{3}} = t^{\frac{7\pi}{8}}$

$t_1 \circ t_7 = t^{\frac{\pi}{8}} \circ t^{\frac{7\pi}{8}} = t^{\frac{\pi}{8}+\frac{7\pi}{8}} = t^{\pi}$; $t_1 \circ t_8 = t^{\frac{\pi}{8}} \circ t^{\pi} = t^{\frac{\pi}{8}+\pi} = t^{\frac{9\pi}{8}}$

$t_1 \circ t_9 = t^{\frac{\pi}{8}} \circ t^{\frac{9\pi}{8}} = t^{\frac{\pi}{8}+\frac{9\pi}{8}} = t^{\frac{5\pi}{4}}$; $t_1 \circ t_{10} = t^{\frac{\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{\pi}{8}+\frac{5\pi}{4}} = t^{\frac{11\pi}{8}}$

$t_1 \circ t_{11} = t^{\frac{\pi}{8}} \circ t^{\frac{11\pi}{8}} = t^{\frac{\pi}{8}+\frac{11\pi}{8}} = t^{\frac{3\pi}{2}}$; $t_1 \circ t_{12} = t^{\frac{\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{\pi}{8}+\frac{3\pi}{2}} = t^{\frac{13\pi}{8}}$

$t_1 \circ t_{13} = t^{\frac{\pi}{8}} \circ t^{\frac{13\pi}{8}} = t^{\frac{\pi}{8}+\frac{13\pi}{8}} = t^{\frac{7\pi}{4}}$; $t_1 \circ t_{14} = t^{\frac{\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{\pi}{8}+\frac{7\pi}{8}} = t^{\frac{15\pi}{4}}$

$t_1 \circ t_{15} = t^{\frac{\pi}{8}} \circ t^{\frac{15\pi}{8}} = t^{\frac{\pi}{8}+\frac{15\pi}{8}} = t^{0}$.

$t_2 \circ t_2 = t^{\frac{\pi}{4}} \circ t^{\frac{\pi}{4}} = t^{\frac{\pi}{8}+\frac{\pi}{4}} = t^{\frac{3\pi}{8}}$;

$t_2 \circ t_3 = t^{\frac{\pi}{4}} \circ t^{\frac{3\pi}{8}} = t^{\frac{\pi}{8}+\frac{3\pi}{8}} = t^{\frac{\pi}{2}}$; $t_2 \circ t_4 = t^{\frac{\pi}{8}} \circ t^{\frac{\pi}{2}} = t^{\frac{\pi}{8}+\frac{\pi}{2}} = t^{\frac{5\pi}{8}}$;

$t_2 \circ t_5 = t^{\frac{\pi}{4}} \circ t^{\frac{5\pi}{8}} = t^{\frac{\pi}{8}+\frac{5\pi}{8}} = t^{\frac{3\pi}{4}}$; $t_2 \circ t_6 = t^{\frac{\pi}{8}} \circ t^{\frac{2\pi}{3}} = t^{\frac{\pi}{8}+\frac{2\pi}{3}} = t^{\frac{7\pi}{8}}$

$t_2 \circ t_7 = t^{\frac{\pi}{4}} \circ t^{\frac{7\pi}{8}} = t^{\frac{\pi}{8}+\frac{7\pi}{8}} = t^{\pi}$; $t_2 \circ t_8 = t^{\frac{\pi}{8}} \circ t^{\pi} = t^{\frac{\pi}{8}+\pi} = t^{\frac{9\pi}{8}}$

$t_2 \circ t_9 = t^{\frac{\pi}{4}} \circ t^{\frac{9\pi}{8}} = t^{\frac{\pi}{8}+\frac{9\pi}{8}} = t^{\frac{5\pi}{4}}$; $t_2 \circ t_{10} = t^{\frac{\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{\pi}{8}+\frac{5\pi}{4}} = t^{\frac{11\pi}{8}}$

$t_2 \circ t_{11} = t^{\frac{\pi}{4}} \circ t^{\frac{11\pi}{8}} = t^{\frac{\pi}{8}+\frac{11\pi}{8}} = t^{\frac{3\pi}{2}}$; $t_2 \circ t_{12} = t^{\frac{\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{\pi}{8}+\frac{3\pi}{2}} = t^{\frac{13\pi}{8}}$

$t_2 \circ t_{13} = t^{\frac{\pi}{4}} \circ t^{\frac{13\pi}{8}} = t^{\frac{\pi}{8}+\frac{13\pi}{8}} = t^{\frac{7\pi}{4}}$; $t_2 \circ t_{14} = t^{\frac{\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{\pi}{8}+\frac{7\pi}{8}} = t^{\frac{15\pi}{4}}$

$t_2 \circ t_{15} = t^{\frac{\pi}{4}} \circ t^{\frac{15\pi}{8}} = t^{\frac{\pi}{4}+\frac{15\pi}{8}} = t^{\frac{\pi}{8}}$.

$t_3 \circ t_3 = t^{\frac{3\pi}{8}} \circ t^{\frac{3\pi}{8}} = t^{\frac{3\pi}{8}+\frac{3\pi}{8}} = t^{\frac{3\pi}{4}}$; $t_3 \circ t_4 = t^{\frac{3\pi}{8}} \circ t^{\frac{\pi}{2}} = t^{\frac{3\pi}{8}+\frac{\pi}{2}} = t^{\frac{7\pi}{8}}$;

$t_3 \circ t_5 = t^{\frac{3\pi}{8}} \circ t^{\frac{5\pi}{8}} = t^{\frac{3\pi}{8}+\frac{5\pi}{8}} = t^{\pi}$; $t_3 \circ t_6 = t^{\frac{3\pi}{8}} \circ t^{\frac{3\pi}{4}} = t^{\frac{3\pi}{8}+\frac{3\pi}{4}} = t^{\frac{9\pi}{8}}$

$t_3 \circ t_7 = t^{\frac{3\pi}{8}} \circ t^{\frac{7\pi}{8}} = t^{\frac{3\pi}{8}+\frac{7\pi}{8}} = t^{\frac{5\pi}{4}}$; $t_3 \circ t_8 = t^{\frac{3\pi}{8}} \circ t^{\pi} = t^{\frac{3\pi}{8}+\pi} = t^{\frac{11\pi}{8}}$

$t_3 \circ t_9 = t^{\frac{3\pi}{8}} \circ t^{\frac{9\pi}{8}} = t^{\frac{3\pi}{8}+\frac{9\pi}{8}} = t^{\frac{11\pi}{8}}$; $t_3 \circ t_{10} = t^{\frac{3\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{3\pi}{8}+\frac{5\pi}{4}} = t^{\frac{13\pi}{8}}$

$t_3 \circ t_{11} = t^{\frac{3\pi}{8}} \circ t^{\frac{11\pi}{8}} = t^{\frac{3\pi}{8}+\frac{11\pi}{8}} = t^{\frac{7\pi}{8}};$ $\quad t_3 \circ t_{12} = t^{\frac{3\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{3\pi}{8}+\frac{3\pi}{2}} = t^{\frac{15\pi}{8}}$
$t_3 \circ t_{13} = t^{\frac{3\pi}{8}} \circ t^{\frac{13\pi}{8}} = t^{\frac{3\pi}{8}+\frac{13\pi}{8}} = t^0;$ $\quad t_3 \circ t_{14} = t^{\frac{3\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{3\pi}{8}+\frac{7\pi}{8}} = t^{\frac{\pi}{8}}$
$t_3 \circ t_{15} = t^{\frac{3\pi}{8}} \circ t^{\frac{15\pi}{8}} = t^{\frac{3\pi}{8}+\frac{15\pi}{8}} = t^{\frac{\pi}{4}}.$

$t_4 \circ t_4 = t^{\frac{\pi}{2}} \circ t^{\frac{\pi}{2}} = t^{\frac{\pi}{2}+\frac{\pi}{2}} = t^{\pi};$
$t_4 \circ t_5 = t^{\frac{\pi}{2}} \circ t^{\frac{5\pi}{8}} = t^{\frac{\pi}{2}+\frac{5\pi}{8}} = t^{\frac{9\pi}{8}};$ $\quad t_4 \circ t_6 = t^{\frac{\pi}{2}} \circ t^{\frac{2\pi}{3}} = t^{\frac{\pi}{2}+\frac{2\pi}{3}} = t^{\frac{5\pi}{4}}$
$t_4 \circ t_7 = t^{\frac{\pi}{2}} \circ t^{\frac{7\pi}{8}} = t^{\frac{\pi}{2}+\frac{7\pi}{8}} = t^{\frac{11\pi}{8}};$ $\quad t_4 \circ t_8 = t^{\frac{\pi}{2}} \circ t^{\pi} = t^{\frac{\pi}{2}+\pi} = t^{\frac{3\pi}{2}}$
$t_4 \circ t_9 = t^{\frac{\pi}{2}} \circ t^{\frac{9\pi}{8}} = t^{\frac{\pi}{2}+\frac{9\pi}{8}} = t^{\frac{13\pi}{8}};$ $\quad t_4 \circ t_{10} = t^{\frac{\pi}{2}} \circ t^{\frac{5\pi}{4}} = t^{\frac{\pi}{2}+\frac{5\pi}{4}} = t^{\frac{7\pi}{4}}$
$t_4 \circ t_{11} = t^{\frac{\pi}{2}} \circ t^{\frac{11\pi}{8}} = t^{\frac{\pi}{2}+\frac{11\pi}{8}} = t^{\frac{15\pi}{8}};$ $\quad t_4 \circ t_{12} = t^{\frac{\pi}{2}} \circ t^{\frac{3\pi}{2}} = t^{\frac{\pi}{2}+\frac{3\pi}{2}} = t^0$
$t_4 \circ t_{13} = t^{\frac{\pi}{2}} \circ t^{\frac{13\pi}{8}} = t^{\frac{\pi}{2}+\frac{13\pi}{8}} = t^{\frac{\pi}{8}};$ $\quad t_4 \circ t_{14} = t^{\frac{\pi}{2}} \circ t^{\frac{7\pi}{4}} = t^{\frac{\pi}{2}+\frac{7\pi}{8}} = t^{\frac{11\pi}{4}}$
$t_4 \circ t_{15} = t^{\frac{\pi}{2}} \circ t^{\frac{15\pi}{8}} = t^{\frac{\pi}{2}+\frac{15\pi}{8}} = t^{\frac{3\pi}{8}}.$

$t_5 \circ t_5 = t^{\frac{5\pi}{8}} \circ t^{\frac{5\pi}{8}} = t^{\frac{5\pi}{8}+\frac{5\pi}{8}} = t^{\frac{5\pi}{4}};$ $\quad t_5 \circ t_6 = t^{\frac{5\pi}{8}} \circ t^{\frac{3\pi}{4}} = t^{\frac{5\pi}{8}+\frac{3\pi}{4}} = t^{\frac{11\pi}{8}}$
$t_5 \circ t_7 = t^{\frac{5\pi}{8}} \circ t^{\frac{7\pi}{8}} = t^{\frac{3\pi}{8}+\frac{7\pi}{8}} = t^{\frac{3\pi}{2}};$ $\quad t_5 \circ t_8 = t^{\frac{5\pi}{8}} \circ t^{\pi} = t^{\frac{5\pi}{8}+\pi} = t^{\frac{13\pi}{8}}$
$t_5 \circ t_9 = t^{\frac{5\pi}{8}} \circ t^{\frac{9\pi}{8}} = t^{\frac{5\pi}{8}+\frac{9\pi}{8}} = t^{\frac{7\pi}{4}};$ $\quad t_5 \circ t_{10} = t^{\frac{5\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{5\pi}{8}+\frac{5\pi}{4}} = t^{\frac{15\pi}{8}}$
$t_5 \circ t_{11} = t^{\frac{5\pi}{8}} \circ t^{\frac{11\pi}{8}} = t^{\frac{5\pi}{8}+\frac{11\pi}{8}} = t^0;$ $\quad t_5 \circ t_{12} = t^{\frac{5\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{5\pi}{8}+\frac{3\pi}{2}} = t^{\frac{\pi}{8}}$
$t_5 \circ t_{13} = t^{\frac{5\pi}{8}} \circ t^{\frac{13\pi}{8}} = t^{\frac{5\pi}{8}+\frac{13\pi}{8}} = t^{\frac{\pi}{4}};$ $\quad t_5 \circ t_{14} = t^{\frac{5\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{5\pi}{8}+\frac{7\pi}{8}} = t^{\frac{3\pi}{2}}$
$t_5 \circ t_{15} = t^{\frac{5\pi}{8}} \circ t^{\frac{15\pi}{8}} = t^{\frac{5\pi}{8}+\frac{15\pi}{8}} = t^2.$

$t_6 \circ t_6 = t^{\frac{3\pi}{4}} \circ t^{\frac{3\pi}{4}} = t^{\frac{3\pi}{4}+\frac{3\pi}{4}} = t^{\frac{3\pi}{2}}$
$t_6 \circ t_7 = t^{\frac{3\pi}{4}} \circ t^{\frac{7\pi}{8}} = t^{\frac{3\pi}{4}+\frac{7\pi}{8}} = t^{\frac{13\pi}{8}};$ $\quad t_6 \circ t_8 = t^{\frac{3\pi}{4}} \circ t^{\pi} = t^{\frac{3\pi}{4}+\pi} = t^{\frac{7\pi}{8}}$
$t_6 \circ t_9 = t^{\frac{3\pi}{4}} \circ t^{\frac{9\pi}{8}} = t^{\frac{3\pi}{4}+\frac{9\pi}{8}} = t^{\frac{15\pi}{8}};$ $\quad t_6 \circ t_{10} = t^{\frac{3\pi}{4}} \circ t^{\frac{5\pi}{4}} = t^{\frac{3\pi}{4}+\frac{5\pi}{4}} = t^0$
$t_6 \circ t_{11} = t^{\frac{3\pi}{4}} \circ t^{\frac{11\pi}{8}} = t^{\frac{3\pi}{4}+\frac{11\pi}{8}} = t^{\frac{\pi}{8}};$ $\quad t_6 \circ t_{12} = t^{\frac{3\pi}{4}} \circ t^{\frac{3\pi}{2}} = t^{\frac{3\pi}{4}+\frac{3\pi}{2}} = t^{\frac{\pi}{8}}$
$t_6 \circ t_{13} = t^{\frac{3\pi}{4}} \circ t^{\frac{13\pi}{8}} = t^{\frac{3\pi}{4}+\frac{13\pi}{8}} = t^{\frac{3\pi}{8}};$ $\quad t_6 \circ t_{14} = t^{\frac{3\pi}{4}} \circ t^{\frac{7\pi}{4}} = t^{\frac{3\pi}{4}+\frac{7\pi}{8}} = t^{\frac{\pi}{2}}$
$t_6 \circ t_{15} = t^{\frac{3\pi}{4}} \circ t^{\frac{15\pi}{8}} = t^{\frac{3\pi}{4}+\frac{15\pi}{8}} = t^{\frac{5\pi}{8}}.$

$t_7 \circ t_7 = t^{\frac{7\pi}{8}} \circ t^{\frac{7\pi}{8}} = t^{\frac{7\pi}{8}+\frac{7\pi}{8}} = t^{\frac{7\pi}{4}};$ $\quad t_7 \circ t_8 = t^{\frac{7\pi}{8}} \circ t^{\pi} = t^{\frac{7\pi}{8}+\pi} = t^{\frac{15\pi}{8}}$
$t_7 \circ t_9 = t^{\frac{7\pi}{8}} \circ t^{\frac{9\pi}{8}} = t^{\frac{7\pi}{8}+\frac{9\pi}{8}} = t^0;$ $\quad t_7 \circ t_{10} = t^{\frac{7\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{7\pi}{8}+\frac{5\pi}{4}} = t^{\frac{\pi}{8}}$
$t_7 \circ t_{11} = t^{\frac{7\pi}{8}} \circ t^{\frac{11\pi}{8}} = t^{\frac{7\pi}{8}+\frac{11\pi}{8}} = t^{\frac{\pi}{4}};$ $\quad t_7 \circ t_{12} = t^{\frac{7\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{7\pi}{8}+\frac{3\pi}{2}} = t^{\frac{3\pi}{8}}$
$t_7 \circ t_{13} = t^{\frac{7\pi}{8}} \circ t^{\frac{13\pi}{8}} = t^{\frac{7\pi}{8}+\frac{13\pi}{8}} = t^{\frac{\pi}{4}};$ $\quad t_7 \circ t_{14} = t^{\frac{7\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{7\pi}{8}+\frac{7\pi}{4}} = t^{\frac{5\pi}{8}}$
$t_7 \circ t_{15} = t^{\frac{7\pi}{8}} \circ t^{\frac{15\pi}{8}} = t^{\frac{7\pi}{8}+\frac{15\pi}{8}} = t^{\frac{3\pi}{4}}.$

$t_8 \circ t_8 = t^{\pi} \circ t^{\pi} = t^{\pi+\pi} = t^{2\pi}$
$t_8 \circ t_9 = t^{\pi} \circ t^{\frac{9\pi}{8}} = t^{\pi+\frac{9\pi}{8}} = t^{\frac{\pi}{8}};$ $\quad t_8 \circ t_{10} = t^{\pi} \circ t^{\frac{5\pi}{4}} = t^{\pi+\frac{5\pi}{4}} = t^{\frac{\pi}{4}}$
$t_8 \circ t_{11} = t^{\pi} \circ t^{\frac{11\pi}{8}} = t^{\pi+\frac{11\pi}{8}} = t^{\frac{3\pi}{8}};$ $\quad t_8 \circ t_{12} = t^{\pi} \circ t^{\frac{3\pi}{2}} = t^{\pi+\frac{3\pi}{2}} = t^{\frac{\pi}{2}}$
$t_8 \circ t_{13} = t^{\pi} \circ t^{\frac{13\pi}{8}} = t^{\pi+\frac{13\pi}{8}} = t^{\frac{5\pi}{8}};$ $\quad t_8 \circ t_{14} = t^{\pi} \circ t^{\frac{7\pi}{4}} = t^{\pi+\frac{7\pi}{8}} = t^{\frac{3\pi}{4}}$
$t_8 \circ t_{15} = t^{\pi} \circ t^{\frac{15\pi}{8}} = t^{\pi+\frac{15\pi}{8}} = t^{\frac{7\pi}{8}}.$

$t_9 \circ t_9 = t^{\frac{9\pi}{8}} \circ t^{\frac{9\pi}{8}} = t^{\frac{9\pi}{8}+\frac{9\pi}{8}} = t^{\frac{9\pi}{4}};$ $\quad t_9 \circ t_{10} = t^{\frac{9\pi}{8}} \circ t^{\frac{5\pi}{4}} = t^{\frac{9\pi}{8}+\frac{5\pi}{4}} = t^{\frac{3\pi}{8}}$
$t_9 \circ t_{11} = t^{\frac{9\pi}{8}} \circ t^{\frac{11\pi}{8}} = t^{\frac{7\pi}{8}+\frac{11\pi}{8}} = t^{\frac{\pi}{2}};$ $\quad t_9 \circ t_{12} = t^{\frac{9\pi}{8}} \circ t^{\frac{3\pi}{2}} = t^{\frac{9\pi}{8}+\frac{3\pi}{2}} = t^{\frac{5\pi}{8}}$
$t_9 \circ t_{13} = t^{\frac{9\pi}{8}} \circ t^{\frac{13\pi}{8}} = t^{\frac{9\pi}{8}+\frac{13\pi}{8}} = t^{\frac{3\pi}{4}};$ $\quad t_9 \circ t_{14} = t^{\frac{9\pi}{8}} \circ t^{\frac{7\pi}{4}} = t^{\frac{9\pi}{8}+\frac{7\pi}{4}} = t^{\frac{7\pi}{8}}$
$t_9 \circ t_{15} = t^{\frac{9\pi}{8}} \circ t^{\frac{15\pi}{8}} = t^{\frac{9\pi}{8}+\frac{15\pi}{8}} = t^0.$

$t_{10}\circ t_{10} = t^{\frac{5\pi}{4}}\circ t^{\frac{5\pi}{4}} = t^{\frac{5\pi}{4}+\frac{5\pi}{4}} = t^{\frac{\pi}{2}}$
$t_{10}\circ t_{11} = t^{\frac{5\pi}{4}}\circ t^{\frac{11\pi}{8}} = t^{\frac{5\pi}{4}+\frac{11\pi}{8}} = t^{\frac{5\pi}{8}};\quad t_{10}\circ t_{12} = t^{\frac{5\pi}{4}}\circ t^{\frac{3\pi}{2}} = t^{\frac{5\pi}{4}+\frac{3\pi}{2}} = t^{\frac{3\pi}{4}}$
$t_{10}\circ t_{13} = t^{\frac{5\pi}{4}}\circ t^{\frac{13\pi}{8}} = t^{\frac{5\pi}{4}+\frac{13\pi}{8}} = t^{\frac{7\pi}{8}};\quad t_{10}\circ t_{14} = t^{\frac{5\pi}{4}}\circ t^{\frac{7\pi}{4}} = t^{\frac{5\pi}{4}+\frac{7\pi}{8}} = t^{\frac{\pi}{8}}$
$t_{10}\circ t_{15} = t^{\frac{5\pi}{4}}\circ t^{\frac{15\pi}{8}} = t^{\frac{5\pi}{4}+\frac{15\pi}{8}} = t^{\frac{5\pi}{8}}.$

$t_{11}\circ t_{11} = t^{\frac{11\pi}{8}}\circ t^{\frac{11\pi}{8}} = t^{\frac{11\pi}{8}+\frac{11\pi}{8}} = t^{\frac{3\pi}{4}};\quad t_{11}\circ t_{12} = t^{\frac{11\pi}{8}}\circ t^{\frac{3\pi}{2}} = t^{\frac{11\pi}{8}+\frac{3\pi}{2}} = t^{\frac{7\pi}{8}}$
$t_{11}\circ t_{13} = t^{\frac{11\pi}{8}}\circ t^{\frac{13\pi}{8}} = t^{\frac{11\pi}{8}+\frac{13\pi}{8}} = t^{\pi};\quad t_{11}\circ t_{14} = t^{\frac{11\pi}{8}}\circ t^{\frac{7\pi}{4}} = t^{\frac{11\pi}{8}+\frac{7\pi}{4}} = t^{\frac{9\pi}{8}}$
$t_{11}\circ t_{15} = t^{\frac{11\pi}{8}}\circ t^{\frac{15\pi}{8}} = t^{\frac{11\pi}{8}+\frac{15\pi}{8}} = t^{\frac{5\pi}{4}}.$

$t_{12}\circ t_{12} = t^{\frac{3\pi}{2}}\circ t^{\frac{3\pi}{2}} = t^{\frac{3\pi}{2}+\frac{3\pi}{2}} = t^{\pi}$
$t_{12}\circ t_{13} = t^{\frac{3\pi}{2}}\circ t^{\frac{13\pi}{8}} = t^{\frac{3\pi}{2}+\frac{13\pi}{8}} = t^{\frac{9\pi}{8}};\quad t_{12}\circ t_{14} = t^{\frac{3\pi}{2}}\circ t^{\frac{7\pi}{4}} = t^{\frac{3\pi}{2}+\frac{7\pi}{8}} = t^{\frac{5\pi}{4}}$
$t_{12}\circ t_{15} = t^{\frac{3\pi}{2}}\circ t^{\frac{15\pi}{8}} = t^{\frac{3\pi}{2}+\frac{15\pi}{8}} = t^{\frac{11\pi}{8}}.$

$t_{13}\circ t_{13} = t^{\frac{13\pi}{8}}\circ t^{\frac{13\pi}{8}} = t^{\frac{11\pi}{8}+\frac{13\pi}{8}} = t^{\frac{5\pi}{4}};\quad t_{13}\circ t_{14} = t^{\frac{13\pi}{8}}\circ t^{\frac{7\pi}{4}} = t^{\frac{13\pi}{8}+\frac{7\pi}{4}} = t^{\frac{11\pi}{8}}$
$t_{13}\circ t_{15} = t^{\frac{13\pi}{8}}\circ t^{\frac{15\pi}{8}} = t^{\frac{13\pi}{8}+\frac{15\pi}{8}} = t^{\frac{3\pi}{2}}.$
$t_{14}\circ t_{14} = t^{\frac{7\pi}{4}}\circ t^{\frac{7\pi}{4}} = t^{\frac{7\pi}{4}+\frac{7\pi}{4}} = t^{\frac{3\pi}{2}};\quad t_{14}\circ t_{15} = t^{\frac{7\pi}{4}}\circ t^{\frac{15\pi}{8}} = t^{\frac{7\pi}{4}+\frac{15\pi}{8}} = t^{\frac{13\pi}{8}}$
$t_{15}\circ t_{15} = t^{\frac{15\pi}{8}}\circ t^{\frac{15\pi}{8}} = t^{\frac{15\pi}{8}+\frac{15\pi}{8}} = t^{\frac{7\pi}{4}}.$

2.8 The Cayley table of the G_T group and the subgroups showing the binary operation and the closure properties.

$\circ$	t^0	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$
t^0	t^0	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$
$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$
$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$
$t^{\frac{3\pi}{8}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$
$t^{\frac{\pi}{2}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$
$t^{\frac{5\pi}{8}}$	$t^{\frac{5\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$
$t^{\frac{3\pi}{4}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$
$t^{\frac{7\pi}{8}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$
t^{π}	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$
$t^{\frac{9\pi}{8}}$	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}
$t^{\frac{5\pi}{4}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$
$t^{\frac{11\pi}{8}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$
$t^{\frac{3\pi}{2}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$
$t^{\frac{13\pi}{8}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$
$t^{\frac{7\pi}{4}}$	$t^{\frac{7\pi}{4}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$
$t^{\frac{15\pi}{8}}$	$t^{\frac{15\pi}{8}}$	$t^{2\pi}$	$t^{\frac{\pi}{8}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{3\pi}{8}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{4}}$	$t^{\frac{7\pi}{8}}$	t^{π}	$t^{\frac{9\pi}{8}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{11\pi}{8}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{13\pi}{8}}$	$t^{\frac{7\pi}{4}}$

2.9 G_{Tsub1}

$\circ$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$
$t^{2\pi}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$
$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$
$t^{\frac{\pi}{2}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$
$t^{\frac{3\pi}{4}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$
t^{π}	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$
$t^{\frac{5\pi}{4}}$	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}
$t^{\frac{3\pi}{2}}$	$t^{\frac{3\pi}{2}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$
$t^{\frac{7\pi}{4}}$	$t^{\frac{7\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{4}}$	$t^{\frac{\pi}{2}}$	$t^{\frac{3\pi}{4}}$	t^{π}	$t^{\frac{5\pi}{4}}$	$t^{\frac{3\pi}{2}}$

2.10 G_{Tsub2}

$\circ$	$t^{2\pi}$	$t^{\frac{\pi}{2}}$	t^{π}	$t^{\frac{3\pi}{2}}$
$t^{2\pi}$	$t^{2\pi}$	$t^{\frac{\pi}{2}}$	t^{π}	$t^{\frac{3\pi}{2}}$
$t^{\frac{\pi}{2}}$	$t^{\frac{\pi}{2}}$	t^{π}	$t^{\frac{3\pi}{2}}$	$t^{2\pi}$
t^{π}	t^{π}	$t^{\frac{3\pi}{4}}$	$t^{2\pi}$	$t^{\frac{\pi}{2}}$
$t^{\frac{3\pi}{2}}$	$t^{\frac{3\pi}{2}}$	$t^{2\pi}$	$t^{\frac{\pi}{2}}$	t^{π}

2.11 G_{Tsub3}

$\circ$	$t^{2\pi}$	t^{π}
$t^{2\pi}$	$t^{2\pi}$	t^{π}
t^{π}	t^{π}	$t^{2\pi}$

2.12 The inverses

The inverse of any $t^{\frac{n_i\pi}{8}} \in G_T$ is denoted by $\hat{t}^{\frac{n_i\pi}{8}}$, call it say $t^{\frac{n_j\pi}{8}}$, defined by $t^{2\pi-\frac{n_j\pi}{8}} = t^{\frac{(16-n_i)\pi}{8}} = t^{\frac{n_j\pi}{8}} \in G_T.$

Remark 2.12.1 t_0 remains the left and right identity for the group.

2.13 Associative property

Take arbitrary elements, $t^{\frac{n_i\pi}{8}}, t^{\frac{n_j\pi}{8}}$ and $t^{\frac{n_k\pi}{8}} \in G_T$, to obtain the associative property of the group:

$$\left(t^{\frac{n_i\pi}{8}} \circ t^{\frac{n_j\pi}{8}}\right) \circ t^{\frac{n_k\pi}{8}} = t^{\left(\frac{n_i\pi}{8}+\frac{n_j\pi}{8}\right)+\frac{n_k\pi}{8}}$$

$$t^{\left(\frac{n_i\pi}{8}+\frac{n_j\pi}{8}+\frac{n_k\pi}{8}\right)} = t^{\frac{n_i\pi}{8}+\left(\frac{n_j\pi}{8}+\frac{n_k\pi}{8}\right)}$$

$$= t^{\frac{n_i\pi}{8}} \circ \left(t^{\frac{n_j\pi}{8}} \circ t^{\frac{n_k\pi}{8}}\right)$$

The structure of G_T is therefore an abelian group, it is therefore amenable. It has the following properties:

a. It is closed under the binary operation.
b. It is associative.
c. It has an identity element.
d. Each element has a unique left and right inverse.
e. The operation of the group is commutative.

2.14 The structural Implications of the $T - GROUP$

The G_T has many features, among the features are:

a. It is a locally compact group.
b. Its points have closures.
c. It is Abelian group.
d. The loops in a class are homotopic (common end points).
e. The identity has a compact neighbourhood.

Theorem 2.14.1 [1] For a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if G is amenable. It follows that $L^1(G_T)$.

3. Non-amenability of $Free - group$ and the Fourier Transform on G_T Grooup

Proposition 3.1 $\mathbb{F}_2$ can be paradoxically decomposed: $\mathbb{F}_2$ is not amenable.

Proof Let $S(a)$ be the set of all strings that start with a and define $S(a^{-1}), S(b),\ S(b^{-1})$ similarly. $S(a)$, $S(a^{-1}),\ S(b)$, $S(b^{-1}) \in 2^{\mathbb{F}_2}$. We need to show that $\mu(aS(a) \cup S(a^{-1})) \nless 1$ where the measure $\mu(\mathbb{F}_2) = 1$. Clearly $\mathbb{F}_2 = e \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$.

The notation $aS(a^{-1})$ means take all the elements in $S(a^{-1})$ and concatenate (a binary operation on $\mathbb{F}_2$) them on the left with a.

$aa^{-1}b \in aS(a^{-1})$ reduces to $S(b)$

$aa^{-1}b^{-1} \in aS(a^{-1})$ reduces to $S(b^{-1})$ which because of the rule that a must not appear next to a^{-1}, reduces to the string b. In this way, $aS(a^{-1})$ contains all the strings that start with $b\ and\ b^{-1}$ and of course it contains all the strings that start with a^{-1}.

Also $\mathbb{F}_2 = aS(a^{-1}) \cup S(a)$ and $\mathbb{F}_2 = bS(b^{-1}) \cup S(b)$

This implies that that $\mu(\ S(a^{-1})) \neq \mu(\ aS(a^{-1}))$. And of course

$\mu(\ S(a^{-1})) < \mu(aS(a^{-1}))$.

Since $\mu\mathbb{F}_2 = aS(a^{-1}) \cup S(a) \cup e = 1$

$\Longrightarrow \mathbb{F}_2 = \mathbb{F}_2 + \mathbb{F}_2$, each of the subsets makes one copy of $\mathbb{F}_2$, [10].

Proposition 3.2 Fourier Transform decomposes convolution in $L^1(G)$to point wise multiplication in $A(G)$ [9].

$\mathcal{F}(f * g)(t) = \mathcal{F}(f) \cdot \mathcal{F}(g)$.

$Proof$

The convolution on $f, g \in L^1(G)$ is

$(f * g)(t) = \int_G g(t-x)f(x)dx, \quad similarly,\ h(t) = \int_G g(t-x)f(x)dx.$

Combining $(f * g)(t) = \int_G g(t-x)f(x)dx$

and

$$\mathcal{F}\{f(t)\} = \hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi st} f(t)dt$$

$\mathcal{F}\{(f * g)(t)\} = \int_G e^{-2\pi st}(f * g)(t)dt, \quad f, g \in G.$

With the necessary substitution we have:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi s(t+x)} g(t)dt\right) f(x)dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi s(t+x)} g(t)f(x)dt\,dx$$
$$= \int_{-\infty}^{\infty} e^{-2\pi st} g(t)dt \cdot \int_{-\infty}^{\infty} e^{-2\pi s(t+x)} f(x)dx$$
$$= \mathcal{F}(g)(t)\ \cdot\ \mathcal{F}(f)(x) = \hat{g}(s) \cdot\ \hat{f}(x).$$

Proposition 3.3 The Fourier algebra $A(G_T)$ is obtained by Fourier transform on G_T.

$Proof$

Let $f, g, h \in L^1(G_T)$ defined $(f * g)(t) = h(t)$ by

$(f * g)(t) = \int_G g(t-x)f(x)dx.$

For $t, x, u \in G_T$, we have $u = s + t$.

Again $h(u) = \int_G g(u-x)f(x)dx.$

The Fourier transform on $h(u)$, denoted by
$\mathcal{F}\{h(u)\} = \hat{h}(s) = \int_G e^{-2\pi isu} h(u)du$
$= \int_G e^{-2\pi isu}(f * g)(t)du$
$= \int_G e^{-2\pi isu} \int_{-\infty}^{\infty} g(u-x)f(x)dx\, du$
$= \int_G \int_G e^{-2\pi isu} g(u - xdu)f(x)dxdu$
$= \int_G \left(\int_G e^{-2\pi isu} g(u - xdu)du\right)f(x)dx.$
Let $u = s + t, t = u - x,\ then\ du = dt,$ and let $t, x \in G_T$ explicitly becomes
$t = t^{\frac{n_i\pi}{8}}\ \ and\ \ x = t^{\frac{n_j\pi}{8}}$, then the above equation becomes
$= \int_G \left(\int_G e^{-2\pi ist^{\frac{n_j\pi}{8}} \circ\, t^{\frac{n_i\pi}{8}}} g(t)dt\right) f(x)dx.$
$= \int_G \int_G e^{-2\pi ist^{\frac{n_j\pi}{8}} \circ\, t^{\frac{n_i\pi}{8}}} g(t)f(x)dtdx$
$= \int_G \int_G e^{-2\pi ist^{\frac{n_j\pi}{8}+\frac{n_i\pi}{8}}} g(t)f(x)dtdx$
$= \int_G \int_G e^{-2\pi ist^{\frac{n_i\pi}{8}}} \cdot e^{-2\pi ist^{\frac{n_j\pi}{8}}} g(t)f(x)dtdx$
$= \int_G e^{-2\pi ist^{\frac{n_i\pi}{8}}} g(t)dt \cdot \int_G e^{-2\pi ist^{\frac{n_j\pi}{8}}} f(x)dx$
$= \int_G e^{-2\pi ist} g(t)dt \cdot \int_G e^{-2\pi isx} f(x)dx$
$= \mathcal{F}(g)(t) \cdot \mathcal{F}(f)(x) = \hat{g}(s) \cdot \hat{f}(x).$
For all $f \in L^1(G_T)$, the function $\hat{f}$ defined on the dual of G_T is the Fourier transform of $f \in L^1(G_T)$. The set of all functions so obtained will be denoted by $A(G_T)$, the Fourier algebra with the sup norm from the group algebra and point wise multiplication. Hence there exists a Fourier algebra $A(G_T)$ from G_T.
The map $f \longrightarrow \hat{f}(s)$ is a complex homomorphism of $L^1(G_T)$ and it is not identically zero. Conversely every non-zero complex homomorphism of $L^1(G_T)$ is obtained in this way, and distinct character induces distinct homomorphism. For all $f \in L^1(G_T)$, the function $\hat{f}$ defined on $\hat{G}_T$ is the Fourier transform of $f \in L^1(G_T)$, with whatever norm with $L^1(G_T)$.
Proposition 3.4 The Fourier algebra $A(G_T)$ is amenable.
$Proof$ Need to show that $A(G_T)$ contains a bounded identity [11]. From [8] a Banach algebra is amenable if and only if its underlying group is amenable. The group G_T has a bounded identity $t^{2\pi}$. The group algebra $f \in L^1(G_T)$, and $\hat{f}t^{2\pi}$
$\frac{-\pi}{16} < t_0 < \frac{\pi}{16}$
$t^{2\pi} = t_0\ is\ bounded, f$ is also bounded. The Fourier transform $\mathcal{F}(f)$ remains the boundedness of the points. $A(G_T)$ contains a bounded identity, $A(G_T)$ is amenable.

Conclusion
Other aspects of amenability, operator amenability [12] and [13], weakly amenability, weakly amenability and ideally can be studied under this algebra as in [14]. This Fourier algebra is the point wise multiplicative member of $C_0(L^1(G_T))$, which is a sub-algebra of Fourier Stieltjes algebra. Initial works done on fundamental groups and other group properties that give rise to amenability can be seen in [15] and [16].

REFERENCES
[1] Johnson, B. E., (1972), *Cohomology in Banach algebra*. Memoirs of the Amer. Maths. Soc. No. 127.
[2] Eymard. P. (1964), *Lalg`ebre de Fourier dun groupe localement compact*. Bull. Soc. Math. France, 92:181236 Soc. Math. France, 92:181-236.
[3] Forrest, B. and Peter, W. (2016), *Cohomology and the operator space structure of the Fourier algebra and its second dual.* Indiana Univ. Math. J., 50(3):1217-1240.
[4] Ghahramani F. and Loy R.J. (2004), *Generalized notions of amenability*. Journal of Functional Analysis 208, 229-260.
[5] Mewomo, O. T. (2011), *Various notions of amenability of Banach algebras*. Expositiones Mathematicae 29, 283 - 299.
[6] Helemskii, Ya. A. (1989), *The homology of Banach and topological algebras*. ISBN 978-94-009.2354.6 Kluwer.

[7] Nemati, M (2017), *On generalized notion of amenability and operator homology of the Fourier algebra.* The Quarterly Journal of athematics, Oxford University press. DOI: 10. 1093/qmath/hax005, 68(3)-: 781-789.
[8] Hatcher A. (2002), Algebraic Topology. Cambridge Univ. Press. Amer. Maths. Soc. No. 127.
[9] Brad, O. (2005), Fourier Transform and its Application. Stanford University Press.
[10] Johnson, B. E.(1994), *Non-amenability of the Fourier algebra of a compact group*. J. London Math. Soc. (2), 50(2): 361374.
[11] Losert V. (1984)b, *Properties of the Fourier algebra of a compact group*. J, London Math. Soc., 91, 347-345.
[12] Masamichi Takesaki. (1979), Theory of operator algebras. Springer-Verlag, NewYork.
[13] Miad, M. S. (2016), *Operator amenability for Fourier algebra.* Proceedings of the Field Institute for Research in Mathematics, Vol. 4, 12-26.
[14] Ogunsola, O. J. and Daniel, I. E. (2018), *Pseudo-amenability and pseudo-contractibility of restricted semigroup algebra.* Annales University Paedagogicae Cracoviensis Studia mathematica. XVII 90 – 102.
[15] Moore, C. C. (1972), *Groups with finite dimensional irreducible representations*. Trans. Amer. Math. Soc., 166:401410.
[16] Higgins, P.J. (1976) *The fundamental groupoid of a graph of groups*. J. London Math. Soc. (2) {13}, (1976) 145–149.