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INVESTIGATING THE SOLUBILITY OF WREATH PRODUCTS GROUP OF DEGREE 4P USING NUMERICAL APPROACH

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Abstract:

Let p be a prime number (p = {3, 5, 7, 11, …}) and G a finite permutation group of degree 4p, generated via wreath products of pairs of permutation groups. We, in this paper discuss the solubility of G using numerical approach. The groups, algorithms and programming (GAP) is used to generate G and also validate our results.

Keywords*: Permutation Group, Solubility, Wreath Products, p-Groups and Sylow p-subgroup.*

1. Introduction

The Wreath product of two permutation groups *C* and *D* denoted by $W = CwrD$ is the semi-direct product of *P* by *D*, so that, $W = \{(f, d) | f \in P, d \in D\}$, with multiplication in *W* defined as $(f_1, d_1)(f_2, d_2) = ((f_1, f_2 d_1^{-1})\ (d_1, d_2))\ \ \forall\ \ f_1, f_2 \in P\ and\ d_1, d_2 \in D$ is a special form of permutation group. When the nature of the Wreath products groups is well Understood it facilitates comprehension of certain types of subgroups of the symmetric groups.

According to [1], if a group G has a sequence of subgroups, say

$G = H_n \supset H_{n-1} \supset \cdot \cdot \cdot \supset H_1 \supset H_0 = \{e\}$,

where each subgroup H_i is normal in H_{i+1} and each of the factor groups H_{i+1}/H_i is abelian, then G is a soluble group. Solvable groups in addition to allowing us to distinguish between certain classes of groups, turn out to be very key to the study of solutions to polynomial equations.

Let p be an arbitrary odd prime number. We intend to obtain more detailed descriptions of the unique structure of Wreath product (permutation) groups of degree 3p that are not p-groups and investigate their solubility using numerical approach.

There are some recent results on the solubility of permutations groups including the following:

Thanos [2] proved that $If\ |G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime number is solvable.

Bello *et al* [3] used the concept of p-groups to construct locally solvable groups using two permutation groups by wreath product.

Gandi *et al.* [4] investigated solvable and Nilpotent concepts on dihedral groups of an even degree regular polygon.

The results from the above papers and other findings on group concepts from the works of the authors in [5], [6] and [7] will be used as valuable references towards achieving our desired objectives.

In Section 2 we give some basic definitions, concepts and results which are required here. In Section 3 we applied groups, algorithms and programming (GAP) [8] to generate and discuss solubility of permutation groups of degree 4p (p = 3, 5, 7,11, ..). The main result of this paper covering all the permutation groups of degree 4p is stated in Section 4.

2. Materials and Methods

2.1 p-group

A finite group G is said to be a p-group if its order is a power of p, where p is prime. A subgroup *H* of a group G is a p-subgroup if it (*H*) is a p-group. By Lagrange's theorem, this is equivalent to the requirement that the order of *H* be a power of p for all $H \in G$.

2.2 Stabilizer

Any subset of G which fix a specified element α is called the stabilizer of α in G and is denoted by $G_\alpha \coloneqq \{x \in G \mid \alpha^x = \alpha\}$.

2.3 Orbit

When a group G acts on a set Ω, a typical point α is moved by elements of G to various other points. The set of these images is called the orbit of α under G, and we denote it by $\alpha^G \coloneqq \{\ \alpha^x |\ x \in G\ \}$.

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2.4 Wreath product [9]

The wreath product of two permutation groups C by D denoted by $W = CwrD$ is the semi-direct product of P by D, so that,

$W = \{(f,d) | f \in P, d \in D\}$,

with multiplication in W defined as

$(f_1, d_1)(f_2, d_2) = ((f_1, f_2 d_1^{-1})(d_1, d_2)) \quad \forall \quad f_1, f_2 \in P \text{ and } d_1, d_2 \in D$

Henceforth, we write $f\,d$ instead of (f, d) for elements of W.

2.5 Theorem [9]

Let D act on P as $f^{d}(\delta) = f(\delta d^{-1})$ where $f \in P, d \in D$ and $\delta \in \Delta$. Let W be group of all juxtaposed symbols fd, with $f \in P, d \in D$ and multiplication given by $(f_1, d_1)(f_2, d_2) = (f_1 f_2 d_1^{-1})(d_1, d_2)$. Then W is a group referred to as the semi-direct product of P by D with the action as defined

2.6 Theorem

If G is a group then the commutator subgroup G' is a normal subgroup of G and G/G' is abelian. If N happens to be a normal subgroup of the group G, then the factor group G/N is abelian if and only if $G' < N'$.

Proof

Let the mapping $f : G \to G$ be any automorphism of a group G. Then by any homomorphism property $f(aba^{-1}b^{-1}) = f(a) f(b) f(a^{-1}) f(b^{-1}) = f(a) f(b) (f(a))^{-1}(f(b))^{-1} \in G'$. Then every element of G' is a finite product of powers of commutators $aba^{-1}b^{-1}$ (where $a, b \in G$) and so $f(G') < G'$. Let f_a be the automorphism of G given by the conjugation by a. Then $aG'a^{-1} = f_a(G') < G'$. So every conjugate $aG'a^{-1}$ is a subgroup of G' and then G' is a normal subgroup of G. Since all a, b $\in G$, we have $a^{-1}b^{-1} \in G$ and so $(a^{-1})^{-1}(b^{-1})^{-1}ab \in G'$ and so $a^{-1}b^{-1}abG' = G^{-1}$ or $abG' = baG'$. But then by delinition of coset multiplication, $(aG')(bG') = abG' = baG' = (bG')(aG')$ and so coset multiplication is commutative and G/G' is abelian.

2.7 Theorem

A permutation group G is said to be a solvable group if and only if it has a solvable series.

Proof

Suppose G is solvable. Then by the definition of “solvable,” in the derived series of commutator subgroups we have $G^{(n)} =$ (1), for some n $\in N$. By Theorem 2.6, in the series $G > G^{(1)} > G^{(2)} > ... > G^{(n)} = (1)$, we have that $G^{(i+1)}$ is normal in $G^{(i)}$ and $G^{(i)}/G^{(i+1)}$) is abelian. Clearly, each subgroup is normal in the preceding subgroup and it follows that G is solvable since the factor groups are abelian.

Now suppose $G = G_0 > G_l > ... > G_n = (l)$ is s solvable series. Then G_i/G_{i+1} is abelian (by definition of solvable series) for $0 \leq i \leq n - l$. By theorem 2.6, $G_{i+l} > (G_i)'$ for $0 \leq i \leq n - 1$. Since in the derived series of commutator subgroups we have $G > G^{(1)} > G^{(2)} > ... > G^{(n)}$, then

$G_1 > G_0' = G' = G^{(1)}$

$G_2 > G_1' = (G^{(1)})' = G^{(2)}$

$G_3 > G_2' = (G^{(2)})' = G^{(3)}$

$G_{i+1} > G'_i = (G^{(i)})' = G^{(i+1)}$

$G_n > G'_{n+1} = (G^{(n-1)})' = G^{(n)}$

But $G_n = (1)$ so it must be that $G^{(n)} = (1)$ and G is solvable.

2.8 Sylow’s Theorems [10]

Let G be a finite group. If $|G| = p^r m$ and $(p, m) = 1$, then

1. There is at least one Sylow p-subgroup H of G.
2. If B is any p-subgroup of G, then $B \subseteq x^{-1}Hx$ for some $x \in G$.
3. If K is any Sylow p-subgroup of G, $K = g^{-1}Hg$ for some $g \in G$
4. If n_p is the number of Sylow p-subgroups of G, then n_p divides m and $n_p \equiv 1 \bmod p$.

2.9 Corollary

A Sylow p-subgroup of a group G is normal if and only if it is unique.

Proof:

Suppose that a Sylow p-subgroup H of a group G is unique. Since all Sylow p-subgroups are conjugate to H, the uniqueness of H implies that $H = g^{-1}Hg$ for all $g \in G$, that is H is normal in G. Conversely, suppose H is normal in G, then $g^{-1}Hg =$ H for all $g \in G$. Let k be any other Sylow p-subgroup of G, then $K = g^{-1}Hg$ for some $g \in G$. that is $K =$ H.

3. Wreath product group of degree 4p (p = 3, 5, 7, 11, .. .)

We shall now construct some p-groups by means of wreath product of two permutations.

3.1 Consider the permutation groups C_1 and D_1

C_1 = {(1),(12345),(13524),(14253),(15432)} and
D_1 = {(1),(6,7)}
acting on the sets Ω_1 = {1,2,3,4,5} and Δ_1 = {6,7} respectively.
Let $P_1 = C_1^{\Delta_1} = \{f:\Delta_1 \rightarrow C_1\}$. Then $|P_1| = |C_1|^{|\Delta_1|} = 5^2 = 25$
The mappings in P_1 are as list below.
f_1: 6→(1), 7→(1)
f_2: 6→(12345), 7→(12345)
f_3: 6→(13524), 7→(13524),
f_4: 6→ (14253), 7→ (14253)
f_5: 6→ (15432), 7→ (15432)
f_6: 6→ (1), 7→ (12345)
f_7: 6→ (1), 7→ (13524)
f_8: 6→ (1), 7→ (14253)
f_9: 6→ (1), 7→(15432)
f_{10}: 6→ (12345), 7→(1)
f_{11}: 6→ (12345), 7→(13524)
f_{12}: 6→ (12345), 7→(14253)
f_{13}: 6→ (12345), 7→(15432)
f_{14}: 6→ (13524), 7→(1)
f_{15}: 6→ (13524), 7→(12345)
f_{16}: 6→ (13524), 7→(14253)
f_{17}: 6→ (13524), 7→(15432)
f_{18}: 6→ (14253), 7→(1)
f_{19}: 6→ (14253), 7→(12345)
f_{20}: 6→ (14253), 7→(13524)
f_{21}: 6→ (14253), 7→(15432)
f_{22}: 6→ (15432), 7→(1)
f_{23}: 6→ (15432), 7→(12345)
f_{24}: 6→ (15432), 7→(13524)
f_{25}: 6→ (15432), 7→(14253)
We can easily verify that P is a group with respect to the operations $(f_1, f_2)(\delta) = f_1(\delta_1) f_1(\delta_1)$, where $\delta_1 \epsilon \Delta_1$
We recall the definition of the action of D_1 on P as $f^{d}(\delta_1) = f(\delta_1 d^{-1})$ where $f \epsilon P$, $d \epsilon D_1$ and $\delta_1 \epsilon \Delta_1$, then D_1 acts on P as a groups.
We also recall the definition $W = C_1\ wr\ D_1$, the semi-direct product of P by D_1 in that order; i.e. $W = \{(f, d) \mid f \epsilon P, \delta_1 \epsilon \Delta_1\}$
Now, W is a group with respect to the operation;
$(f_1, d_1)(f_2, d_2) = (f_1, f_2^{d_1^{-1}})(d_1, d_2)$, and
accordingly, $d_1 = (1)$, $d_2 = (6,7)$.
Then the elements of W_1 are
(f_1,d_1), (f_2,d_1), (f_3,d_1), (f_4,d_1), (f_5,d_1), (f_6,d_1), (f_7,d_1), (f_8,d_1), (f_9,d_1), (f_{10},d_1), (f_{11},d_1), (f_{12},d_1),
(f_{13},d_1), (f_{14},d_1), (f_{15},d_1), (f_{16},d_1), (f_{17},d_1), (f_{18},d_1), (f_{19},d_1), (f_{20},d_1), (f_{21},d_1), (f_{22},d_1), (f_{23},d_1) (f_{24},d_1), (f_{25},d_1), (f_1,d_2), (f_2,d_2), (f_3,d_2), (f_4,d_2), (f_5,d_2), (f_6,d_2), (f_7,d_2), (f_8,d_2), (f_9,d_2), (f_{10},d_2), (f_{11},d_2), (f_{12},d_2), (f_{13},d_2), (f_{14},d_2), (f_{15},d_2), (f_{16},d_2), (f_{17},d_2), (f_{18},d_2), (f_{19},d_2), (f_{20},d_2), (f_{21},d_2), (f_{22},d_2), (f_{23},d_2), (f_{24},d_2), (f_{25},d_2)
Now, define action of W_1 on $\Omega_1 \times \Delta_1$ as
$(\beta, \delta_1)fd = (\beta f(\delta), d\delta)$ where $\beta \epsilon \Omega_1$ and $\delta_1 \epsilon \Delta_1$
Further, $\Omega_1 \times \Delta_1$ = {(1,6), (1,7), (2,6), (2,7), (3,6), (3,7), (4,6), (4,7), (5,6), (5,7)}
We obtain the following permutation by action of W_1 on $\Omega_1 \times \Delta_1$
$(1,6)f_1d_1 = (1f_1(6), d_1) = (1(1), 6(1)) = (1,6)$
$(1,7)f_1d_1 = (1f_1(7), d_1) = (1(1), 7(1)) = (1,7)$
$(2,6)f_1d_1 = (2f_1(6), d_1) = (2(1), 6(1)) = (2,6)$
$(2,7)f_1d_1 = (2f_1(7), d_1) = (2(1), 7(1)) = (2,7)$
$(3,6)f_1d_1 = (3f_1(6), d_1) = (3(1), 6(1)) = (3,6)$

$(3,7)f_1d_1 = (3f_1(7), d_1) = (3(1), 7(1)) = (3,7)$
$(4,6)f_1d_1 = (4f_1(6), d_1) = (4(1), 6(1)) = (4,6)$
$(4,7)f_1d_1 = (4f_1(7), d_1) = (4(1), 7(1)) = (4,7)$
$(5,6)f_1d_1 = (5f_1(6), d_1) = (5(1), 6(1)) = (5,6)$
$(5,7)f_1d_1 = (5f_1(7), d_1) = (5(1), 7(1)) = (5,7)$
And in summary,

$$(\Omega_1 \times \Delta_1)f_1d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,6)\ (1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_2d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 2,6)\ (2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_3d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,6)\ (3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_4d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,6)\ (4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_5d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,6)\ (5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_6d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,6)\ (2,7)(2,6)(3,7)(3,6)(4,7)(4,6)(5,7)(5,6)(1,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_7d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,6)\ (3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_3d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,7)\ (3,6)(4,7)(4,6)(5,7)(5,6)(1,7)(1,6)(2,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_4d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,7)\ (4,6)(5,7)(5,6)(1,7)(1,6)(2,7)(2,6)(3,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_5d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,7)\ (5,6)(1,7)(1,6)(2,7)(2,6)(3,7)(3,6)(4,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_6d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,6)\ (1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_7d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,7)\ (3,6)(2,7)(4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_8d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 1,7)\ (4,6)(2,7)(5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_9d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,7)(5,6)(2,7)(1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{10}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 2,7)\ (1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{11}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 2,7)\ (3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{12}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 2,7)\ (4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{13}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 2,7)\ (5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{14}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,7)\ (1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{15}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,7)\ (2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{16}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,7)\ (4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{17}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 3,7)\ (5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{18}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,7)\ (1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6)(3,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)f_{19}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,7)\ (2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6)(3,7)(1,6) \end{pmatrix}$$

$(\Omega_1 \text{ x } \Delta_1)f_{20}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,7)\ (3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6)(3,7)(2,6) \end{pmatrix}$

$(\Omega_1 \text{ x } \Delta_1)f_{21}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 4,7)\ (5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6) \end{pmatrix}$

$(\Omega_1 \text{ x } \Delta_1)f_{22}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,7)\ (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6) \end{pmatrix}$

$(\Omega_1 \text{ x } \Delta_1)f_{23}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,7)\ (2,6)(1,7)(3,6)(2,7)(4,6)(3,7)(5,6)(4,7)(1,6) \end{pmatrix}$

$(\Omega_1 \text{ x } \Delta_1)f_{24}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,7)\ (3,6)(1,7)(4,6)(2,7)(5,6)(3,7)(1,6)(4,7)(2,6) \end{pmatrix}$

$(\Omega_1 \text{ x } \Delta_1)f_{25}d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ 5,7)\ (4,6)(1,7)(5,6)(2,7)(1,6)(3,7)(2,6)(4,7)(3,6) \end{pmatrix}$

Renaming the symbols as
(1,6)→1, (1,7)→2, (2,6)→3, (2,7)→ 4, (3,6)→5, (3,7)→6, (4,6)→ 7, (4,7)→8, (5,6)→9, (5,7)→10,
The permutations in cyclic form are as follows.
G_1 = {(1), (6,7,8,9,10), (6,8,10,7,9), (6,9,7,10,8), (6,10,9,8,7), (1,2,3,4,5), (1,2,3,4,5) (6,7,8,9,10), (1,2,3,4,5)(6,8,10,7,9), (1,2,3,4,5)(6,9,7,10,8), (1,2,3,4,5)(6,10,9,8,7), (1,3,5,2,4), (1,3,5,2,4) (6,7,8,9,10), (1,3,5,2,4)(6,8,10,7,9), (1,3,5,2,4)(6,9,7,10,8), (1,3,5,2,4)(6,10,9,8,7), (1,4,2,5,3), (1,4,2,5,3)(6,7,8,9,10), (1,4,2,5,3)(6,8,10,7,9), (1,4,2,5,3)(6,9,7,10,8), (1,4,2,5,3)(6,10,9,8,7), (1,5,4,3,2), (1,5,4,3,2)(6,7,8,9,10), (1,5,4,3,2)(6,8,10,7,9), (1,5,4,3,2)(6,9,7,10,8), (1,5,4,3,2) (6,10,9,8,7), (1,6)(2,7)(3,8)(4,9)(5,10), (1,6,2,7,3,8,4,9,5,10), (1,6,3,8,5,10,2,7,4,9), (1,6,4,9,2,7,5,10,3,8), (1,6,5,10,4,9,3,8,2,7), (1,7,2,8,3,9,4,10,5,6), (1,7,3,9,5,6,2,8,4,10), (1,7,4,10,2,8,5,6,3,9), (1,7,5,6,4,10,3,9,2,8), (1,7)(2,8)(3,9)(4,10)(5,6), (1,8,3,10,5,7,2,9,4,6), (1,8,4,6,2,9,5,7,3,10), (1,8,5,7,4,6,3,10,2,9), (1,8)(2,9)(3,10)(4,6)(5,7), (1,8,2,9,3,10,4,6,5,7), (1,9,4,7,2,10,5,8,3,6), (1,9,5,8,4,7,3,6,2,10), (1,9)(2,10)(3,6)(4,7)(5,8), (1,9,2,10,3,6,4,7,5,8), (1,9,3,6,5,8,2,10,4,7), (1,10,5,9,4,8,3,7,2,6), (1,10)(2,6)(3,7)(4,8)(5,9), (1,10,2,6,3,7,4,8,5,9), (1,10,3,7,5,9,2,6,4,8), (1,10,4,8,2,6,5,9,3,7)}
The degree of the wreath product (W_1) = $|C_1|$ x $|D_1|$ = 10, while the order is given by
$|W_1| = |C_1|^{|\Delta 1|} \text{ x } |D_1| = 5^2 \text{ x } 2 = 50$

3.2 Consider the permutation groups C_2 and D_2

Let C_6 be a group of degree 6 and D_6 a group of degree 2 acting on the sets $\Omega_6 = \{1,2,3,4,5,6\}$ and $\Delta_6 = \{7,8\}$ respectively. Let $P_6 = C_6^{\Delta_6} = \{f{:}\Delta_6 \rightarrow C_3\}$. Then $|P_6| = |C_6|^{|\Delta 6|} = 6^2 = 36$. Then Wreath product $W_2 = G_2$ is soluble.

Proof:
After following the same procedure as in 3.1, we obtained the permutations group G_2 with order $|W_6| = |C_6|^{|\Delta 6|} \text{ x } |D_6| = 72 = 2^3 3^2$
G_2 has Sylow 2-subgroups of order 8 and large number of Sylow 3-subgroups of order 9.
This implies that the subgroups of G_2 include: H_1 of order 1, H_2 of order 2, H_3 of 3, H_4 of order 6, H_4 of order 12, H_5 of order 24 and H_6 of order 72.
G_2 is solvable by theorem 2.7, since it has solvable series
$G_6 = H_6 \rhd H_5 \rhd H_4 \rhd H_3 \rhd H_2 \rhd H_1 = (1)$
with cyclic factor groups C_3, C_2, $C_{2,}$ C_2 and C_2, therefore the factor groups are abelian. Thus G_6 solvable.

3.3 Consider the permutation groups C_3 and D_3

Let C_7 be a group of degree 10 and D_7 a group of degree 2 acting on the sets $\Omega_7 = \{1,2,3,4,5,6,7,8,9,10\}$ and $\Delta_7 = \{11,12\}$ respectively. Let $P_7 = C_7^{\Delta_7} = \{f{:}\Delta_7 \rightarrow C_7\}$. Then $|P_7| = |C_7|^{|\Delta 7|} = 10^2 = 100$. Then Wreath product $W_3 = G_3$ is soluble.
Proof:
After following the same procedure as in 3.1, we obtained the permutations group G_3
with order $|G_3| = |C_7|^{|\Delta 7|} \text{ x } |D_7| = 200 = 2^3 5^2$.
G_3 has Sylow 2-subgroups of order 8 and large number of Sylow 5-subgroups of order 25.
This implies that the subgroups of G_3 include: H_1 of order 1, H_2 of order 5, H_3 of order 25, H_4 of order 50, H_4 of order 100 and H_5 of order 200.
G_3 is solvable by theorem 2.7, since it has solvable series
$G_7 = H_6 \rhd H_5 \rhd H_4 \rhd H_3 \rhd H_2 \rhd H_1 = (1)$
with cyclic factor groups C_3, C_2, $C_{2,}$ $C_5$and C_5, therefore the factor groups are abelian. Thus G_6 solvable.
The main results obtain from the investigation of wreath product groups are as follows:

3.4 Proposition

Let G be the Wreath product of pairs arbitrary permutation groups C and D of degree 4p (p $\geq$ 3) and H the Sylow p-subgroup of G. Then (i) H is normal in G and is soluble (ii) G/H is soluble and (iii) G is soluble.

Proof

Now, the order of G that is, $|G| = 8\times p^2$.
Let $n_p(G)$ be the number of Sylow p-subgroups of the group G.
By Sylow theorem 2.8, we have
$n_p(G) \equiv 1 \pmod p$ and $n_p(G) \mid 8$.
It follows from this constraints that we have $n_p(G) = 1$.
Let H be the unique p-Sylow subgroup of G.
The subgroup H is normal in G as is the unique p-sylow subgroup by corollary 2.9 proving (i).
Then consider the subnormal series
$G \triangleright H \triangleright \{e\}$.
Note, e here, is the identity element of the group G.
Then the factor groups G/H, $H/\{e\}$ have order 2^3 and p^2 respectively, and hence these are cyclic groups and in particular abelian by theorem 2.7 proving (ii).
Therefore the group G of order $8p^2$ has a subnormal series whose factor groups are abelian groups, and thus G is a solvable group by theorem 2.7 proving (iii).

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