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NET (SEQUENTIAL) CHARACTERIZATION OF EQUICONTINUITY

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Abstract

Net (Sequential) characterization of continuity is almost as old as the subject of General Topology (GT) itself, but no such characterization is known for equicontinuity. This paper furnishes one for equicontinuity. An application when the codomain space is a pseudo- metric space is given.

Keywords and Phrases uniformity, neighbourhood, equicontinuous set of functions, net convergence.
Subject Classification General Topology **(GT)**

1 LANGUAGE AND NOTATION Our language and notation shall be pretty standard, as found, for example, in [1], [2], [3], [4] and [5]. $\mathbb{R}$ denotes the *real numbers*, $\mathbb{N}$ denotes the *natural numbers* 1, 2, 3, ……..., by $X \neq \emptyset$ or $\emptyset \neq X$ we mean that the set X is a non-empty set, and we indicate by /// the end or absence of a proof.

The reader is assumed familiar with the rudiments of General Topology (**GT**) up to Uniform Spaces — *neighbourhood, neighbour- hood system of a point, local base of neighbourhoods of a point, filter,*

filterbase, net, net convergence, first countable topological space, continuity, uniformity, uniform space, peseudometric space etc, etc.

2 CONTINUITY I *Neighbourhood* Let $X \neq \emptyset$ and τ a topology on X. We call the pair (X, τ) a *topological space*. if $x_0 \in G \in \tau$ and $G \subseteq N \subseteq X$, N is called a *neighbourhood of* x_0. The family of non-empty subsets of X.

$\mathrm{N}_{x_0}(\tau) \equiv \{N : N \text{ is a neighbourhood of } x_0\}$,

is called the *neighbourhood system of* x_0; it is a filter, and so also called the *filter of neighbourhoods of* x_0. A subfamily, $\mathcal{R}_{x_0}(\tau)$, of $\mathrm{N}_{x_0}(\tau)$ such that for every $U \in \mathrm{N}_{x_0}(\tau)$ there exists $V \in \mathcal{R}_{x_0}(\tau)$ included in U, is called a *local base of neighbourhoods of* x_0.

II *Directed Set* Let $I \neq \emptyset$ and $\leq$ a reflective and transitive relation on I. We call $\leq$ *a partial order* (or an *order*) on I [2], and call the pair $(I, \leq)$ a *partially ordered set*. Of course, by a *relation* ρ *on* $X \neq \emptyset$ is simply meant a subset of XxX. If $(a, b) \in \rho$ then we may write $a\rho b$ and say that " a is ρ-related to b" and also that "a precedes b" or that "b dominates a".

Now, continuing, the order $\leq$ on I is said to *direct I* if for any $\delta_1, \delta_2 \in I$ there exists $\delta \in I$ such that $\delta \geq \delta_1$ and $\delta \geq \delta_2$; and the partially ordered set $(I, \leq)$ then called a *directed set*.

III *Net* If $X \neq \emptyset$ and $(I, \leq)$ is a directed set, a map $f : (I, \leq) \to X$ of the directed set $(I, \leq)$ into X, is called a *net in X based on the directed set* $(I, \leq)$. We write the net f as

$f = (f(\delta))_{\delta \in (I, \leq)}$.

And if, for an instance, we denote $f(\delta)$, for $\delta \in I$, by x_δ, then we may write

$f = (f(\delta))_{\delta \in (I, \leq)} = (x_\delta)_{\delta \in (I, \leq)}$

Hence, the language "let $(x_\delta)_{\delta \in (I, \leq)}$ be a net in X" is unambiguous. We give a popular theoretical example.

Example 1 Let (X, τ) be a topological space, $x_0 \in X$ and $I = \mathrm{N}_{x_0}(\tau)$. Direct I by $\leq$ defined as follows:

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$$\left.\begin{array}{l}\delta_1, \delta_2 \in I \\ \text{and} \\ \delta_1 \leq \delta_2\end{array}\right\} \quad \text{means } \delta_1 \supseteq \delta_2.$$

Then, $(I, \leq) = (\mathrm{N}_{x_0}(\tau), \leq)$ is a directed set. Now, for $\delta \in I = \mathrm{N}_{x_0}(\tau)$, let $x_\delta \in \delta$. Then, $(x_\delta)_{\delta\in(I,, \leq)}$ is a net in X.

IV *Sequence* The *natural order*, i.e., the *less than or equal to* order, $\leq$, on $\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$, directs $\mathbb{N}$, and so, $(\mathbb{N}, \leq)$ is a directed set. Let $X \neq \varnothing$. A net $(x_\delta)_{\delta\in(\mathbb{N}, \leq)}$ in X is called a *sequence in X*. We may also write $(x_\delta)_{\delta\in(\mathbb{N}, \leq)}$ as $(x_n)_{n\in(\mathbb{N}, \leq)}$,

V *Net* (*Sequential*) *Convergence* Let $X \neq \varnothing$, $(x_\delta)_{\delta\in(I, \leq)}$ a net in X, and $\varnothing \neq A \subseteq X$. We say that the net $(x_\delta)_{\delta\in(I, \leq)}$ is *eventually in A* if there exists $\delta_0 = \delta_0(A) \in I$ such that $x_\delta \in A$ for all $\delta \geq \delta_0$. Now, let (X, τ) be a topological space, $x_0 \in X$ and $(x_\delta)_{\delta\in(I, \leq)}$ a net in X. We say that $(x_\delta)_{\delta\in(I, \leq)}$ *converges to* x_0, and write $x_\delta \xrightarrow{\tau} x_0$, provided $(x_\delta)_{\delta\in(I, \leq)}$ is eventually in every neighbourhood of x_0. That is, if W is a neighbourhood of x_0, there exists $\delta_0 = \delta_0(x_0, W)$ such that $x_\delta \in W$ for all $\delta \geq \delta_0$. Hence, if $(x_n)_{n\in(\mathbb{N}, \leq)}$ is a sequence in X, it converges to x_0 provided it is eventually in every neighboUrhood of x_0, and we may write $x_n \xrightarrow{\tau} x_0$. That is, given a neighboUrhood W of x_0, there exists a positive integer $N = N(x_0, W)$ such that $x_n \in W$ for all $n \geq N$.

Example 2 [1, (ii) of p. 119] Let language and notation be as in Example 1. The net $(x_\delta)_{\delta\in(I, \leq)}$ of that Example 1 converges to x_0.

We here recall that if (X, τ) and (X', τ') are topological spaces, $x_0 \in X$, and $f : (X, \tau) \to (X', \tau')$ a map, f is said to be *continuous at* x_0 provided for every τ'-neighbourhood W of $f(x_0)$ there exists a τ-neigh- bourhood N of x_0 such that $f(N) \subseteq W$. If f is continuous at every point of X, f is simply said to be *continuous*.

Net (sequential) Characterization of Continuity 3 [1, Proposition 4. 19, p. 120] Let (X, τ) and (X', τ') be topological spaces, $x_0 \in X$, and $f : (X, \tau) \to (X', \tau')$ a map. f is continuous at x_0 if and only if for every net $(x_\delta)_{\delta\in(I, \leq)}$ in X τ-converging to x_0, the net $(f(x_\delta))_{\delta\in(I, \leq)}$ in X, τ'-converges to $f(x_0)$. If (X, τ) is first countable, then f is continuous at x_0 if and only if for every sequence $(x_n)_{n\in(\mathbb{N}, \leq)}$ in X τ-converging to x_0, the sequence $(f(x_n))_{n\in(\mathbb{N}, \leq)}$ in X', τ'-converges to $f(x_0)$. ///

3 UNIFORM SPACE Let $X \neq \varnothing$ and consider the Cartesian product XxX. The subset $\Delta_X = \{(x, x) \in X\text{x}X : x \in X\}$ of XxX is called its *diagonal*. If $\varnothing \neq A \subseteq X\text{x}X$, $A^{-1} = \{(a, b) \in X\text{x}X : (b, a) \in A\}$ is called the *inverse* of A, and A is said to be *symmetric* if $A = A^{-1}$. One almost effortlessly verifies.

FACT 1 Let $X \neq \varnothing$.

(i) If $\varnothing \neq A \subseteq X\text{x}X$, then, $(A^{-1})^{-1} = A$.

(ii) If $\varnothing \neq A \subseteq B \subseteq X\text{x}X$, then, $A^{-1} \subseteq B^{-1}$.

(iii) If $\varnothing \neq A,\ B \subseteq X\text{x}X$, $(A \cap B)^{-1} = A^{-1} \cap B^{-1}$.

(iv) If $\varnothing \neq A \subseteq X\text{x}X$, then $A \cap A^{-1}$ is symmetric. ///

Let $X \neq \varnothing$ and $\varnothing \neq A, B \subseteq X\text{x}X$. Define $A\text{o}B = \{(p, q) \in X\text{x}X$: there exists $r \in X$ such that $(p, r) \in B$ and $(r, q) \in A\}$.

Let $X \neq \varnothing$. A *filter U* in Xxλ [|i.e., a non-empty collection of non-empty subsets of XxX closed under *finite intersections* and the *taking of supersets* |] is called a *uniformity* on X if every member U of U has the properties

UFT 1 $U \supseteq \Delta_X$

UFT 2 $U^{-1} \in U$

UFT 3 There exists $V \in U$ such that $V\text{o}V \subseteq U$.

If U is a uniformity on X, the pair (X, U) is called a *uniform space*. The members of U are called its *entourages* [3, p. 22].

Let $X \neq \varnothing$ and U a uniformity on X. A *filterbase* $\mathcal{B}$ in XxX [|i.e., a non-empty family $\mathcal{B}$ of non-empty subsets of XxX such that, for $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ included in $A \cap B$.|] *generating U* [|i.e., $\mathcal{B} \subseteq U$ and supersets of members of $\mathcal{B}$ constitute U |] is called a *base* for U. We have

FACT 2 Let $X \neq \varnothing$, U a uniformity on X and $\mathcal{B}$ a filterbase in XxX which is a base for U. Then, every member U of $\mathcal{B}$ satisfies

BUFT 1 $U \supseteq \Delta_X$

BUFT 2 There exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$

BUFT 3 There exists $V \in \mathcal{B}$ such that $VoV \subseteq U$.

We also have

FACT 3 Let $X \neq \emptyset$, and $\mathcal{B}$ a filterbase in XxX every member U of which satisfies BUFT 1, BUFT 2 and BUFT 3. Then, $\mathcal{B}$ is a base for some uniformity on X. ///

Example 4 Let $X \neq \emptyset$ and d a pseudometric on X. Let $\varepsilon > 0$ and define

$U_{d,\,\varepsilon} = \{(a, b) \in X\text{x}X : d(a, b) < \varepsilon\}$.

Clearly, one verifies almost trivially that

(i) $U_{d,\,\varepsilon} \supseteq \Delta_X$

(ii) $U_{d,\,\varepsilon}{}^{-} = U_{d,\,\varepsilon} \subseteq U_{d,\,\varepsilon}$

(iii) $U_{d,\,(1/2)\varepsilon}$ o $U_{d,\,(1/2)\varepsilon} \subseteq U_{d,\,\varepsilon}$.

Immediate from these, therefore, is that $\mathcal{B}_d = \{ U_{d,\,\varepsilon} : \varepsilon > 0\}$meets the conditions of FACT 3 and so is a base for some uniformity on X. We denote this uniformity by $\mathcal{U}_d$; it is called the *pseudometric uniformity* of d or of the pseudometric space (X, d).

Let $X \neq \emptyset$, $\emptyset \neq U \subseteq X\text{x}X$ and $x_0 \in X$. Define

$U(x_0) = \{y \in X : (x_0, y) \in U\}$.

An almost trivial *set-inclusion proof* shows that if $\emptyset \neq U$, $V \subseteq X\text{x}X$ and $x_0 \in X$, then

$(U \cap V)(x_0) = U(x_0) \cap V(x_0)$.

We have

FACT 5 [2, last paragraph, p. 202] Let $(X, \mathcal{U})$ be a uniform space. Then,

$\tau_{\mathcal{U}} = \{\emptyset\} \cup \{\emptyset \neq G \subseteq X$: For every $x \in G$,

there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$ }

is a topology on X. ///

FACT 6 With notation as in FACT 5, if $\mathcal{B}$ is a base for $\mathcal{U}$, then,

$\tau_{\mathcal{U}} = \{\emptyset\} \cup \{\emptyset \neq G \subseteq X$: For every $x \in G$,

there exists $U \in \mathcal{B}$ such that $U(x) \subseteq G\}$.

Proof Immediate from FACT 5 and definition of a base. ///

With language and notation as in FACT 5 and FACT 6, the topo- logy $\tau_{\mathcal{U}}$ is called a *uniform topology*, more precisely, the *unif*orm *topology of* the uniformity $\mathcal{U}$ or *of* the uniform space $(X, \mathcal{U})$.

Example 7 Let (X, d) be a pseudometric space. Let $\varepsilon > 0$. From Example 4, $U_{d,\varepsilon} = \{(a, b) \in X\text{x}X : d(a, b) < \varepsilon\}$ and $\mathcal{B}_d = \{ U_{d,\varepsilon} : \varepsilon > 0\}$ is a base for $\mathcal{U}_d$. Clearly, for $x_0 \in X$, $U_{d,\varepsilon}(x_0) = B_d(x_0, \varepsilon)$ – the ball of radius ε centred on x_0. Hence, by FACT 6, $\emptyset \neq G \in \tau_{\mathcal{U}d}$ if and only if G is a union of balls, which, if and only if, G is a non-empty open set of the pseudometric topology τ_d of d. Hence, $\tau_{\mathcal{U}d} = \tau_d$.

Note 8 When reference is made to a uniform space $(X, \mathcal{U})$ as if it is a topological space, then its topology is $\tau_{\mathcal{U}}$.

FACT 9 Let $(X, \mathcal{U})$ be a uniform space and $x_0 \in X$. Then,

(i) $U(x_0) \in \mathrm{N}_{x_0}(\tau_{\mathcal{U}})$ for $U \in \mathcal{U}$,

and

(ii) $N \in \mathrm{N}_{x_0}(\tau_{\mathcal{U}}) \Rightarrow N = U(x_0)$ for some $U \in \mathcal{U}$. ///

One sees easily that

FACT 10 Let (Z, τ) be a topological space, $(X, \mathcal{U})$ a uniform space, $z_0 \in Z$ and $f : (Z, \tau) \to (X, \mathcal{U})$ a map. Then, f is continuous at z_0 provided for every entourage U of $\mathcal{U}$, there exists a τ-neighbourhood $N = N(U, z_0)$ of z_0 such that $f(N) \subseteq U(f(z_0))$.

That is, $(f(z_0), f(n)) \in U$ for all $n \in N$.

4 EQUICONTINUITY Let (X, τ) be a topological space, $(Y, \mathcal{U})$ a uniform space, $x_0 \in X$, and F a collection of maps $f : (X, \tau) \to (Y, \mathcal{U})$.

DEFINITION 1 [2, p. 288] The collection F is said to be *equi- continuous at* x_0 provided for every entourage W of $\mathcal{U}$, there exists a neighbourhood $N = N(x_0, W)$ of x_0 in (X, τ) such that

$(f(x_0), f(x)) \in W$ for all $x \in N$ and all $f \in F$

That is, such that

$f(N) \subseteq W(f(x_0))$ for all $f \in F$. [| Compare 3.9 and 3.10|]

Now let $\mathcal{B}_{\mathcal{U}}$ be a base for the uniformity $\mathcal{U}$. Following [3] let us call members of $\mathcal{B}_{\mathcal{U}}$ *basic entourages*. Then, DEFINITION 1 can be recast as follows.

DEFINITION 2 The collection F of maps $f : (X, \tau) \to (Y, \mathcal{U})$ is equicontinuous at x_0 provided for every basic entourage W, there exists a neighbourhood $N = N(x_0, W)$ of x_0 in (X, τ) such that

$(f(x_0), f(x)) \in W$ for all $x \in N$ and all $f \in F$.

That is, such that

$f(N) \subseteq W(f(x_0))$ for all $f \in F$.

Immediate from DEFINITION 1 and 3.10 is

FACT 3 Each member f of the equicontinous family F is also $(\tau,\tau_{\mathcal{U}})$-continuous at x_0; hence, perhaps, the label *equi*-continuous. ///

DEFINITION 4 If the collection F is equicontinuous at every point of X it is simply said to be *equicontinuous*,

Example 5 Let (Z, d) be a pseudometric space. By 3.4,

$\mathcal{B}_d = \{U_{d,\varepsilon} : \varepsilon > 0\}$

is a base for the pseudometric uniformity, $\mathcal{U}_d$, of d. Hence, we can take $U_{d,\varepsilon}$ $(\varepsilon > 0)$ as a basic entourage in DEFINITION 2. By 3.7, also for $z_0 \in Z$,

$U_{d,\varepsilon}(z_0) = B_d(z_0, \varepsilon)$

where $B_d(z_0, \varepsilon)$ is the ball in (Z, d) of radius ε centred on z_0.

Immediate, therefore, from Example 5 and DEFINITION 2 is

FACT 6 [5, Definition 3.1.41, p. 165] Let (X, τ) be a topological space $x_0 \in X$, and (Z, d) a pseudometric space. A collection F of maps $f : (X, \tau) \to (Z, d)$ is equicontinuous at x_0 if and only if for every $\varepsilon > 0$, there exists a neighbourhood $N = N(x_0, \varepsilon)$ of x_0 such that

$f(N) \subseteq B_d(f(x_0), \varepsilon)$ for all $f \in F$.

where $B_d(f(x_0), \varepsilon)$ is the ball in (Z, d) of radius ε centered on $f(x_0)$. ///

Definition 7, *NEC at* x_0 Let (X, τ) be a topological space, $(Y, \mathcal{U})$ a uniform space, $x_0 \in X$ *and* F is a collection of maps $f : (X, \tau) \to (Y, \mathcal{U})$. We shall say that F is *NEC at* x_0 if for every net $(x_\delta)_{\delta\in(I,\leq)}$ in (X, τ) converging to x_0 and every entourage W of $\mathcal{U}$, there exists $\delta_0 = \delta_0(x_0, W) \in I$ such that

$(f(x_0), f(x_\delta)) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$ …(Σ)

By now familiar arguments we can recast DEFINITION 7 as follows.

Recast 8 Let (X, τ) be a topological space, $(Y, \mathcal{U})$ a uniform space, $x_0 \in X$ and F a collection of maps $f : (X, \tau) \to (Y, \mathcal{U})$. Then, F is *NEC at* x_0 if and only if for every net $(x_\delta)_{\delta\in(I,\leq)}$ in (X, τ) converging to x_0 and every basic entourage W of $\mathcal{U}$, there exists $\delta_0 = \delta_0(x_0, W) \in I$ such that

$(f(x_0), f(x_\delta)) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$. ///

We now come to the first of the contributions of this paper.

Net Characterization of Equicontinuity 9 Let (X, τ) be a topolo- gical space, (Y, U) a uniform space, $x_0 \in X$ and F a collection of

maps $f : (X, \tau) \to (Y, U)$. Then, F is equicontinuous at x_0 if and only if F is *NEC at* x_0.

Proof ⇒: *Hypothesis* F is equicontinuous at x_0.

Hence, if given an entourage W of U, there exists a neighbourhood $N = N(x_0, W)$ of x_0 in (X, τ) such that

$(f(x_0), f(x)) \in W$ for all $x \in N$ and all $f \in F$ ….(1)

Let $(x_\delta)_{\delta\in(I, \leq)}$ be a net in (X, τ) converging to x_0. Hence, this net is

eventually in N, and so there exists $\delta_0 = \delta_0(x_0, N) \in I$ such that

$x_\delta \in N$ for all $\delta \geq \delta_0$ ....(2)

From (1) and (2), therefore, follow that

$(f(x_0), f(x_\delta)) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$ ….(3)

But by DEFINITION 7, (3) means F is *NEC at* x_0.

⇐: Assume that F is not equicontinuous at x_0, and so there exists a "stubborn" entourage, W, say, of U for which we cannot find a "good" neighbourhood of x_0. Hence, for this "stubborn" entourage W, and any given neighbourhood N of x_0 in (X, τ), there exists $x_N \in N$ and $f_N \in F$ such that

$(f_N(x_0), f_N(x_N)) \notin W$ ….(4)

Let I be the neighbourhood system at x_0. That is, $I = \mathrm{N}_{x_0}(\tau)$. Direct I by $\leq$ as in 2.1. By (4), for any $\delta \in I$, there exist $x_\delta \in \delta$ and $f_\delta \in F$ such that

$(f_\delta(x_0), f_\delta(x_\delta)) \notin W$ ….(5)

By 2.2,

$(x_\delta)_{\delta\in(I, \leq)}$ converges to x_0 ….(6)

Clearly, by (5) and (6) and DEFINITION 7, F **cannot be** *INEC at* x_0. ///

Definition 10 *SEC at* x_0 Let (X, τ) be a topological space, (Y, U) a uniform space, $x_0 \in X$ and F a collection of maps $f : (X, \tau) \to (Y, U)$. We shall say that F is *SEC at* x_0 if for every sequence $(x_n)_{n\in(\mathbb{N},\leq)}$ in (X, τ), converging to x_0, and every entourage W of U, there exists a positive integer $N = N(x_0, W)$ such that

$(f(x_0), f(x_n)) \in W$ for all $n \geq N$ and all $f \in F$.

Of course, immediate is

Recast 11 Let (X, τ) be a topological space, (Y, U) a uniform space, $x_0 \in X$ and F a collection of maps $f : (X, \tau) \to (Y, U)$.

Then, F is *SEC at* x_0 if and only if for every sequence $(x_n)_{n\in(\mathbb{N}, \leq)}$ in (X, τ), converging to x_0, and every *basic entourage* W of U, there exists a positive integer $N = N(x_0, W)$ such that

$(f(x_0), f(x_n)) \in W$ for all $n \geq N$ and all $f \in F$. ///

Immediate is

***NEC at* x_0 ⇒*SEC at* x_0 12** Let (X, τ) be a topological space, (Y, U) a uniform space, $x_0 \in X$ and F a collection of maps $f : (X, \tau) \to (Y, U)$. Then, F is *NEC at* $x_0 \Rightarrow F$ is *SEC at* x_0. ///

We have, for first countable topological spaces,

THEOREM 13 For first countable topological space (X, τ), an arbitrary uniform space (Y, U) and $x_0 \in X$, a collection F of maps $f : (X, \tau) \to (Y, U)$ is *NEC at* x_0 if and only if it is *SEC at* x_0.

Proof The implication ⇒ is the preceding THEOREM 12.

⇐ : *Hypotheses* **Hyp (1)** (X, τ) is first countable

Hyp (ii) F is *SEC at* x_0.

We want to show that F is *NEC at* x_0. Therefore, suppose

$(x_\delta)_{\delta\in(I,\leq)}$ is a net in (X, τ) converging to x_0. ….(1)

By HYp(i), (X, τ) is first countable, and so has a decreasing local base $N_1 \supseteq N_2 \supseteq$ …….. of neighbourhoods of x_0. Therefore, since $(x_\delta)_{\delta\in(I,\leq)}$ converges in (X, τ) to x_0, there exists $\delta_1 \in I$ such that

$x_\delta \in N_1$ for all $\delta \geq \delta_1$

Since $(I, \leq)$ is a directed set, it also follows that there exists $\delta_2 \in I$, $\delta_2 \geq \delta_1$, and

$x_\delta \in N_2$ for all $\delta \geq \delta_2$.

Similarly, there exists $\delta_3 \geq \delta_2$, such that

$x_\delta \in N_3$ for all $\delta \geq \delta_3$.

Continuing, we shall obtain a sequence $(x_{\delta_1}, x_{\delta_2}, x_{\delta_3}, \ldots)$ in (X, τ) with $x_{\delta_k} \in N_k$ for all $k = 1, 2,$ …….. and such that

$x_\delta \in N_k$ for all $\delta \geq \delta_k$ …(*)

Since $N_1 \supseteq N_2 \supseteq$ …….. is a decreasing local base at x_0, the sequence $(x_{\delta_1}, x_{\delta_2}, \ldots)$ converges to x_0. Clearly, also again by (*) any sequence

$(x_{\alpha_1}, x_{\alpha_2}, \ldots)$ with terms from $(x_\delta)_{\delta\in(I,\leq)}$
such that $\alpha_1 \geq \delta_1$, $\alpha_2 \geq \delta_2$, ……., converges to x_0. …(2)

Now, assume the opposite that F is not *NEC at* x_0. By (1) and Definition 7, therefore, there exists a "stubborn" entourage W of U for which we cannot find a $\delta_0 = \delta_0(x_0, W)$ meeting the condition (Σ) of Definition 7. In particular, there exist

$\mu_1 \in I$, $\mu_1 \geq \delta_1$, $f_{\mu_1} \in F$

such that

$(f_{\mu_1}(x_0)$ and $f_{\mu_1}(x_{\mu_1})) \notin W$

Similarly, there exist

$\mu_2 \in I$, $\mu_2 \geq \delta_2$ and $f_{\mu_2} \in F$

such that

$(f_{\mu_2}(x_0), f_{\mu_2}(x_{\mu_2})) \notin W$

Continuing, we shall come up with a sequence

$(x_{\mu_k})_{k\in(\mathbb{N},\leq)}$ with terms from the net
$(x_\delta)_{\delta\in(I,\leq)}$, and $f_{\mu_k} \in F$, such that $\mu_k \geq \delta_k$,
$(f_{\mu_k}(x_0), f_{\mu_k}(x_{\mu_k})) \notin W$, $k = 1, 2,$ …. …(3)

By (2), $(x_{\mu_k})_{k\in(\mathbb{N},\leq)}$ converges to x_0. But by (3), F is **not** *SEC at* x_0 — a contradiction to Hyp (ii). ///

We therefore now have

Sequential Characterization of Equicontinuity 14 For a first countable topological space (X, τ), an arbitrary uniform space (Y, U) and $x_0 \in X$, a collection F of maps $f : (X, \tau) \to (Y, U)$ is equicontinuous at x_0 if and only if it is *SEC at* x_0.

Proof THEOREM 13 and THEOREM 9. ///

Immediate from the preceding THEOREM 14 noting that the topology τ_d of a pseudometric space (X, d) is first countable, that $\mathcal{B}_d = \{U_{d,\varepsilon} : \varepsilon > 0\}$ is a base for the uniformity U_d, and that, for $x_0 \in X$, $U_{d,\varepsilon}(x_0) = B_d(x_0, \varepsilon)$ – the ball of radius ε centred on x_0, is

THEOREM 15 Let (X, d) and (X', d') be pseudometric spaces, $x_0 \in X$ and F a collection of maps $f : (X, \tau_d) \to (X', U_{d'})$. Then, F is equicontinuous at x_0 if and only if for every sequence $(x_n)_{n\in(\mathbb{N},\leq)}$ in (X, τ_d) converging to x_0, and $\varepsilon > 0$, there exists a positive integer $N = N(x_0, \varepsilon)$ such that $f(x_n) \in B_d'(f(x_0), \varepsilon)$ for all $n \geq N$ and all $f \in F$. ///

We apply the above THEOREM 15 in the next section.

5 THE KARL STROMBERG'S EXAMPLE For $n \in \mathbb{N}$, con- sider the Euclidean metric space $(\mathbb{R}^n, d_n)$, $\mathbb{R}^n$ with its Euclidean metric d_n. Of course, for $\varnothing \neq D \subseteq \mathbb{R}^n$, $(D, d_n|D)$ is also a metric space. Karl Stromberg in [5] proved

The Karl Stromberg's Example [5, Example 3.142, p.165] 1 Let $D = [0, 1] \subseteq \mathbb{R}^1 = \mathbb{R}$ and $d_1 = d_1|D$. Let $F = \{ f_m : m = 1, 2, \ldots\ldots\}$ be a collection of maps $f_m : (D, \tau_{d_1}) \to (\mathbb{R}, \tau_{d|\,|})$, $f_m(x) = x^m$, $x \in D$, $m = 1, 2, \ldots\ldots$. F is **not** equicontinuous at 1. ///

We here furnish another proof of the above, using our 4.15.

A second Proof By the Density Theorems of *Elementary Real Analysis* (*ERA*), there exists a sequence $(ir_n)_{n\in(N,\leq)}$, of irrationals, ir_n, $0 < ir_n < 1$, converging to 1. By another result of *ERA* for a given such $i\gamma_n$, there exists a positive integer $N(n)$ such that

$$(i\gamma_n)^{N(n)} < \frac{1}{4}.$$

Hence,

$$d_1(f_{N(n)}(1), f_{N(n)}(ir_n)) = d_1(1, (ir_n)^{N(n)}) = |1 - (ir_n)^{N(n)}|$$

$$= 1 - (ir_n)^{N(n)} > 1 - \frac{1}{4} = \frac{3}{4} > \frac{1}{2}.$$

That is,

$$d_1(f_{N(n)}(1), f_{N(n)}(i\gamma_n)) > \frac{1}{2} \text{ for all } n \qquad \ldots.(\Sigma)$$

From (Σ) follows that 4.15 cannot be satisfied with $\varepsilon = \frac{1}{2}$, say. ///

6 EQUICONTINUITY AND THE SUP-UNIFORMITY Let $X \neq \varnothing$ and ξ a collection of non-empty subsets of XxX each of which contain the diagonal Δ_X that have the Finite Intersection Property (the FIP). Then, the collection, $\mathcal{B}(\xi)$, of finite intersections of members of ξ, is a filterbase. If $\mathcal{B}(\xi)$ is a base for uniformity, U, say, on X, we shall say that ξ, *generates U*.

We have

FACT 1 Let $X \neq \varnothing$ and $(U_k)_{k\in I}$ a family of uniformities on X. Let $\mathcal{B}_k$, $k \in I$, be a base for U_k. Then,

(i) $\bigcup_{k\in I} U_k$ generates a uniformity, U^{ent}, say on X,

(ii) $\bigcup_{k\in I} \mathcal{B}_k$ generates a uniformity, U^{base}, say on X,

and

(iii) $U^{\text{ent}} = U^{\text{base}}$.///

Notation 2 With language and notation as in the preceding, U^{ent} is called the *supremum of the uniformities* U_k, $k \in I$. And so we may write $U^{\text{ent}} = \sup_{k\in I} U_k$.

Language and Notation 3 Let (X, τ) be a topological space, $x_0 \in X$, (Y, U) a uniform space and F a collection of maps $f : (X, \tau) \to (Y, U)$. If F is equicontinuous at x_0, we may say that, and write, F is (τ,U)-equicontinuous at x_0. Similarly, F is (τ,U) *NEC/SEC at* x_0 is unambiguous.

Now we have

THEOREM 4 Let (X, τ) be a topological space, $x_0 \in X$, $Y \neq \emptyset$, $(U_k)_{k\in I}$ a collection of uniformities on Y, and F a collection of maps $f : (X, \tau) \to Y$. Then, F is $(\tau, \sup_{k\in I} U_k)$-equicontinuous at x_0 if and only if it is (τ, U_k)-equicontinuous at x_0 for each $k \in I$.

Proof

$\Leftarrow$ **:** *Hypothesis* F is (τ, U_k)-equicontinuous at x_0 for each $k \in I$.

And so, by 4.9

F is (τ, U_k)-NEC at x_0 for each $k \in I$ …..(Δ^1)

Let $(x_\delta)_{\delta\in(D,\le)}$ be a net in (X, τ) converging to x_0, and suppose $W \in \sup_{k\in I} U_k = U^{ent}$. By the construction of U^{ent}, $W \supseteq W_1 \cap W_2 \cap \ldots\ldots \cap W_n$ for some $n \in \mathbb{N}$, where $W_i \in U_i$, $i = 1, 2, \ldots\ldots, n$. By ($\Delta^1$), F is (τ, U_i)-NEC at x_0 for each $i = 1, 2, \ldots\ldots, n$. Hence for each i there exists $\delta_0(i) = \delta_0(i, W_i)$ such that

$(f(x_0), f(x_\delta)) \in W_i$ for all $\delta \ge \delta_0(i)$ and all $f \in F$.

Since the poset $(D, \le)$ on which the net $(x_\delta)_{\delta\in(D,\le)}$ is based is a directed set, there exists, therefore, $\delta_0^* \in D$ such that

$(f(x_0), f(x_\delta)) \in W_1 \cap W_2 \cap \ldots\ldots \cap W_n \subseteq W$ for all

$\delta \ge \delta_0^*$ and all $f \in F$ ….(Δ^2)

Since $W \in \sup_{k\in I} U_k$ was arbitrary, it follows from (Δ^2) that F is $(\tau, \sup_{k\in I} U_k)$-NEC at x_0, and so, by 4.9,

F is $(\tau, \sup_{k\in I} U_k)$-equicontinuous at x_0.

$\Rightarrow$ **:** *Hypothesis* F is $(\tau, \sup_{k\in I} U_k)$-equicontinuous at x_0.

Therefore, by 4.9,

F is $(\tau, \sup_{k\in I} U_k)$-NEC at x_0 ….(Δ^3)

But observe from the construction of $\sup_{k\in I} U_k$ that if $W \in U_k$ $(k \in I)$ then, also $W \in \sup_{k\in I} U_k$. From (Δ^3) therefore follows that

F is (τ, U_k)-NEC at x_0 for each $k \in I$ ….(Δ^4)

Clearly, (Δ^4) and 4.9 give F is (τ, U_k)-equicontinuous at x_0 for each $k \in I$. ///

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