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NOTES ON EQUICONTINUITY IN TOPOLOGICAL VECTOR SPACES

Sunday Oluyemi

Odo-Koto, Aiyedaade, Ilorin South LGA, Kwara State, NIGERIA.

Abstract

Equicontinuity is a Uniform Space (General Topology (GT)) concept that has assumed some notoriety in Locally Convex Space theory (Topological Vector Spaces (TVS)). We here

(i) Trace the link from GT to TVS, and

(ii) Exploit the link to assemble some notes.

In addition, from the notes, we show that:

(iii) The T-limited sets of John Webb are equicontinous sets of linear functionals. [7].

Keywords: Null net, equicontinuous set of linear maps (functionals).

1. LANGUAGE AND NOTATION

For the rudiments of General Topology (**GT**) we assume the reader is familiar with [1]. $\mathbb{R}$ denotes the *real numbers*, $\mathfrak{C}$ denotes the *complex numbers*, and by K we denote either of $\mathbb{R}$ and $\mathfrak{C}$. Our vector space $E = (E, +, \theta)_K$ is an additive Abelian group with an external multiplication (scalar multiplication) by the elements of K, the additive identity of our vector space $E = (E, +, \theta)_K$ is the element θ called its *zero*. *Note*: (K, +, ·, 0, 1) is itself a vector space over itself with its zero the element 0. A *topological vector space* is a topological space (E,τ) where E is a vector space (over K) and τ is a topology on E compatible with the addition and scalar multiplication of E. We assume familiarity with the rudiments of **TVS**, that can be gleaned from the first few pages of [2], [3], [4] and [5].

We indicate by /// the end or absence of a proof.

2 THE UNIFORMITY OF A TVS [8] Let $E_K = (E, +, \theta)_K$ be a vector space, and A a non-empty subset of E. A is called a *balanced set* if $\lambda A \subseteq A$ for all $\lambda \in K$, $|\lambda| \le 1$.

FACT 1 Let $E_K = (E, +, \theta)_K$ be a vector space and $\varnothing \ne A \subseteq E$. If A is balanced, then $-A = (-1)A = A$. More generally, $\lambda A = A$ for all $\lambda \in K$, $|\lambda| = 1$. ///

Let $(E,\tau) = ((E, +, \theta)_K,\tau)$ be a topological vectors space. We denote by $N_\theta(\tau)$ the neighbourhood system of zero, θ.

FACT 2 Let $(E, \tau) = ((E, +, \theta)_K, \tau)$ be a topological vector space.

(i) For every $U \in N_\theta(\tau)$, there exists a balanced $V \in N_\theta(\tau)$ such that $V \subseteq U$.

(ii) For $U \in N_\theta(\tau)$, there exists a balanced $V \in N_\theta(\tau)$ such that $V + V \subseteq U$.

(iii) [2, Proposition 2.3.1, p.81]. There exists a local base of balanced neighbourhoods of zero. ///

Now recall from [1] that if $X \ne \varnothing$, the subset $\Delta_X = \{(x, x) \in X \text{ x } X : x \in X\}$ of X x X is called the *diagonal* of X x X; if $\varnothing \ne A \subseteq X \text{ x } X$, A^{-1}

$= \{(a, b) \in X \text{ x } X : (b, a) \in A\}$ is called the *inverse* of A; if $\varnothing \ne A, B \subseteq X \text{ x } X$, the *nought product* $A\text{o}B = \{(p, q) \in X \text{ x } X :$ there exists $r \in X$ such that $(p, r) \in B$ and $(r, q) \in A\}$.

Let $(E, \tau) = ((E, +, \theta)_K, \tau)$ be a topological vector space and $W \in N_\theta(\tau)$. Define

$B_W = \{(x, y) \in E \text{ x } E : x - y \in W\}$.

Then, with notation as above, we have

Corresponding Author**:** Sunday O., Email: soluyemi19@yahoo.com, Tel: +2348160865176

FACT 3 (i) $B_W \supseteq \Delta_E$,

(ii) $(B_{-W})^{-1} = B_W \subseteq B_W$

(iii) For some balanced $U \in N_\theta(\tau)$, $B_{(1/2)U} \circ B_{(1/2)U} \subseteq B_W$.

Proof (i): $\theta \in W$, and for any $x \in E$, $x - x = \theta$. Hence, $B_w \supseteq \Delta_E$

(ii): Let $(x, y) \in (B_{-W})^{-1}$ which means $(y, x) \in B_{-W}$, which in turn means $y - x \in -W$. Hence, $x - y = -(y - x) \in -(-W) = W$. And so, $(x, y) \in B_W$. Thus, we have shown that $(B_{-W})^{-1} \subseteq B_W$. $B_W \subseteq (B_{-W})^{-1}$ can similarly be proved

(iii): By FACT 2(ii), there exists a balanced $U \in N_\theta(\tau)$ such that

$$U + U \subseteq W \qquad \text{…..(1)}$$

Hence, since $\left|\frac{1}{2}\right| \le 1$, by the definition of balanced, it follows from (1) that

$$(1/2)U + (1/2)U \subseteq U + U \subseteq W \qquad \text{…..(2)}$$

Now let $(x, y) \in B_{(1/2)U} \circ B_{(1/2)U}$. Then, there exists $z \in E$ such that

$(x, z) \in B_{(1/2)U}$ and $(z, y) \in B_{(1/2)U}$

That is,

$x - z \in (1/2)U$ and $(z - y) \in (1/2)U$.

And so, from (2) it follows that

$x - y = (x - z) + (z - y) \in (1/2)U + (1/2)U \subseteq U + U \subseteq W$.

That is, $x - y \in W$. And so, $(x, y) \in B_W$. ///

Again, let $X \neq \emptyset$, and recall from [1] that a filter $\mathcal{U}$ in X xX is called a *uniformity* on X if every $U \in \mathcal{U}$ has the properties

UFT 1 $U \supseteq \Delta_X$

UFT 2 $U^{-1} \in \mathcal{U}$

UFT 3 There exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Also, a filterbase $\mathcal{B}$ in XxX is a *base for some uniformity* (i.e., generates a uniformity) if every $U \in \mathcal{B}$ has the following properties.

BUFT 1 $U \supseteq \Delta_X$

BUFT 2 There exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$.

BUFT 2 There exists $V \in \mathcal{B}$ such that $V \circ V \subseteq U$.

FACT 4 Let $(E, \tau) = ((E, +, \theta)_K, \tau)$ be a topological vector space. Then, $\mathcal{B}_{(E,\tau)} = \{B_W : W \in N_\theta(\tau)\}$ is a base for a uniformity on E.

Proof Clearly, $\mathcal{B}_{(E,\tau)}$ is a non-empty family of non-empty subsets of ExE. Clearly, for $W, W' \in N_\theta(\tau)$, $B_W \cap B_{W'} = B_{W \cap W'}$, from which follows that $\mathcal{B}_{(E,\tau)}$ is a filterbase in ExE. That $\mathcal{B}_{(E,\tau)}$ is a base for a uniformity on E is upheld by FACT 3. ///

Let (E, τ) be a topological vector space and let us denote by $\mathcal{U}_{(E,\tau)}$ the uniformity on E for which $\mathcal{B}_{(E,\tau)}$ is a base.

FACT 5 [4, (11.10), p50][5, First paragraph, p. 134] If (E, τ) is a topological vectors space, then $\tau = \tau_{\mathcal{U}(E,\tau)}$. ///

For the topological vector space (E, τ), the uniformity $\mathcal{U}_{(E,\tau)}$, which we deal with throughout, is *the uniformity of* the topological vector space (E, τ).

3 EQUICONTINUITY We recall from [1] that a collection F of maps $f : (X, \tau) \to (Y, \mathcal{U})$ from a topological space (X, τ) into a uniform space $(Y, \mathcal{U})$ is said to be equicontinuous at a point $x_0 \in X$ if for every entourage W of the uniformity $\mathcal{U}$, there exists $N \in N_{x_0}(\tau)$ such that $(f(x_0), f(x)) \in W$ for all $x \in N$ and all $f \in F$. And noted is

FACT 1 F is equicontinuous at x_0 if and only if for every *basic entourage* W of $\mathcal{U}$, there exist $N = N(x_0) \in N_{x_0}(\tau)$ such that $(f(x_0), f(x)) \in W$ for all $x \in N$ and all $f \in F$. ///

Let (E, τ^*) be a topological vector space . Then, $(E, \mathcal{U}_{(E,\tau^*)})$ is a uniform space, and so can be used in place of $(Y, \mathcal{U})$ in either of the definition or FACT 1 above. If used in FACT 1, then the members of $\mathcal{B}_{(E,\ \tau^*)}$ may be chosen as our *basic entourages*. Therefore, it follows from the definition of $\mathcal{B}_{(E,\ \tau^*)}$ that we can give the following.

DEFINITION 2 Let (X, τ) be topological space, $(E, \tau^*) = ((E, +, \theta)_K, \tau^*)$ be a topological vector space, $a \in X$, and F a collection of maps $f : (X, \tau) \to (E, \tau^*)$. F is equicontinuous at a if for every $W \in N_\theta(\tau^*)$ there exists $N \in N_a(\tau)$ such that $(f(a), f(x)) \in B_W$ for all $x \in N$ and all $f \in F$.

Observation 3 We have followed the tradition in the literature by writing

"…..of maps $f : (X, \tau) \to (E, \tau^*)$".

instead of appropriately writing

"…..of maps $f : (X, \tau) \to (E, U_{(E,\ \tau^*)})$"

We continue to do this in deference to a well-established practice.

For the discussions that follow, we fix the notation of DEFINI- TION 2 above.

Notation 4 (X, τ) is a topological space

$a \in X$

$(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ is a topological vector space

$U_{(E,\ \tau^*)}$ is the uniformity of (E, τ^*)

$\mathcal{B}_{(E,\ \tau^*)} = \{B_W : W \in N_{\theta^*}(\tau^*)\}$

and

$B_W = \{(x, y) \in ExE : x - y \in W\}$

For $W \in N_{\theta^*}(\tau^*)$ fix balanced $W' \in N_{\theta^*}(\tau^*)$ such that $W' \subseteq W$ (2.2(i) and (iii)).

From the definition of B_W, one sees easily that

$$\left.\begin{array}{l} W_1, W_2 \in N_{\theta^*}(\tau^*) \\ \text{and} \\ W_1 \subseteq W_2 \end{array}\right\} \Rightarrow B_{W_1} \subseteq B_{W_2}.$$

Hence, for $W \in N_{\theta^*}(\tau^*)$ and its fixed balanced $W' \in N_{\theta^*}(\tau^*)$, we have

$$B_{W'} \subseteq B_W \qquad \text{……}(\Sigma)$$

Therefore, from (Σ) and DEFINITION 2, it follows that :

If (X, τ) is a topological space, $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ a topolo- gical vector space, $a \in X$, and F a collection of maps $f : (X, \tau) \to (E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$(See Observation 3), then the following are equivalent.

(1) F is equicontinuous at a.

(2) For every $W \in N_{\theta^*}(\tau^*)$ there exists $N = N(a, W', W) \in N_a(\tau)$ such that $(f(a), f(x)) \in B_{W'} \subseteq B_W$ for all $x \in N$ and all $f \in F$.

(3) For every $W \in N_{\theta^*}(\tau^*)$ there exists $N = N(a, W', W) \in N_a(\tau)$ such that $f(x) - f(a) \in -W' = W' \subseteq W$ for all $x \in N$ and all $f \in F$.

(4) For every $W \in N_{\theta^*}(\tau^*)$ there exists $N = N(a, W', W) \in N_a(\tau)$ such that $f(x) \in f(a) + W' \subseteq f(a) + W$ for all $x \in N$ and all $f \in F$.

(5) For every $W \in N_{\theta^*}(\tau^*)$ there exists $N = N(a, W', W) \in N_a(\tau)$ such that $f(x) \in f(a) + W$ for all $x \in N$ and all $f \in F$.

Thus, we have proved.

THEOREM 5 ***Note* 1** [2, last paragraph, p. 198] Let (X, τ) be a topological space, $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ a topological vector space, $a \in X$, and F a collection of maps $f : (X, \tau) \to (E, \tau^*)$. Then, F is equicontinuous at a if and only if for every $W \in N_{\theta^*}(\tau^*)$ there exists $N \in N_a(\tau)$ such that $f(x) \in f(a) + W$ for all $x \in N$ and all $f \in F$. That is,

$$f(N) \subseteq f(a) + W \text{ for all } f \in F \qquad \ldots\ldots(*)$$

.///

Now let $(X, \tau) \equiv ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ be both topological vector spaces. Let $f: ((X, +, \theta)_K, \tau) \to ((E, +, \theta^*)_K, \tau^*)$ be a linear map. Suppose $a \in X$, $W \in N_{\theta^*}(\tau^*)$, $N \in N_a(\tau)$ and

$$f(N) \subseteq f(a) + W \qquad \ldots.(1)$$

Since (X, τ) is a topological vector space and $N \in N_a(\tau)$, then

$$N = a + V \qquad \ldots..(2)$$

for some $V \in N_\theta(\tau)$. By the linearity of f, we have therefore, that

$f(N) = f(a + V) = f(a) + f(V)$.

That is

$$f(N) = f(a) + f(V) \qquad \ldots.(3)$$

Clearly, (3) and (1) now give

$$f(a) + f(V) \subseteq f(a) + W \qquad \ldots.(4)$$

which is equivalent to

$$f(V) \subseteq W \qquad \ldots.(5)$$

which in turn is equivalent to

$$f(b) + f(V) \subseteq f(b) + W \qquad \ldots.(6)$$

for any other $b \in X$. Hence, from the preceding and THEOREM 5 with its (*), we now have

THEOREM 6 ***Note* 2** For topological vector spaces $(X, \tau) = ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$, $a \in X$, and F a collection of *linear maps* $f : (X, \tau) \to (E, \tau^*)$, the following are equivalent.

(i) F is equicontinuous at a.

(ii) [3, Definition 9 – 1 – 1, p.128]. For every $W \in N_{\theta^*}(\tau^*)$ there exists $V \in N_\theta(\tau)$ such that $f(V) \subseteq W$ for all $f \in F$

(iii) F is equicontinuous at every other point $b \in X$.

(iv) F is equicontinuous at θ.

(v) F is equicontinuous. ///

Observation 7 If $(X, +, \theta)_K$ and $(X', +, \theta')_K$ are vector spaces and $f : (X, +, \theta)_K \to (X', +, \theta')_K$ is a linear map, then $f(\theta) = \theta'$.

If

$$((X, +, \theta)_K, \tau) \qquad \ldots.(\text{tvs})$$

is a topological vector space, we shall call a net in (tvs) converging to the zero, θ, of (tvs) a *null* net. We now have from [1] and ***Note* 2**, taking cognizance of the balanced $W' \subseteq W$ of **Notation 4**.

THEOREM 8 ***Note* 3** For topological vector spaces $(X, \tau) = ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$, and F a collection of linear maps $f : (X, \tau) \to (E, \tau^*)$, the following are equivalent.

(i) F is equicontinuous.

(ii) F is equicontinuous at the zero, θ, of (X, τ).

(iii) F is *NEC at the zero*, θ, of (X, τ).

(iv) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in $((X, +, \theta)_K, \tau)$ and every $W \in N_{\theta^*}(\tau^*)$, there exists $\delta_0 = \delta_0(W) \in I$ such that $f(\theta) - f(x_\delta) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$.

(v) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in $((X, +, \theta)_K, \tau)$ and every $W \in N_{\theta^*}(\tau^*)$, there exists $\delta_0 = \delta_0(W', W) \in I$ such that $f(\theta) - f(x_\delta) \in W' \subseteq W$ for all $\delta \geq \delta_0$ and all $f \in F$.

(vi) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in $((X, +, \theta)_K, \tau)$ and every $W \in N_{\theta^*}(\tau^*)$, there exists $\delta_0 = \delta_0(W', W) \in I$ such that $f(x_\delta) - f(\theta) = -(f(\theta) - f(x_\delta)) \in -W' = W' \subseteq W$ for all $\delta \geq \delta_0$ and all $f \in F$

(vii) For every null net $((x_\delta)_{\delta \in (I, \leq)}$ in $((X, +, \theta)_K, \tau)$ and every $W \in N_{\theta^*}(\tau^*)$, there exists a $\delta_0 = \delta_0(W', W) \in I$ such that $f(x_\delta) \in W' \subseteq W$ for all $\delta \geq \delta_0$ and all $f \in F$

(viii) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in $((X, +, \theta)_K, \tau)$ and every $W \in N_{\theta^*}(\tau^*)$, there exists a $\delta_0 = \delta_0(W) \in I$ such that $f(x_\delta) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$. ///

The definition of a *null sequence* is clear. We have, for domain space first countable,

THEOREM 9 ***Note* 4** For topological vector spaces $(X, \tau) = ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$, (X, τ) first countable, and F a collection of linear maps $f : (X, \tau) \to (E, \tau^*)$, the following are equivalent.

(i) F is equicontinuous.

(ii) F is equicontinuous at the zero, θ, of (X, τ).

(iii) F is *SEC at the zero*, θ, of (X, τ).

(iv) For every null sequence $(x_n)_{n \in (\mathbb{N}, \leq)}$ in (X, τ) and every $W \in N_{\theta^*}(\tau^*)$, there exists a positive integer $N = N(W)$ such that $f(\theta) - f(x_n) \in W$ for all $n \geq N$ and all $f \in F$.

(v) For every null sequence $(x_n)_{n \in (\mathbb{N}, \leq)}$ in (X, τ) and every $W \in N_{\theta^*}(\tau^*)$, there exists a positive integer $N = N(W)$ such that $f(x_n) \in W$ for all $n \geq N$ and all $f \in F$. ///

4 EQUICONTINUOUS SET OF LINEAR FUNCTIONALS Suppose that in 3.8 we take (E, τ^*) as (K, τ_K), that is, as K with its usual topology [6]. And so, F is a collection of *linear functionals*. Hence, we have the two theorems that follow.

THEOREM 1 ***Note* 5** For a topological vector space $(X, \tau) = ((X, +, \theta)_K, \tau)$ and a collection F of linear functionals on (X, τ), the following are equivalent

(i) F is equicontinuous.

(ii) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in (X, τ) and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) \in I$ such that $f(x_\delta) \in B_d(0, \varepsilon)$ for all $\delta \geq \delta_0$ and all $f \in F$ [$|B_d(0, \varepsilon)$ is the ball in K of radius ε, centered on 0$|$].

(iii) For every null net $(x_\delta)_{\delta \in (I, \leq)}$ in (X, τ) and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) \in I$ such that $\sup_{\substack{\delta \geq \delta_0 \\ f \in F}} |f(x_\delta)| \leq \varepsilon$. ///

THEOREM 2 ***Note* 6** For a first countable topological vector space $(X, \tau) = ((X, +, \theta)_K, \tau)$ and a collection F of linear functionals on (X, τ), the following are equivalent.

(i) F is equicontinuous.

(ii) For every null sequence $(x_n)_{n \in (\mathbb{N}, \leq)}$ in (X, τ) and $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $f(x_n) \in B_d(0, \varepsilon)$ for all $n \geq N$ and all $f \in F$ [$|B_d(0, \varepsilon)$ is the ball in K of radius ε, centered on 0$|$].

(iii) For every null sequence $(x_n)_{n \in (N, \leq)}$ in (X, τ) and $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\sup_{\substack{n \geq N \\ f \in F}} |f(x_n)| \leq \varepsilon$.

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A topological vector space (E, τ) is called a *locally convex space*, if it has a local base of neighbourhoods at zero comprising convex sets.

John Webb in [7] calls a set F of linear functionals on a Hausdorff locally convex space (E, τ) a *T-limited set* provided for every null sequence $(x_n)_{n \in (\mathbb{N}, \leq)}$ in (E, τ), $\lim_{n \to \infty} (\sup_{f \in F} |f(x_n)| = 0$; and so for every $\varepsilon > 0$, there exists a positive integer $N =$

$N(\varepsilon)$ such that

$\sup_{f \in F} |f(x_n)| < \varepsilon$ for all $n \geq N$.

And so, $\sup_{\substack{n \geq N \\ f \in F}} |f(x_n)| \leq \varepsilon$. That is, for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\sup_{\substack{n \geq N \\ f \in F}} |f(x_n)| \leq \varepsilon$.

We therefore, have from THEOREM 2 ***Note*** 6 that

THEOREM 3 ***Note*** **7** (i) If (E, τ), is a first countable Hausdorff locally convex space, then its T-limited sets are equicontinuous sets of linear functionals.

(ii) For a metrizable local convex space (E, τ), its T-limited sets are equicontinuous sets of linear functionals . ///

REMARK 4 (i) This paper results from the successful attempt of replacing John Webb's *null* sequence in his T-limited sets by a null net.

(ii) Another such successful attempt in replacing John Webb's null sequence by *bounded null nets* results in a description of the *continuous dual* of Person's [8] mixed topology of a bitopological space and a consequent characterization of separated locally convex space with complete strong dual. We report these elsewhere.

REMARK 5 We report elsewhere, also, an application of 3.8 ***Note*** 3 (i) ⇔ (v) to supremum $V\Phi$ of vector topologies Φ on the range space $E = (E, +, \theta^*)_K$.

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