Transactions of the Nigerian Association of Mathematical Physics
***Volume 17, (October – December, 2021 Issue*), pp35 – 40**
© Trans. of NAMP

A_0-STABLE RATIONAL INTEGRATOR FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

O. A. Elakhe,[1] A.O. Isere[2] and F. Ebhohimen[3]

Faculty of Physical Sciences, Ambrose Alli University, Ekpoma. Edo State. Nigeria.

Abstract

This research work is concerned with the determination of solution to different classes of problems in Ordinary Differential Equations (ODEs). We derived an A_0-Stable rational integrators for the solution of ordinary differential equations. We establish the convergence, consistency and the stability of our scheme in the interpolants of order m = 3, through the rational integrator. The stability analysis of the method was carried with the use of MAPLE-18 and MATLAB softwares. We compared our new and solve real-life problems which ascertain the convergence and consistency of scheme. Our result shows that our integrator is stable analytically and computationally.

1. Introduction

In Science and Engineering, mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields such as economics, medicine, psychology, operation research and even in areas such as biological simulations and anthropology, population models, electrical, network, biological, nuclear reactors mechanical oscillations, chemical kinetics, engineering work, process control [1,2]. Interestingly, differential equations arising from the modeling of physical phenomena often often do not have analytic solutions. Hence, the development of solutions becomes necessary, [3]
This research work is concerned with finding a suitable numerical solution to the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0, \;\; a \le x \le b \tag{1}$$

where$f(x,y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region D ⊂ [a,b] that are either Stiff or Singular, [4]. Also to determine its stability from both analytical and computational method.
In the course of this work, we shall make use of some terms, which are carefully defined for more understanding.

Definition 1: Elakhe [5]: A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $Re\, h\lambda < 0$.
However, A-stable is a severe requirement to ask of a numerical method for stiffness to be satisfied.

Definition 2: [6]: A one-step numerical method is said to be L-stable if it is A –stable and, in addition, when applied to the scalar test equation $y' = \lambda y,\ \lambda$ is a complex constant with $Re\, h\lambda < 0$, it yields $y_{n+1} = Re\,(h\lambda)y_n$,where $|Re\,(h\lambda)| \to 0$ as $Re\,(h\lambda) \to \infty$.

Definition 3: A_0-Stable by Fatunla (1987): An integration procedure is A_0-Stable if its region of absolute stability contains the negative real axis.

$$\text{i.e}\{\bar{h} * Im(h) = 0.\ \ Re(\bar{h}) < 0\} \ \subset RAS. \tag{2}$$

Definition 4: A_α-Stable by [7]: A numerical integration method is said to beA_α-Stable for some $\alpha \in [0, \frac{\pi}{2}]$ if the infinite wedge $S\alpha = \{\bar{h}: |Arg(-\bar{h})| < \alpha,\ \ \bar{h} \neq 0\}$ is contained in its region of absolute stability. A method is said to be A_0-Stable if it A_α-Stable for some $\alpha \in [0, \frac{\pi}{2}]$

2. DERIVATION OF THE METHED

The rational interpolant U : R → R is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{m} q_i x_{n+1}^i} \tag{3}$$

Corresponding Author**:** Elakhe F., Email: ebhohimenfidelis@yahoo.com, Tel: +2348066282372

where p_i *and* q_i are called the integrator parameters, and r is a constant parameter.

At m = 3, from (3) we have

$$y_{n+1} = r + \frac{\sum_{i=0}^{2} p_i x_{n+1}^{i}}{1+\sum_{i=1}^{3} q_i x_{n+1}^{i}} \tag{4}$$

$$y_{n+1} = r + [p_0 + p_1 x + p_2 x^2][1 + q_1 x + q_2 x^2 + q_3 x^3]^{-1}$$

$$= r + [p_0 + p_1 x + p_2 x^2]\{\sum_{i=0}^{\infty}(-1)^i [\, q_1 x + q_2 x^2 + q_3 x^3]^i\}$$

By the use of binomial expansion for rational functions and comparing with taylor series, we obtain the values of p_0, p_1, p_2, q_1, q_2, q_3 and r.

$$y_n = r + p_0 \tag{5}$$

$$r = y_n - p_0 \tag{6}$$

$$p_1 = \frac{hy_n^{(1)}}{1!\, x_{n+1}} + p_0 q_1$$

$$p_1 x_{n+1} = hy_n^{(1)} + p_0 q_1 x_{n+1} \tag{7}$$

$$p_2 = \frac{h^2 y_n^{(2)}}{2!\, x_{n+1}^2} + \frac{hy_n^{(1)}}{1!\, x_{n+1}} + p_0 q_2$$

$$p_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + hy_n^{(1)} q_1 x_{n+1} + p_0 q_2 x_{n+1}^2 \tag{8}$$

$$p_0 q_3 = \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_1 + \frac{h^1 y_n^{(1)}}{1! x_{n+1}} q_2$$

$$p_0 = \frac{-1}{q_3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right] \tag{9}$$

$$r = y_n - \frac{1}{q_3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right] \tag{10}$$

Now from (3), the rational integrator is expanded to give

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + r + r q_1 x_{n+1} + r q_2 x_{n+1}^2 + r q_3 x_{n+1}^3}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3} \tag{11}$$

Then

$$p_0 + r = r + p_0 = y_n \tag{12}$$

$$p_1 x_{n+1} + r q_1 x_{n+1} = hy_n^{(1)} - \frac{q_1 x_{n+1}}{q_3 x_{n+1}^3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right]$$

$$+ q_1 x_{n+1} y_n + \frac{q_1 x_{n+1}}{q_3 x_{n+1}^3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right]$$

$$p_1 x_{n+1} + r q_1 x_{n+1} = hy_n^{(1)} + q_1 x_{n+1} y_n \tag{13}$$

$$p_2 x_{n+1}^2 + r q_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + hy_n^{(1)} q_1 x_{n+1} - \frac{q_2 x_{n+1}^2}{q_3 x_{n+1}^3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right]$$

$$+ q_2 x_{n+1}^2 y_n + \frac{q_2 x_{n+1}^2}{q_3 x_{n+1}^3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right]$$

$$p_2x_{n+1}^2 + rq_2x_{n+1}^2 = \frac{h^2y_n^{(2)}}{2!} + hy_n^{(1)}q_1x_{n+1} + q_2x_{n+1}^2y_n \quad (14)$$

$$rq_3x_{n+1}^3 = q_3x_{n+1}^3y_n + \frac{q_3x_{n+1}^3}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x_{n+1}^2\right] \quad (15)$$

Substituting equs. (13), – (15) into (11), we obtain our general integrator

$$= \frac{y_n + hy_n^{(1)} + q_1x_{n+1}y_n + \frac{h^2y_n^{(2)}}{2!} + hy_n^{(1)}q_1x_{n+1} + q_2x_{n+1}^2y_n + \frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x_{n+1}^2 + q_3x_{n+1}^3y_n}{1 + q_1x_{n+1} + q_2x_{n+1}^2 + q_3x_{n+1}^3}$$

Now, we let

A= $q_1x_{\mathrm{n}+1}$, B = $q_2x_{n+1}^2$, and C = $q_3x_{n+1}^3$ (16)

Therefore, our integrator formula becomes

$$y_{n+1} = \frac{\sum_{i=0}^{3}\frac{h^iy_n^{(i)}}{i!} + A\sum_{i=0}^{2}\frac{h^iy_n^{(i)}}{i!} + B\sum_{i=0}^{1}\frac{h^iy_n^{(i)}}{i!} + Cy_n}{1+A+B+C} \quad (17)$$

4. STABILITY CONSIDERATION

To effectively solve initial value problems in ordinary differential equations which is stiff, we need numerical integrators that possess special stability properties such as A_0-Stability. To test for stability, we apply the integrator to the test equation.

$$y' = \lambda y \quad (18)$$

where by induction, will give $y_n^{(r)} = \lambda^r y_n$ and $\lambda h = \bar{h}$. We have

$$\zeta(\bar{h}) = \frac{y_{n+1}}{y_n} = \frac{120+60\bar{h}+30\bar{h}^2+\bar{h}^3}{120-60\bar{h}+30\bar{h}^2-\bar{h}^3} \quad (19)$$

Our new integrator is said to be Absolutely Stable if $\left|\zeta(\bar{h})\right| \le 1$,

If we let $\zeta(\bar{h}) = \frac{y_{n+1}}{y_n} = \frac{120+60\bar{h}+30\bar{h}^2+\bar{h}^3}{120-60\bar{h}+30\bar{h}^2-\bar{h}^3}$

It implies $\left|\zeta(\bar{h})\right| \le 1 \iff \left\|\frac{\phi(\bar{h})}{\Psi(\bar{h})}\right\| \le 1$

To anaylse the formula, we set $\bar{h} = u + iv$ where $i^2 = -1$,

Hence $\|\phi(\bar{h})\| \le \|\Psi(\bar{h})\| \therefore \|\phi(\bar{h})\| - \|\Psi(\bar{h})\| \le 0$

Where $\phi(\bar{h}) = 120 + 60\bar{h} + 30\bar{h}^2 + \bar{h}^3$ and

$\Psi(\bar{h}) = 120 - 60\bar{h} + 30\bar{h}^2 - \bar{h}^3$

Let $\phi(\mathrm{u},\mathrm{v}) = A(u,v) + iB(u,v)$ and set $\bar{h} = u + iv$ where $i^2 = -1\Psi(\mathrm{u},\mathrm{v}) = C(u,v) + iD(u,v)$

$\Rightarrow \phi(\mathrm{u},\mathrm{v}) = 120 + 60(u+iv) + 30(u+iv)^2 + (u+iv)^3$

$\phi(\mathrm{u},\mathrm{v}) = 120 + 60(u+iv) + 30(u^2 + 2uvi + v^2i^2) + (u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3)$

$Also \ \Psi(\mathrm{u},\mathrm{v}) = 120 - 60(u+iv) + 30(u+iv)^2 - (u+iv)^3$

$\Psi(\mathrm{u},\mathrm{v}) = 120 - 60(u+iv) + 30(u^2 + 2uvi + v^2i^2) - (u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3)$

On expansion, we get

$A(\mathrm{u},\mathrm{v}) = u^3 - 3uv^2 + 30u^2 - 30v^2 + 60u + 120$

$B(\mathrm{u},\mathrm{v}) = 3u^2v - v^3 + 60uv + 60v$

$C(u,v) = -u^3 + 3uv^2 + 30u^2 - 30v^2 - 60u + 120$

$D(u,v) = -3u^2v + v^3 + 60uv - 60v$

Hence, the inequality becomes

$\left|\zeta(\bar{h})\right| \le 1 \iff |A(u,v) + iB(u,v)| \le |C(u,v) + iD(u,v)|$

$\iff A(u,v)^2 + iB(u,v)^2 \le A(u,v)^2 + iB(u,v)^2$

$\Leftrightarrow A(u,v)^2 + iB(u,v)^2 - A(u,v)^2 + iB(u,v)^2 \leq 0$
Simplifying further, the inequality above holds if and only if
$A^2 = u^6 - 6u^4v^2 + 9u^2v^4 + 60u^5 - 240u^3v^2 + 180uv^4 + 1020u^4 - 2160u^2v^2 + 900v^4$
$+3840u^4 - 4320uv^2 + 10800u^2 - 7200v^2 + 14400u + 14400$
$B^2 = 9u^4v^2 - 6u^4v^2 + 360u^3v^2 - 120uv^4 + 3960u^2v^2 - 120v^4 + 7200uv^2 + 3600v^2$
$A^2 + B^2 = u^6 + 3u^4v^2 + 3u^2v^4 + v^6 + 60u^5 + 120u^3v^2 + 60uv^4 + 1020u^4 + 1800u^2v^2$
$+780v^4 + 380u^3 + 2880uv^2 + 10800u^2 - 3600v^2 + 14400u + 14400$
$C^2 = u^6 - 6u^4v^2 + 9u^2v^4 - 60u^5 + 240u^3v^2 - 180uv^4 + 1020u^4 - 2160u^2v^2 + 900v^4$
$-3840u^3 + 4320uv^2 + 10800u^2 - 7200v^2 - 14400u + 14400$
$D^2 = 9u^4v^2 - 6u^4v^2 - 360u^3v^2 + 120uv^4 + 3960u^2v^2 - 120v^4 - 7200uv^2 + 3600v^2$
$C^2 + D^2 = u^6 + 3u^4v^2 + 3u^2v^4 + v^6 - 60u^5 - 120u^3v^2 - 60uv^4 + 1020u^4 + 1800u^2v^2$
$+780v^4 - 380u^3 - 2880uv^2 + 10800u^2 - 3600v^2 - 14400u + 14400$
$A^2 + B^2 - (C^2 + D^2) = 120u^5 + 240u^3v^2 + 120uv^4 + 7680u^3 + 5760uv^2 + 28800u \leq 0$
Which shows that the integrator is $A_0 -$ Stable. If and only if $u < 0$

5. NUMERICAL EXPERIMENTS

We shall be concerned with the application of our new integrator in solving a number of problems in this section. This integrator have been designed for and also determine the convergence and consistency structure of the method.

Problem 1: [9]
$y' = -y; \quad y(0) = 0.5; \ 0 \leq x \leq 1, \quad h = 0.1$
The exact solution is $y(x_n) = \frac{1}{e^{x_n}}$

Table 1. Error in Numerical Integration

XN	TSOL	YN (4th-stage)	New Method
.1D+00	0.90D+00	-.81D-07	1.864e-08
.2D+00	0.81D+00	-.14D-06	3.374e-08
.3D+00	0.74D+00	-.20D-06	4.579e-08
.4D+00	0.67D+00	-.24D-06	5.525e-08
.5D+00	0.60D+00	-.27D-06	6.249e-08
.6D+00	0.54D+00	-.29D-06	6.785e-08
.7D+00	0.49D+00	-.31D-06	7.163e-08
.8D+00	0.44D+00	-.32D-06	7.407e-08
.9D+00	0.40D+00	-.33D-06	7.540e-08
1D+00	0.36D+00	-.33D-06	7.581e-08

The table above shows the performance of our new numerical integrator. Our new integrator compete favorably well with the explicit fourth-stage, fourth-order Runge-Kutta methods of [9] with a very rate of convergence.

Problem 2: [10] [Real Life Problem]

A new cereal product is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years? The differential equation for this problem is given by $\frac{dy}{dt} = k(1-y); \quad y = 1 - e^{-0.693t}$where y represents the number (in millions) of people at time t who have heard of the product. This means that $(1-y)$ is the number of people who have not heard, and dy/dt is the rate at which the population hears about the product. The exact solution of this problem can easily be obtained as $y = 1 - e^{-kt}$, where $k = 0.693$.

Table 2: Error in Numerical Integration

XN (Time)	TSOL	Elakhe et al (2020)	New method
1.0e+00	4.9992e-01	-7.3595e-05	2.2665e-06
2.0e+00	7.4992e-01	-5.2051e-05	7.4408e-06
3.0e+00	8.7494e-01	-3.6816e-05	2.0639e-06
4.0e+00	9.3746e-01	-2.6044e-05	2.0664e-06
5.0e+00	9.6872e-01	-1.8447e-05	1.5976e-06
6.0e+00	9.8436e-01	-1.3038e-05	1.0928e-06
7.0e+00	9.9217e-01	-9.2181e-06	6.9643e-07
8.0e+00	9.9608e-01	-6.5177e-06	4.2395e-07
9.0e+00	9.9804e-01	-4.6085e-06	2.5002e-07
10.e+00	9.9902e-01	-3.2586-e06	1.4407e-07

The result which is shown in the numerical solution in table 2 agree excellently with the result at 2 years.

6. CONCLUSION

The use of numerical integrators in solving initial value problems have been demonstrated using A_0-Stable rational integrator and have proven more effective. The numerical tables shows that our new integrator converges quickly to the analytic solution at each corresponding meh point than its counterparts. The integrator compares favourably well with the recent work Esekhaigbe, (2017) and Elakhe and Ehika (2020).

References

[1] Abhulimen, C. E. (2009): An exponential fitting predictor- corrector formula for stiff systems of Ordinary Differential Equations. International Journal of Computational and Applied Mathematics, vol.4, No 2, 115 – 126.

[2] Ebhohimen F. and Anetor O. (2017);The Stability of the Rational Interpolation Method in Ordinary Differential Equations at k = 6.Transactions of the Nigerian Association of Mathematical physics Volume 5, (September and November, (2017);pp 33-38.

[3] Elakhe O. A. and Aashikpelokhai, U.S.U. (2011): On A Singulo Oscillatory-Stiff Rational Integrator. International Journal of Natural and Applied Science, Vol. 8, pp1703-1715.

[4] Elakhe O. A. and Aashikpelokhai, U.S.U. (2013): A High Accuracy Order Three and Four Numerical Integrator for initial value problems, International Journal of Numerical Mathematics,, Vol. 6, pp57 – 71.

[5] Elakhe O. A. (2011):Singulo-Oscillatory Stiff Rational Integrator.Ph.D Thesis, Ambrose Alli University, Ekpoma. 173pp.

[6] Aashikpelokhai, U.S.U (2010): A general [L, M] One-step integrator for Initial Value Problems. International Journal of Computer Mathematics, vol. 1 – 12.

[7] Fatunla, S.O. (1978); "An implicit Two-part numerical integration Formula for linear and non-linear stiff system of ODE", Mathematics of Computation 32,1-11.

[8] Lambert J.D. (1995): Numerical Methods for Ordinary Differential Systems. John Wiley and Sons Limited, England, 293 pages.

[9] Esekhaigbe, C. A. (2017): On the component analysis and transformation of an explicit fourth-stage fourth-order Runge-Kutta methods.Ph.D Thesis, Ambrose Alli University, Ekpoma. 85pp.

[10] Elakhe O. A ,Ehika, E and Ehika S. (2020): A quartic based denominator of order six rational integrator. J, Physical & Applied Sciences, Vol.2, No 1, Ambrose Alli University, Ekpoma.

[11] Aashikpelokhai, U.S.U (2010): A general [L, M] One- step integrator for Initial Value Problems. International Journal of Computer Mathematics, vol. 1 – 12.

[12] Aashikpelokhai, U.S.U (2014): Renaissance of Mathematics in our Time. Pub. FNS AAU Ekpoma

[13] Ebhohimen F. and Anetor O. (2017);The Stability of the Rational Interpolation Method in Ordinary Differential Equations at k = 6.Transactions of the Nigerian Association of Mathematical physics Volume 5, (September and November, (2017);pp 33-38.

[14] Elakhe O. A ,Ehika, E and Ehika S. (2020): A quartic based denominator of order six rational integrator. J, Physical & Applied Sciences, Vol.2, No 1, Ambrose Alli University, Ekpoma.

[15] Elakhe O.A. and Aashikpelokhai, U.S.U. (2010): On ASingulo-Stiff Rational Integrator. International Journal of Natural and Applied Science, Vol. 2, pp43-53.

[16] Elakhe O. A. (2011):Singulo-Oscillatory Stiff Rational Integrator.Ph.D Thesis, Ambrose Alli University, Ekpoma. 173pp.

[17] Elakhe O. A. and Aashikpelokhai, U.S.U. (2013): On A Singulo Oscillatory-Stiff Rational Integrator. International Journal of Natural and Applied Science, Vol. 8, pp1703-1715.

[18] Esekhaigbe, C. A. (2017): Transformation and Implementation of a Highly Efficient Fully Implicit Forth-Order Runge-Kutta Method. International Journal of Innovative Research, vol 5(1), 171 – 183.

[19] Fatunla, S.O. (1978); "An implicit Two-part numerical integration Formula for linear and non-linear stiff system of ODE", Mathematics of Computation 32,1-11.

[20] Lambert J.D and Shaw B. (1965): "On the Numerical Solution of y' = f(x,y) by a class of formulae based on Rational Approximation". Mathematics of Computation19:456- 462.