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SOLUTION OF ORDINARY DIFFERENTIAL EQUATION: PERTURBATION ITERATION METHOD APPROACH

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Abstract

A Perturbation iteration algorithm for solving differential equations of first order is proposed. The applications of the new method to systems of first order ordinary differential equations are highlighted with four perturbation parameters considered.
The results obtained using the model were compared to the exact solution of a first order ordinary differential equation problem after five iterations were carried out, a minimal error was obtained in the four perturbation parameters considered. Graphical representations of the results clearly show the relationship between the exact solutions and the approximate solutions at each iteration stage.
Based on the results presented, it is concluded that the lower the perturbation parameter, the greater the efficiency of this model. Nevertheless, as the perturbation parameter increases, more iterations is expected to be carried out to get an accurate result. However, the model is efficient in solving first order differential equation.

***Keywords*:** Perturbation Methods, Perturbation Iteration Algorithms, First Order Differential Equations

INTRODUCTION

Perturbation method is one of the pioneering techniques to obtain approximate analytical solutions for mathematical models. It was introduced by S.D. Poisson and extended by J.H. Poincare. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear equations was used only a bit later. The most significant efforts were focused on celestial mechanics, fluid mechanics and aerodynamics. It has also been successfully applied to differential equations and algebraic equations. Many different perturbation techniques such as the method of averaging, the method of multiple scales, the renormalization method, the Lindstedt-Poincare method, the method of matched asymptotic expansion, and their variants were developed within time [1]. One of the deficiencies in applying perturbation methods is that a small parameter is needed in the equations or the small parameter should be introduced artificially to the equations. The solutions therefore have a limited range of validity. Nevertheless, the solved problem is a weak nonlinear problem and it becomes hard to obtain a valid approximate solution for strongly nonlinear systems.

Perturbation iteration method has been successfully applied to different types of equations but there is a need to check its efficiency on ordinary differential equations. With an inspiration from the work on algebraic equations, the systematic approach of combining perturbation and iterations was applied to ordinary differential equations. The new algorithm developed would be applied to first order ordinary differential equations.

Perturbation theory has been successfully applied in different ways to different types of equations. Many researchers have used perturbation iteration algorithm to produce various root finding schemes while others combined it with other methods to solve different problems. The review of its applications in different areas is presented below.
Dolapci developed an iteration algorithm PIA(1,1) and applied to some fredholm and volterra type of integral equations for the first time. Their numerical results show that method PIA (1,1) is an effective perturbation-iteration technique producing

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successful analytic results for integral equations. Aksoy and Pakdemirli used perturbation iteration method to solve Bratu-type equations. In their work:

a) A symmetric algorithm approach for developing new perturbation iteration is presented.

b) The perturbation iteration algorithm developed do not require a "small perturbation parameter" assumptions for prerequisite for valid solutions.

c) The perturbation iteration algorithms are applied successfully to Bratu-type nonlinear problems and iterations solutions with a few steps converge to numerical ones.

With the systematic approach used in their study, new algorithm with PIA(n,m) (n: number of correction terms in the perturbation expansion; m: order of derivatives in the taylors series expansion; n≤m) can be constructed easily.

In [1] effort, various root finding schemes are produced by employing perturbation theory. Depending on the number of correction terms, number of terms in the Taylors expansions and separation of equations, many different algorithms are produced. Some of those algorithms are the well-known formulas such as Newton-Raphson and Householders iteration and some are higher order iterations. The formulas as well as two recent algorithms are contrasted with each other. As expected as the number of correction terms in the perturbation expansion increases, the iteration schemes perform better and less iterations are needed. As far as the convergence intervals of a specific root are considered, a gain is not detected by additional correction terms. Pakdemirli et al. showed that one may take n correction term in the perturbation expansion and m additional terms in the Taylors expansion. Obviously m≥n for all unknowns to be solved. From his paper, one may conclude that the performance becomes better as n increases with an optimum selection of m=n. in his paper, m=n=4 is the best algorithm selected compared to the m=4, n=3 and m=4, n=2 algorithms.

METHODS

Generally, perturbation iteration method explore the Taylors series expansion to obtain approximate analytical solutions of n^{th} order ordinary differential equation. However, this study is limited to first order ordinary differential equation.

For the roots of the nonlinear equation

$$f(x) = 0 \quad (1)$$

A perturbation expansion of the below form with n correction terms might be assumed

$$x = x_o + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots + \varepsilon^n x_n \quad (2)$$

Inserting (2) into (1) and expanding in a taylors series up to m^{th} order derivative terms yields

$$f(x_o + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots + \varepsilon^n x_n) \cong f(x_0) + f'(x_0)(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots + \varepsilon^n x_n) + \frac{f''(x_0)}{2!}(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots + \varepsilon^n x_n)^2 + \cdots + \frac{f^m(x_0)}{m!}(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots + \varepsilon^n x_n)^m = 0 \quad (3)$$

Note that since n terms in the perturbation expansion and m^{th} order derivatives in the Taylors series are considered, the perturbation iteration algorithm developed will be named $PIA(n, m)$.

n should be always less than or equal to m, otherwise the unknowns (correction terms in the perturbation expansion) cannot be determined. Equation (3) should be grouped with respect to the orders of ε, then separated and solved for the unknown correction terms. Substituting back the correction terms into (2) yields an iteration algorithm for solution of (1). Note that separations may not be unique and there might be different ways of separating (3). Below are the details of the algebraic equations.

$$f(x) = 0$$

$$x = x_o + \varepsilon x_1$$

Taylors expansion

$$f(x_0 + \varepsilon x_1) + f(x_0) + f'(x_0)\varepsilon x_1 = 0$$

$$f(x_0) + f'(x_0)\varepsilon x_1 = 0$$

$$\varepsilon x_1 = -\frac{f(x_0)}{f'(x_0)}$$

(Newton-Raphson Equation)

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (4)$$

$[PIA(1,1)]$

In this work, (PIA 1,1) is applied to first order differential equation.

Consider the general first order differential equation

$$F(u, \dot{u}, \varepsilon) = 0, \quad (5)$$

with $u = u(t)$ *and* ε the perturbation parameter, only one correction term is taken in the perturbation expansion.

$u_{n+1} = u_n + \varepsilon(u_c)_n + \cdots$ (6)

Upon substitution of (5)into (6) and expanding in a Taylor series with first derivative only yields

$F(u,\dot{u},0) + F_u(u,\dot{u},0)\varepsilon u + F_{\dot{u}}(u,\dot{u},0)\varepsilon\dot{u} + F\varepsilon(u,\dot{u},0)\varepsilon = 0$

$F + F_u\varepsilon u + F_u\varepsilon\dot{u} + F_{\dot{u}}\varepsilon\dot{u} + F_{\dot{\varepsilon}}\varepsilon$=0, (7)

where subscripts denote differentiation with respect to the variable .

Note that in this method, the function and its derivatives are considered to be independent variables.

Rearranging the equation:

$$\dot{u}_c + \frac{F_u}{F_{\dot{u}}}u_c = -\left[\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{\dot{u}}}\right] \quad (8)$$

$$= e^{\int\frac{F_u}{F_{\dot{u}}}dx}. \quad (9)$$

Multiply through by $e^{\int\frac{F_u}{F_{\dot{u}}}dx}$

$$d\left[ue^{\int\frac{F_u}{F_{\dot{u}}}dx}\right] = \left[-\left[\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_u}\right]e^{\int\frac{F_u}{F_{\dot{u}}}dx}\right]dx\ . \quad (10)$$

Integrate both sides to have

$$u_c = ce^{\left(-\int\frac{F_u}{F_{\dot{u}}}dx\right)} - \left[\left[\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_u}\right]e^{\int\frac{F_u}{F_{\dot{u}}}dx}\right]dxe^{\int\frac{F_u}{F_{\dot{u}}}dx}. \quad (11)$$

Substituting equation (11) into equation (6) and constructing the iteration scheme yields

$$u_{n+1} = u_n + \varepsilon c_n e\left[-\int\frac{F_u(u_{n,\dot{u}_{n,0}})}{F_{\dot{u}}(u_n,\dot{u}_n,0)}dt\right] - \varepsilon\left[\int\frac{\frac{F_\varepsilon(u_n,\dot{u}_n,0)+F(u_n,\dot{u}_n,0)}{\varepsilon}}{F_{u(u_n,\dot{u}_n,0)}}e\left[\int\frac{F_u(u_n,\dot{u}_n,0)}{F_{\dot{u}(u_n,\dot{u}_n,0)}}dt\right]dt\right]e\left[-\int\frac{F_u(u_n,\dot{u}_n,0)}{F_{\dot{u}(u_n,\dot{u}_n,0)}}dt\right] \quad (12)$$

EXAMPLE PROBLEM

Consider the differential equation with the condition

$\dot{u} + \varepsilon u^2 = 0$ $\quad u(0) = 1$ (Boyaci and Pakdemirli, 2007) (13)

whose exact solution is $u = \frac{1}{1+\varepsilon t}$

$\dot{u} + \varepsilon u^2 = 0$

$\frac{du}{u^2} = -\varepsilon dt.$ (14)

Integrate both sides to have

$-\frac{1}{u} = -\varepsilon t + c$ (15)

$since\ u(0) = 1$

$u = \frac{1}{\varepsilon t+1} \quad or \quad u = \frac{1}{1+\varepsilon t}$ (16)

Using equation (12)

$u_{n+1} = \varepsilon c_n - \varepsilon\int {u_n}^2 dt$ (17)

In applying the iteration formula, an initial guess satisfying the initial conditions should be selected and at each step c_n coefficient have to be determined from the initial condition. Selecting $u_0 = 1$

When $n = 0$

$u_1 = \varepsilon c_n - \varepsilon\int(u_0)^2 dt$ (18)

$u_1 = \varepsilon c_n - \varepsilon t$

$using\ u(0) = 1$

$1 = \varepsilon c_n - \varepsilon(0)$

$u_1 = 1 - \varepsilon t$ (19)

When $n = 1$

$u_2 = \varepsilon c_n - \varepsilon\int {u_1}^2 dt$ (20)

$u_2 = 1 - \varepsilon t + \varepsilon^2 t^2 - \frac{\varepsilon^3 t^3}{3}$ (21)

When $n = 2$

$u_3 = \varepsilon c_n - \varepsilon\int {u_2}^2 dt$ (22)

$u_3 = 1 - \varepsilon t + \varepsilon^2 t^2 - \varepsilon^3 t^3 + \frac{2\varepsilon^4 t^4}{3} - \frac{\varepsilon^5 t^5}{3} + \frac{\varepsilon^6 t^6}{9} - \frac{\varepsilon^7 t^7}{63}$ (23)

When $n = 3$

$u_4 = \varepsilon c_n - \varepsilon \int {u_3}^2 dt$ (24)

$u_4 = 1 - \varepsilon t + \varepsilon^2 t^2 - \varepsilon^3 t^3 + \varepsilon^4 t^4 - \frac{13\varepsilon^5 t^5}{15} + \frac{2\varepsilon^6 t^6}{3} - \frac{38\varepsilon^7 t^7}{63} + \frac{71\varepsilon^8 t^8}{252} - \frac{86\varepsilon^9 t^9}{567} + \frac{22\varepsilon^{10} t^{10}}{315}$
$- \frac{55\varepsilon^{11} t^{11}}{2079} + \frac{\varepsilon^{12} t^{12}}{126} - \frac{\varepsilon^{13} t^{13}}{567} + \frac{\varepsilon^{14} t^{14}}{3969} - \frac{\varepsilon^{15} t^{15}}{59535}$ (25)

When $n = 4$

$u_5 = \varepsilon c_n - \varepsilon \int {u_4}^2 dt$ (26)

$u_5 = 1 - \varepsilon t + \varepsilon^2 t^2 - \varepsilon^3 t^3 + \varepsilon^4 t^4 - \varepsilon^5 t^5 + \frac{86\varepsilon^6 t^6}{90} - \frac{111\varepsilon^7 t^7}{105} + \frac{1348\varepsilon^8 t^8}{2520} - \frac{3677\varepsilon^9 t^9}{5670} + \frac{20303\varepsilon^{10} t^{10}}{28350} -$
$\frac{17447\varepsilon^{11} t^{11}}{44500} + \frac{19459\varepsilon^{12} t^{12}}{68040} - \frac{2921\varepsilon^{13} t^{13}}{14742} + \frac{491\varepsilon^{14} t^{14}}{3780} - \frac{72874\varepsilon^{15} t^{15}}{893025} + \frac{73732\varepsilon^{16} t^{16}}{1587600} - \frac{1222093\varepsilon^{17} t^{17}}{1686825} + \frac{80657\varepsilon^{18} t^{18}}{6429780} -$
$\frac{702325339\varepsilon^{19} t^{19}}{1.212179\times 10^{11}} + \frac{3517\varepsilon^{20} t^{20}}{1428840} - \frac{272431457\varepsilon^{21} t^{21}}{8.934179\times 10^{10}} + \frac{6257\varepsilon^{22} t^{22}}{18336780} - \frac{6890899\varepsilon^{23} t^{23}}{6.262927\times 10^{10}} + \frac{16987\varepsilon^{24} t^{24}}{540101520} - \frac{58453\varepsilon^{25} t^{25}}{7426395900} +$
$\frac{53\varepsilon^{26} t^{26}}{46437300} - \frac{2\varepsilon^{27} t^{27}}{6751269} + \frac{13\varepsilon^{28} t^{28}}{315059220} - \frac{29\varepsilon^{29} t^{29}}{6852538035} + \frac{2\varepsilon^{30} t^{30}}{7088832450} - \frac{\varepsilon^{31} t^{31}}{1.098769\times 10^{11}}$ (27)

INTRODUCTION OF THE PERTURBATION PARAMETERS

The exact solution for the problem in consideration ($\dot{u} + \varepsilon u^2 = 0$) is $u = \frac{1}{1+\varepsilon t}$

In this work, the perturbation parameter ε is considered to be 0.001, 0.005, 0.01 and 0.05

At $\varepsilon = 0.001$

$u_1 = 1 - 0.001t$ (28)

$u_2 = 1 - 0.001t + 0.001^2 t^2 - \frac{1}{3} 0.001^3 t^3$ (29)

$u_3 = 1 - 0.001t + 10^{-6} t^2 - 10^{-9} t^3 + \frac{2}{3} 10^{-12} t^4 - \frac{1}{3} 10^{-15} t^5 + \frac{1}{9} 10^{-18} t^6 - \frac{1}{63} 10^{-21} t^7$ (30)

$u_4 = 1 - 0.001t + 10^{-6} t^2 - 10^{-9} t^3 + 10^{-12} t^4 - \frac{13}{5} 10^{-15} t^5 + \frac{2}{3} 10^{-18} t^6 - \frac{38}{63} 10^{-21} t^7 + \frac{71}{252} 10^{-24} t^8 - \frac{89}{567} 10^{-27} t^9 +$
$\frac{22}{315} 10^{-30} t^{10} - \frac{55}{2079} 10^{-33} t^{11} \ldots$ (31)

$u_5 = 1 - 0.001t + 10^{-6} t^2 - 10^{-9} t^3 + 10^{-12} t^4 - 10^{-15} t^5 + \frac{89}{90} 10^{-18} t^6 - \frac{111}{105} 10^{-21} t^7 + \frac{1348}{2520} 10^{-24} t^8 - \frac{3677}{5670} 10^{-27} t^9 +$
$\frac{20303}{28350} 10^{-30} t^{10} - \frac{17447}{44550} 10^{-33} t^{11} + \frac{19459}{68040} 10^{-36} t^{12} - \frac{2921}{14742} 10^{-39} t^{13} \ldots$ (32)

At $\varepsilon = 0.005$

$u_1 = 1 - 0.005t$ (33)

$u_2 = 1 - 0.005t + 2.5 \times 10^{-5} t^2 - \frac{1}{3} 1.25 \times 10^{-7} t^7$ (34)

$u_3 = 1 - 0.005t + 2.5 \times 10^{15} t^2 - 1.25 \times 10^{-7} t^3 + \frac{2}{3} 6.25 \times 10^{-4} t^4 - \frac{1}{3} 6.25 \times 10^{-10} t^5 + \frac{1}{9} 1.5625 \times 10^{-14} t^6 -$
$\frac{1}{63} 7.8125 \times 10^{-7} t^7$ (35)

$u_4 = 1 - 0.005t + 2.5 \times 10^{-5} t^2 - 1.25 \times 10^{-7} t^3 + 6.25 \times 10^{-10} t^4 - \frac{13}{15} 3.125 \times 10^{-12} t^5 + \frac{2}{3} 3.125 \times 10^{-12} t^6 -$
$\frac{38}{63} 7.8125 \times 10^{-17} t^7 + \frac{71}{252} 3.90625 \times 10^{-19} t^8 - \frac{89}{567} 1.953125 \times 10^{-21} t^9 + \frac{22}{315} 9.765625 \times 10^{-24} t^{10} - \frac{55}{2079} 4.8828 \times$
$10^{-26} t^{11} \ldots$ (36)

$u_5 = 1 - 0.005t + 2.5 \times 10^{-5} t^2 - 1.25 \times 10^{-7} t^3 + 6.25 \times 10^{-10} t^4 - 3.125 \times 10^{-12} t^5 + \frac{89}{90} 1.5625 \times 10^{-14} t^6 -$
$\frac{111}{105} 7.8125 \times 10^{-17} t^7 + \frac{1348}{2520} 3.9063 \times 10^{-19} t^8 - \frac{3677}{5670} 1.9531 \times 10^{-21} t^9 + \frac{20303}{28350} 9.7656 \times 10^{-24} t^{10} - \frac{17447}{44550} 4.8828 \times$
$10^{-26} t^{11} + \frac{19459}{68040} 2.4414 \times 10^{-28} t^{12} - \frac{2921}{14742} 1.2207 \times 10^{-30} t^{13} \ldots$ (37)

At $\varepsilon = 0.01$

$u_1 = 1 - 0.01t$ (38)

$u_2 = 1 - 0.01t + 10^{-4} t^2 - \frac{1}{3} 10^{-6} t^3$ (39)

$u_3 = 1 - 0.01t + 10^{-4} t^3 - 10^{-6} t^3 + \frac{2}{3} 10^{-12} t^4 - \frac{1}{3} 10^{-10} t^5 + \frac{1}{9} 10^{-12} t^6 - \frac{1}{63} 10^{-14} t^7$ (40)

$u_4 = 1 - 0.01t + 10^{-4} t^2 - 10^{-6} t^3 + 10^{-8} t^4 - \frac{13}{15} 10^{-10} t^5 + \frac{2}{3} 10^{-6} t^6 - \frac{38}{63} 10^{-14} t^7 + \frac{71}{252} 10^{-16} t^8 - \frac{89}{567} 10^{-18} t^9 +$
$\frac{22}{315} 10^{-20} t^{10} - \frac{55}{2079} 10^{-22} t^{11} \ldots$ (40)

$u_5 = 1 - 0.01t + 10^{-4}t^2 - 10^{-6}t^3 + 10^{-8}t^4 - 10^{-10}t^5 + \frac{89}{90}10^{-12}t^6 - \frac{111}{105}10^{-14}t^7 + \frac{1348}{2520}10^{-16}t^8 - \frac{3677}{5670}10^{-18}t^9 + \frac{20303}{28350}10^{-20}t^{10} - \frac{17447}{44550}10^{-22}t^{11} + \frac{19459}{68040}10^{-24}t^{12} - \frac{2921}{14742}10^{-26}t^{13} \ldots$ (41)

At $\varepsilon = 0.05$

$u_1 = 1 - 0.05t$ (42)

$u_2 = 1 - 0.05t + 2.5 \times 10^{-3}t^2 - \frac{1}{3}1.25 \times 10^{-4}t^3$ (43)

$u_3 = 1 - 0.05t + 2.5 \times 10^{-3}t^2 - 1.25 \times 10^{-4}t^3 + \frac{2}{3}6.25 \times 10^{-6}t^4 - \frac{1}{3}3.125 \times 10^{-7}t^5 + \frac{1}{9}1.5625 \times 10^{-8}t^6 - \frac{1}{63}7.8125 \times 10^{-10}t^7$ (44)

$u_4 = 1 - 0.05t + 2.5 \times 10^{-3}t^2 - 1.25 \times 10^{-4}t^3 + 6.25 \times 10^{-6}t^4 - \frac{13}{15}3.125 \times 10^{-7}t^5 + \frac{2}{3}1.562510^{-8}t^6 - \frac{38}{63}7.8125 \times 10^{-10}t^7 + \frac{71}{252}3.9063 \times 10^{-11}t^8 - \frac{89}{567}1.9531 \times 10^{-12}t^9 + \frac{22}{315}9.7666 \times 10^{-14}t^{10} - \frac{55}{2079}4.8828 \times 10^{-15}t^{11} \ldots$ (45)

$u_5 = 1 - 0.05t + 2.5 \times 10^{-3}t^2 - 1.25 \times 10^{-4}t^3 + 6.25 \times 10^{-6}t^4 - 3.125 \times 10^{-7}t^5 + \frac{86}{90}1.5625 \times 10^{-9}t^6 - \frac{111}{105}7.8125 \times 10^{-10}t^7 + \frac{1348}{2520}3.9063 \times 10^{-11}t^8 - \frac{3677}{5670}1.9531 \times 10^{-12}t^9 + \frac{20303}{28350}9.7656 \times 10^{-14}t^{10} - \frac{17447}{44550}4.8828 \times 10^{-15}t^{11} + \frac{19459}{68040}2.4414 \times 10^{-16}t^{12} - \frac{2921}{14742}1.2207 \times 10^{-17}t^{13}$ (46)

RESULT AND DISCUSSION

The result of the problem is presented here where t is considered at 0.1, 0.2, 0.3, 0.4, 0.5 are summarized in the tables and figures below.

PERTURBATION PARAMETER $\boldsymbol{\varepsilon}$ AT 0.001

Table 1: *Relationship between u exact,* u_1, u_2, u_3, u_4 *and* u_5 *when the perturbation parameter* $\varepsilon = 0.001$

T	Exact	Approximate Solution									
		u_1	Error	u_2	Error	u_3	Error	u_4	Error	u_5	Error
0.0	1	1	0	1	0	1	0	1	0	1	0
0.1	0.99990001	0.9999	9.0001×10^{-4}	0.99990001	0	0.99990001	0	0.99990001	0	0.99990001	0
0.2	0.99980004	0.9998	4.0×10^{-8}	0.99980004	0	0.99980004	0	0.99980004	0	0.99980004	0
0.3	0.99970009	0.9997	9.0×10^{-8}	0.99970009	0	0.99970009	0	0.999700089	1.0×10^{-9}	0.99970009	0
0.4	0.999600159	0.9996	1.59×10^{-7}	0.99960016	-1.0×10^{-9}	0.999600159	0	0.999600158	1.0×10^{-9}	0.999600159	0
0.5	0.999500249	0.9995	2.49×10^{-7}	0.99950025	-1.0×10^{-9}	0.999500249	0	0.999500237	1.2×10^{-8}	0.999500249	0

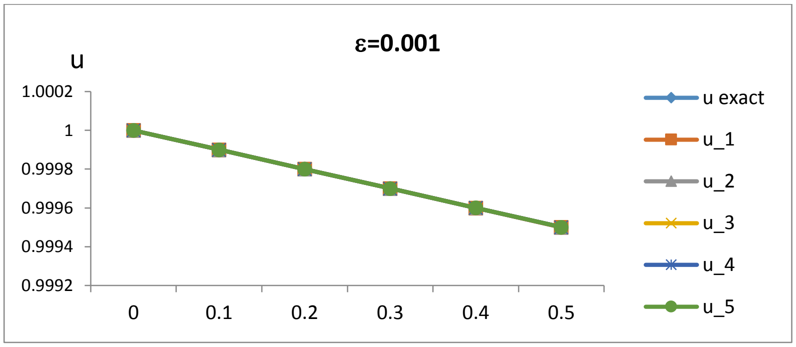

Figure 1: Relationship between $u - exact, u_1, u_2, u_3, u_4$ $and\ u_5$ $when\ the\ perturbation\ parameter\ \varepsilon = 0.001$

PERTURBATION PARAMETER ε AT 0.005

Table 2: $Relationship\ between\ u\ exact, u_1, u_2, u_3, u_4\ and\ u_5\ when\ the\ perturbation\ parameter\ \varepsilon = 0.005$

t	Exact Solution	Approximate Solution									
		u_1	Error	u_2	Error	u_3	Error	u_4	Error	u_5	Error
0.0	1	1	0	1	0	1	0	1	0	1	0
0.1	0.999500249	0.9995	2.49×10^{-7}	0.99950025	-1.0×10^{-9}	0.999500249	0	0.999500249	0	0.999500249	0
0.2	0.999000999	0.999	9.99×10^{-7}	0.999000999	0	0.999000999	0	0.999000999	0	0.999000999	0
0.3	0.998502246	0.9985	2.246×10^{-6}	0.998502251	-5.0×10^{-9}	0.998502246	0	0.998502246	0	0.998502246	0
0.4	0.998003992	0.998	3.992×10^{-6}	0.998003997	-5.0×10^{-9}	0.998003992	0	0.998003992	0	0.998003992	0
0.5	0.997506234	0.9975	6.234×10^{-6}	0.997506244	-1×10^{-8}	0.997506234	0	0.997506234	0	0.997506234	0

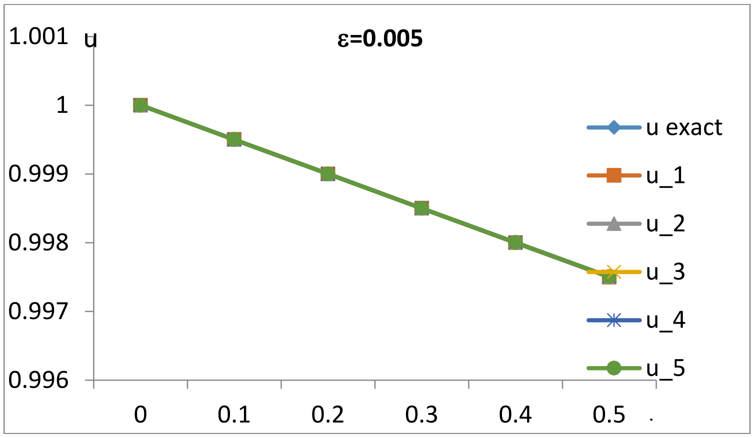

Figure 2: Relationship between $u - exact, u_1, u_2, u_3, u_4$ *and* u_5 *when the perturbation parameter* $\varepsilon = 0.005$

4.5 PERTURBATION PARAMETER ε AT 0.01

Table 3: *Relationship between u exact,* u_1, u_2, u_3, u_4 *and* u_5 *when the perturbation parameter* $\varepsilon = 0.01$

t	Exact Solution	Approximate Solution									
		u_1	Error	u_2	Error	u_3	Error	u_4	Error	u_5	Error
0.0	1	1	0	1	0	1	0	1	0	1	0
0.1	0.999000999	0.999	9.99×10^{-7}	0.999000999	0	0.999000999	0	0.999000999	0	0.999000999	0
0.2	0.998003992	0.998	3.992×10^{-6}	0.998003997	-5×10^{-9}	0.998003992	0	0.998003992	0	0.998003992	0
0.3	0.997008973	0.997	8.973×10^{-6}	0.997008973	-1.8×10^{-8}	0.997008973	0	0.997008973	0	0.997008973	0
0.4	0.996015936	0.996	1.5936×10^{-5}	0.996015978	-4.2×10^{-8}	0.996015936	0	0.996015936	0	0.996015936	0
0.5	0.995024875	0.995	2.4875×10^{-5}	0.995024958	-8.3×10^{-8}	0.995024875	0	0.995024875	0	0.995024875	0

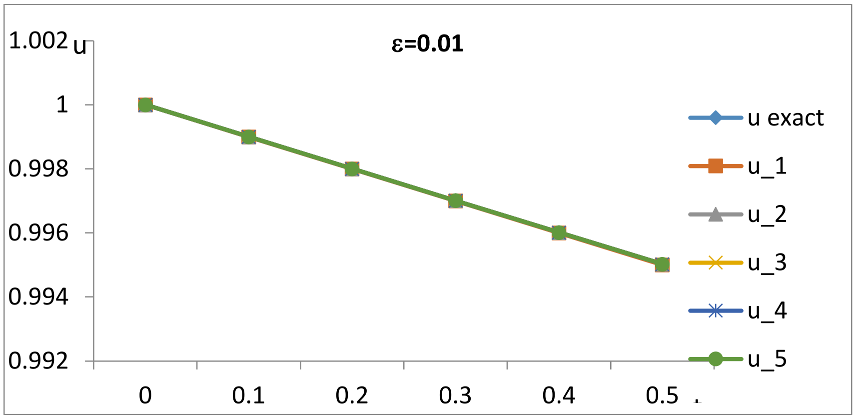

Figure 3: Relationship between $u - exact, u_1, u_2, u_3, u_4$ and u_5 $when$ the $perturbation$ $parameter$ $\varepsilon = 0.01$

PERTURBATION PARAMETER ε AT 0.05

Table 4: $Relationship$ $between$ u $exact, u_1, u_2, u_3, u_4$ and u_5 $when$ the $perturbation$ $parameter$ $\varepsilon = 0.05$

t	Exact Soluti on	Approximate Solution									
		u_1	Error	u_2	Error	u_3	Error	u_4	Error	u_5	Error
0.0	1	1	0	1	0	1	0	1	0	1	0
0.1	0.9950 24875	0.995	2.4875×10^{-5}	0.9950 24958	-8.3×10^{-8}	0.9950 24875	0	0.9950 24875	0	0.995 0248 75	0
0.2	0.9900 99009	0.99	9.9009×10^{-5}	0.9900 99666	-6.57×10^{-7}	0.9900 99006	3.9×10^{-9}	0.9900 99009	0	0.990 0990	0
0.3	0.9852 21674	0.985	2.2164×10^{-4}	0.9852 23875	-2.201×10^{-6}	0.9852 21658	1.6×10^{-8}	0.9852 21675	1.0×10^{-9}	0.985 2216	0
0.4	0.9800 392156	0.98	3.92156×10^{-5}	0.9803 97333	3.58117×10^{-4}	0.9803 92105	-3.528×10^{-4}	0.9803 92157	-3.529×10^{-4}	0.980 3921	-3.529×10^{-4}
0.5	0.9756 09756	0.975	6.0975×10^{-4}	0.9756 19791	-1.0035×10^{-5}	0.9756 09632	1.24×10^{-7}	0.9756 09698	5.8×10^{-8}	0.975 6097	$10 \times^{-9}$

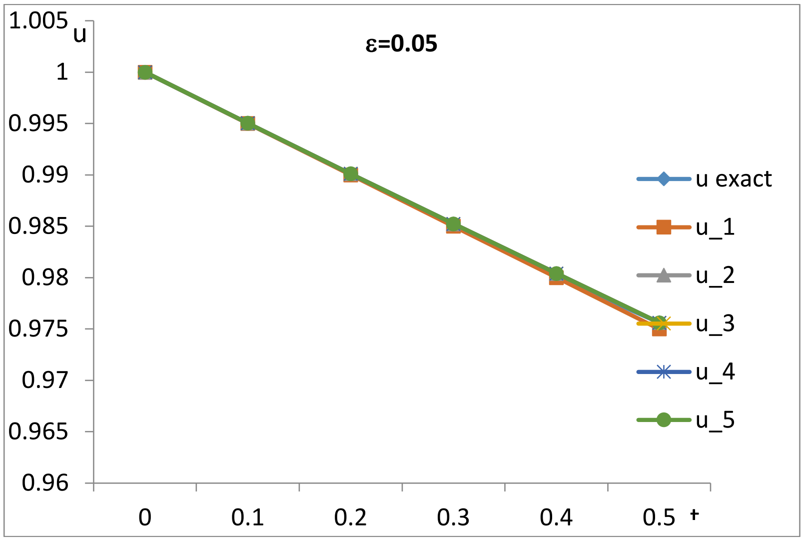

Figure 4: Relationship between $u - exact, u_1, u_2, u_3, u_4$ *and* u_5 *when the perturbation parameter* $\varepsilon = 0.05$

SUMMARY

The efficiency of this model is deduced after comparing the result of the exact solution to the approximate solution. It is clear that when our perturbation parameter is set at 0.001 and 0.005, the approximate solution matches the exact solution which indicates convergence, meanwhile there is a very negligible difference in other cases, but the error has to be highlighted for the sake of accuracy of this work. At ε = 0.001, 0.005 and 0.01, a little difference occurs u_1 and u_2 while convergence occurs from u_3 upward which shows that the more iterations carried out, the more accurate the result is.

A graphical illustration is shown from a careful look at figure 4.1, figure 4.2 and figure 4.3 which show the graph of $u_1, u_2, u_{3,} u_{4,} u_5$ lying on the same path thereby making it look like a single line. This shows the accuracy of the model at $\varepsilon = 0.001, 0.005$ and 0.01. but figure 4.4 shows clearly at the tail end that the line is more than one and also table 4.4 shows the more t increases, the farther we are from the exact solution.

CONCLUSION

Hence, it is concluded that the lower the value of ε the more accurate the result is. Nevertheless, as ε increases more iteration is expected to be carried out for convergence to take place. However, this method is recommended to solve a first order ordinary differential equation.

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