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THE SUP-EQUICONTINUITY THEOREM

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Abstract

We establish two theorems on equicontinuity involving the supremum of vector/locally convex topologies on the codomain space. With the help of these we establish a third theorem, here labeled, The Sup-Equicontinuity Theorem.

Keywords: equicontinuous, supremum of vector/locally convex topologies.

Subject Classification General Topology **(GT)**, Topological vector Spaces **(TVS)**.

1. LANGUAGE AND NOTATION

Our language and notation shall be pretty standard, as found, for example, in [1], [2], [3], [4] and [5]. By $\mathbb{N}$ we denote the natural numbers 1, 2, …… $\mathbb{R} \equiv$ the real numbers, and $\mathfrak{C} \equiv$ the complex numbers. $\mathbb{K} = \mathbb{R}$ or $\mathfrak{C}$, and by /// we signify the end or absence of a proof. *Equicontinuity* of a collection of maps is one of a number of General Topology (**GT**)(Uniform Spaces) concepts that have assumed some notoriety in Topological Vector Spaces (**TVS**)(Locally Convex Spaces). For, *A vector Topology is Uniformizable* [6]. So, in what follows we cite results from both **GT** and **TVS**. We assume the reader familiar with [1] and [2], and so language and notation there shall be used without citation.

2. SUPREMUM TOPOLOGIES

Let X be a non-empty set and Φ a collection of topologies on X. Following [3, p. 89][4, Theorem 1-6-8, p.11] we denote by $\vee\Phi$ the *supremum* of Φ.

FACT 1 (GT) Let $X \neq \varnothing$ and $\Phi = \{\tau_I : i \in I\}$ a collection of topologies on X. Let $x \in X$, and suppose that for $i \in I$, $\mathcal{B}_i$ is a τ_i-local base of neighbourhoods of x. Then, a $\vee\Phi$-local base of neighbourhoods of x is the family of finite intersections.

$\{ A_{1_X} \cap A_{2_X} \cap \ldots\ldots \cap A_{n_X} : A_{k_X} \in \mathcal{B}_{k_X}, k = 1, 2, \ldots\ldots, n, \ n \in \mathbb{N}, \mathcal{B}_{k_X}$ is a $\mathcal{B}_i$ for some $i \in I \}$. ///

FACT 2 (TVS) Let $E_{\mathbb{K}}$ be a vector space and Φ a collection of vector topologies on $E_{\mathbb{K}}$. Then, $\vee\Phi$ is a vector topology on $E_{\mathbb{K}}$. ///

From FACT 1 now follows that

FACT 3 (TVS) Let $(E, +, \theta)_{\mathbb{K}}$ be a vector space and Φ a collection of vector topologies on $(E, +, \theta)_{\mathbb{K}}$. If $W \in N_\theta(\vee\Phi)$, then, W contains a finite intersection, $\bigcap_{i=1}^{n} V_i \in N_\theta(\tau_i)$, $i = 1, 2, \ldots\ldots, n$, $\tau_1, \tau_2, \ldots..\tau_n \in \Phi$, $n \in \mathbb{N}$. ///

FACT 4 [2, 3.8, Note 3] Let $((X, +,\theta)_{\mathbb{K}}, \tau) = (X, \tau)$ and $((E, +, \theta^*)_{\mathbb{K}}, \tau^*) = (E, \tau^*)$ be topological vector spaces and F a collection of linear maps $f : (X, \tau) \to (E, \tau^*)$. The following are equivalent.

(i) F is equicontinous.

(ii) For every null net $(x_\delta)_{\delta\in(I, \leq)}$ in (X, τ) and every $W \in N_{\theta^*}(\tau^*)$ there exists $\delta_0 = \delta_0(W) \in I$ such that $f(x_\delta) \in W$ for all $\delta \geq \delta_0$ and all $f \in F$. ///

Observation 5 (GT) [4, last assertion of Theorem 1 – 6 – 8, p.11] Let $X \neq \varnothing$, Φ a collection of topologies on X, (Z, ξ) a topological space, and $f : Z \to X$ a map. Then, f is $(\xi, \vee\Phi)$ continuous if and only, if f is (ξ, τ)-continuous for each $\tau \in \Phi$. ///

Our next theorem, one of the two advertized in the abstract, is a generalization of Observation 5 which we have stated only for ease of comparison.

THEOREM 6 Let (X, τ) be a topological vector space, and Φ a collection of vector topologies on the vector space $(X', +, \theta')_{\mathbb{K}}$. Then, a collection, F of linear maps $f : X \to X'$ is $(\tau, \vee\Phi)$-equicontinous if and only if it is (τ, τ')-equicontinuous for each $\tau' \in \Phi$.

Proof The forward implications is immediate from FACT 4 and [2, 3.6 Note 2(ii), taking cognizance of the fact that $N_{\theta'}(\tau') \subseteq N_{\theta'}(\vee\Phi)$ for each $\tau' \in \Phi$.

FACT 3 and FACT 4 prove the reverse implication $\Leftarrow$, taking cognizance, of the fact that the set $(I, \leq)$ on which a net $(x_\delta)_{\delta\in(I, \leq)}$ *is based* is a directed set. ///

We move on to state and prove the second of the advertised theorems of the abstract. First, some language and notation. If $\varnothing \neq X$ is a vector space and p is a seminorm on X, we denote the pseudometric topology of p by τ_p, which is a locally convex topology. The supremum of a collection of locally convex topologies is a locally convex topology. Hence, if P is a collection of seminorms on X, then $\vee\tau_p$ denoted simply τ_p is also a locally convex topology.

FACT 7 If τ is a locally convex topology on a vector space, then $\tau = \tau_p$ for some collection P of seminorms. ///

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Observation 8 [5, Proposition 5.15, p.159] Let X and Y be vector spaces with topologies defined on them by the families $\{p_\alpha : \alpha \in A\}$and $\{q_\beta : \beta \in B\}$, of seminorms, respectively, and $T : X \to Y$ a linear map. Then, T is continuous if and only if for each $\beta \in B$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, $n \in \mathbb{N}$, and $C > 0$ such that

$$q_\beta(T(x)) \leq C\sum_{i=1}^{n} p_{\alpha_1}(x) \text{ for all } x \in X$$

[4, Theorem 7 – 2 – 5, p.94] Let X be a vector space, P a collection of seminorms on X and $f : (X, \tau_p) \to K$ a linear functional. Then, f is continuous if and only if there exist $M > 0$ and $p_1, p_2, \ldots, p_n \in P$, $n \in \mathbb{N}$, such that

$$| f(x)| \leq M \sum_{i=1}^{n} p_i(x) \text{ for all } x \in X. \ ///$$

Notation 9 If $(E, +, \theta)_K$ is a vector space and p is a seminorm on $(E, +, \theta)_K$, we write $p(\leq 1)$ for $\{v \in E : p(v) \leq 1\}$.

FACT 10 (TVS) With notation as in the preceding, $p(\leq 1) \in N_\theta(\tau_p)$. ///

Notation 11 Let $\varepsilon > 0$ and $p_1, p_2, \ldots p_n (n \in \mathbb{N})$ be seminorms on the vector space $E_K = (E, +, \theta)_K$. Write $Bp_1p_2 \ldots p_n, \varepsilon$ for the set $\{v \in E_K : p_1(v) < \varepsilon, p_2(v) < \varepsilon, \ldots, p_n(v) < \varepsilon \}$.

FACT 12 [4, Theorem 7 – 2 – 4, p.94]. Let P be a collection of seminorms on the vector space $(E, +, \theta)_K$. Then, the family **{**$Bp_1p_2 \ldots p_n, \varepsilon : p_1, p_2, \ldots, p_n \in P, n \in \mathbb{N}, \varepsilon > 0$**}**
is a τ_p-local base of neighbourhoods of θ. ///

FACT 13 (TVS) Let E_K be a vector space.

(i) If p is a seminorm on E_K and $\alpha > 0$, then αp is also a seminorm on E_K

(ii) If $p_1, p_2, \ldots, p_n \in P$, $n \in \mathbb{N}$, are seminorms on E_K, then, their sum $p_1 + p_2 + \ldots + p_n$ is also a seminorm on E_K. ///

FACT 14 [4, Theorem 1 – 6, 8, p.22] Let P be a collection of seminorms on the vector space $(X, +, \theta)_K$. Then, a net is null in (X, τ_P) if and only if it is null in (X, τ_p) for each $p \in P$. ///

The second of the advertized theorems simply generalizes Observation 8 to an equicontinuous collection of near maps $f : X \to Y$.

THEOREM 15 Let $X_K = (X, +, \theta)_K$ and $Y_K = (Y, +, \theta^*)_K$ be vector spaces with topologies defined on them by families $\{p_\alpha : \alpha \in A\}$ and $\{q_p : \beta \in B\}$of seminorms, respectively, and F a collection of linear maps $f : X_K \to Y_K$. Then, F is equicontinous if and only if for each $\beta \in B$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, $n \in \mathbb{N}$, and $C > 0$ such that

$$q_\beta(f(x)) \leq C\sum_{i=1}^{n} p_{\alpha_i}(x) \text{ for all } x \in X \text{ and all } f \in F.$$

Proof Let $P = \{ p_\alpha : \alpha \in A\}$ and $Q = \{q_p : \beta \in B\}$.

⇒: *Hypothesis* F is (τ_P, τ_Q)-equicontinuous.

By THEOREM 6, therefore, F is (τ_P, τ_{q_β})-equicontinous for each $\beta \in B$. By FACT 10, $q_\beta(\leq 1) \in N_{\theta^*}(\tau_{q_\beta})$. By [2, 3.6Note 2(i) ⇔ (ii)] and FACT 12,

$$\bigcap_{f \in F} f^{-1}(q_\beta(\leq 1)) \supseteq \overline{B} p_1, p_2, \ldots, p_n, \varepsilon \qquad \ldots(\Delta^1)$$

for some $p_1, p_2, \ldots, p_n \in P$, $n \in \mathbb{N}$, and $\varepsilon > 0$.

Let $p = p_1 + p_2 + \ldots + p_n$. Then, if $x \in X$, $p(x) < \varepsilon \Rightarrow p_1(x) < \varepsilon, p_2(x) < \varepsilon, \ldots, p_n(x) < \varepsilon$, which means $x \in Bp_1p_2 \ldots p_n, \varepsilon$ and so by (Δ^1), $x \in \bigcap_{f \in F} f^{-1}(q_\beta(\leq 1))$. And so. $q_\beta(f(x)) \leq 1$ for all $f \in F$. Thus, we have shown that

$$p(x) < \varepsilon \Rightarrow \sup_{f \in F} q_\beta(f(x)) \leq 1 \qquad \ldots(\rho)$$

We *CLAIM* that

$$\sup_{f \in F} q_\beta(f(x)) \leq \frac{1}{\varepsilon} p(x) \text{ for all } x \in X \qquad \ldots(\Delta^3)$$

which proves our theorem.

Assume the opposite that for some $a \in X$

$$\sup_{f \in F} q_\beta(f(a)) > \frac{1}{\varepsilon} p(a) \qquad \ldots(\sigma^1)$$

By the Density Theorems of Elementary Real Analysis it follows from (σ^1) that there exists a rational t such that

$$\sup_{f \in F} q_\beta(f(a)) > t > \frac{1}{\varepsilon} p(a) \qquad \ldots(\sigma^2)$$

From the second inequality of (σ^2) follows that

$$p((1/t)a) < \varepsilon \qquad \ldots(1)$$

From the first follows that

$$\frac{1}{t}\sup_{f\in F} q_\beta(f(a)) > 1$$

from which follows that

$$\sup_{f\in F} q_\beta(f((1/t)a)) > 1 \qquad \text{…..(2)}$$

Clearly, (1) and (2) jointly contradict (ρ).

$\Leftarrow$ **:** *Hypothesis* For each $\beta \in B$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, $n \in \mathbb{N}$, and $C > 0$ such that

$$q_\beta(f(x)) \le C\sum_{i=1}^{n} p_{\alpha_i}(x) \text{ for all } x \in X \text{ and all } f \in F.$$

Let $W \in N_\theta^*(\tau_{q_\beta})$. Then, $W \supseteq q_\beta(<\varepsilon) = \{y \in Y : q_\beta(y) < \varepsilon\}$ for some $\varepsilon > 0$. With a view to employ [2, 3.8 Note 3(i) ⇔(viii)], let $(x_\delta)_{\delta\in(I,\le)}$ be a τ_p-null net. Then, $(x_\delta)_{\delta\in(I,\le)}$ by FACT 14 is τ_p-null for each $p \in P$. Taking cognizance of the fact that the set $(I, \le)$ on which the net is based is directed, it shall follow from the *Hypothesis* that there exists a $\delta_0 = \delta_0(q_\beta(<\varepsilon))$ such that $q_\beta(f(x_\delta)) < \varepsilon$ for all $\delta \ge \delta_0$ and all $f \in F$. That is,

$f(x_\delta) \in q_\beta(<\varepsilon) \subseteq W$ for all $\delta \ge \delta_0$ and all $f \in F$.

Hence, F is (τ_P, τ_{q_β})-equicontinuous. By theorem 6, therefore, F is (τ_P, τ_Q)-w equicontinuous. ///

COROLLARY 16 Let X_K be a vector space, P a collection of seminorms on X_K, and F a collection of linear functionals $f : (X_K, \tau_p) \to K$. Then, F is equicontinuous if and only if there exist $M > 0$ and $p_1, p_2, \ldots, p_n \in P$, $n \in \mathbb{N}$, such that $|f(x)| \le M\sum_{i=1}^{n} p_i(x)$ for all $x \in X_K$ and all $f \in F$. ///

3 THE SUP-EQUICONTINUITY THEOREM

FACT 1 (TVS) Suppose $((X, +, \theta)_K, \tau)$ is a topological vector space and $V \in N_\theta(\tau)$. Then, for $n \in \mathbb{N}$, $n \ge 2$, there exists $V' \in N_\theta(\tau)$ such that

$$\underbrace{V' + V' + \ldots\ldots + V'}_{n\ summands} \subseteq V.$$

Notation 2 Let $\varnothing \ne X, Y$, and suppose F is a collection of maps $f : X \to Y$. If $\varnothing \ne U \subseteq X$, we shall denoted the union $\bigcup_{f\in F} f(U)$ simply by $F(U)$.

With the Notation 2 we can recast [2, 3.6 Note 2(i) ⇔ (ii)] as follows.

FACT 3 For topological vector spaces $(X, \tau) = ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ and F a collection of linear maps $f : (X, \tau) \to (E, \tau^*)$, the following are equivalent.

(i) F is (τ, τ^*)-equicontinuous.

(ii) For every $W \in N_{\theta^*}(\tau^*)$, there exists $V \in N_\theta(\tau)$ such that $F(V) \subseteq W$. ///

We can now state and prove the first of four *Observations* heralding *The Sup-Equicontinuity Theorem*.

Observation I 4 Let $(X, \tau) = ((X, +, \theta)_K, \tau)$ and $(E, \tau^*) = ((E, +, \theta^*)_K, \tau^*)$ be topological vector spaces, and $F_1, F_2, \ldots, F_n$, $n \in \mathbb{N}$, $n \ge 2$, sets of (τ, τ^*)-equicontinuous linear maps. Then, their sum $F_1 + F_2 + \ldots + F_n = \{f_1 + f_2 + \ldots + f_n : f_k \in F_k, k = 1, 2, \ldots, n\}$ is also (τ, τ^*)-equicontinuous.

Proof Let $W \in N_{\theta^*}(\tau^*)$. By FACT 1, there exists $V \in N_{\theta^*}(\tau^*)$ such that

$$\underbrace{V + V + \ldots\ldots + V}_{n\ summands} \subseteq W \qquad \text{……}(\sigma^1)$$

By the hypothesis and FACT 3, for each $k \in \{1, 2, \ldots, n\}$ there exist $U_k \in N_\theta(\tau)$ such that

$F_k(U_k) \subseteq V$ ……(σ^2)

Clearly, (σ^1) and (σ^2), therefore, give

$F_1(U_1) + F_2(U_2) + \ldots + F_n(U_n) \subseteq V + V + \ldots + V \subseteq W$

Clearly, $U = \bigcap_{k=1}^{n} U_k \in N_\theta(\tau)$. Also,

$F_1(U) + F_2(U) + \ldots + F_n(U) \subseteq V + V + \ldots + V \subseteq W.$

Hence,

$(F_1 + F_2 + \ldots + F_n)(U) \subseteq W.$ ///

Observation II 5 Let X_K be a vector space, τ_1, τ_2 vector topologies on X_K, Φ a collection of vector topologies on X_K, (X_K', τ') a topological vector space and F a collection of linear maps $f : X_K \to (X_K', \tau')$. Then,

(i) $\tau_1 \leq \tau_2$ and F (τ_1, τ')-equicontinuous $\Rightarrow F$ is (τ_2, τ')-equicon- tenuous.
(ii) F is (τ, τ')-equicontinuous for each $\tau \in \Phi$ implies F is $(\vee\Phi, \tau')$-equicontinuous.
Proof Clearly, (ii) is immediate from (i), and (i) is immediate from FACT 3. ///
Observation III 6 Let (X, τ) be a topological vector space, and F a collection of τ-equicontinuous set of linear functionals. If $\emptyset \neq A \subseteq F$, then A is also equicontinuous.

Proof Clear. ///
Notation 7 Let X_K be a vector space. Following [4], we denote by $X_K^{\#}$ the set of all the linear functionals on X_K, and call it the *algebraic dual* of X_K.
Observation IV 8 [4, Lemma 7 – 2 – 15, p.96] Let p, q be semi- norms on the vector space X_K, and suppose $f \in X_K^{\#}$ satisfy $|f(x)| \leq p(x) + q(x)$ for all $x \in X_K$. Then, there exist $g, h \in X_K^{\#}$ such that $|g(x)| \leq p(x)$, $|h(x)| \leq q(x)$ for all $x \in X_K$, and $f = g + h$. ///
One extends the preceding easily to
LEMMA 9 Let $n \in \mathbb{N}$, $n \geq 2$, $p_1, p_2, \ldots, p_n$ seminorms on the vector space X_K, and $f \in X_K^{\#}$ satisfies $|f(x)| \leq p_1(x) + p_2(x) + \ldots + p_n(x)$ for all $x \in X_K$. Then, there exist $g_1, g_2, \ldots, g_n \in X_K^{\#}$ such that $|g_1(x)| \leq p_1(x)$, $|g_2(x)| \leq p_2(x)$, ……, $|g_n(x)| \leq p_n(x)$ for all $x \in X_K$, and $f = g_1 + g_2 + \ldots + g_n$. ///

Notation 10 Let (X_K, τ) be a topological vector space. Again, following [4], we denote by $(X_K, \tau)'$ the set of all the τ-continuous members of
$X_K^{\#}$. So, $(X_K, \tau)' \subseteq X_K^{\#}$. At times, $(X_K, \tau)'$ is shortened to X_K' if τ is understood already.

FACT 11 [4, Theorem 7 – 2 – 16, p.96/97] Let Φ be a collection of locally convex topologies on the vector space X_K. Then $f \in (X_K, \vee\Phi)'$ if and only it there exists $\tau_1, \tau_2, \ldots, \tau_n \in \Phi$, $n \in \mathbb{N}$, $g_1 \in (X_K,$
$\tau_1)'$ $g_2 \in (X_K, \tau_2)'$, ……, $g_n \in (X_K, \tau_n)'$, and $f = g_1 + g_2 + \ldots + g_n$. ///
We generalize FACT 11 from *one* $\vee\Phi$-*continuous* linear functional to a *collection* (set) of $\vee\Phi$-equicontinuous linear functionals, thus giving us the theorem here labeled

The Sup-Equicontinuity Theorem 12 Let Φ be a collection of locally convex topologies on the vector space X_K. A collection F of linear functionals on X_K is $\vee\Phi$-equicontinuous if and only if there exist $n \in \mathbb{N}$, $\tau_1, \tau_2, \ldots, \tau_n, \in \Phi$, $F_1, F_2, \ldots, F_n \subseteq X_K^{\#}$, F_k τ_k-equicontinuous, $k = 1, 2, \ldots, n$ and $F \subseteq F_1 + F_2 + \ldots + F_n$.

Proof The reverse implication $\Leftarrow$ is immediate from Observation I 4, Observation II 5 and Observation III 6.
The proof of the forward implication $\Rightarrow$ is a modification of the proof of FACT 11.
$\Rightarrow$*Hypothesis* $F \subseteq X_K^{\#}$ is $\vee\Phi$-equicontinuous.
Let [4, Problem 7 – 2 – 2, p.97] P be the union of he seminorms generating the members of Φ. By 2.16, there exist $n \in \mathbb{N}$, $p_1, p_2, \ldots, p_n \in P$, and $M > 0$ such that

$$|f(x)| \leq M \sum_{i=1}^{n} p_i(x) \text{ for all } x \in X \text{ and all } f \in F.$$

In particular, for each $f \in F$,

$$|f(x)| \leq M \sum_{i=1}^{n} p_i(x) = \sum_{i=1}^{n} M\, p_i(x) \text{ for all } x \in X.$$

That is,

$$|f(x)| \leq \sum_{i=1}^{n} M\, p_i(x) \text{ for all } x \in X. \qquad \ldots\ldots(\Delta^2)$$

Clearly, (Δ^1) and Lemma 9 give a decomposition $f = f_1 + f_2 + \ldots + f_n$ where
$|f_i(x)| \leq Mp_i(x)$ for all $x \in X$, $i = 1, 2, \ldots, n$ ……(Δ^2)
Define, for $i \in \{1, 2, \ldots, n\}$, $F_i = \{ f_i : f \in F\}$. Clearly, (Δ^1) can be read as
$|f_i(x)| \leq Mp_i(x)$ for all $x \in X$ and all $f_i \in F_i$.
And so, by 2.16, again, F_i is $\tau_i = \tau_{p_i}$ -equicontinuous. Clearly, $F \subseteq F_1 + F_2 + \ldots + F_n$. ///
Remark 13 A good many vector topologies are supremum topologies, and so it is hoped that the Sup-Equicontinuity Theorem shall have many applications. We furnish one such application elsewhere.

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