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AN EOQ MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEM WITH TWO-PHASE DEMAND RATES, TIME-DEPENDENT LINEAR HOLDING COST AND TIME-DEPENDENT PARTIAL BACKLOGGING RATE UNDER TRADE CREDIT POLICY

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Abstract

In this article, an EOQ model for non-instantaneous deteriorating item with two-phase demand rates, time- dependent linear holding cost and time-dependent partial backlogging rate under trade credit policy has been considered. The demand rate before deterioration sets in is assumed to be time-dependent quadratic after which it is considered as constant. Shortages are allowed and partially backlogged. When shortages occur, some customers may wait for backorders to be fulfilled and others may opt to buy from other sellers. For most items, such as fashionable goods, electronics, photographic films, seasonal products, automobiles and its spare parts and so on, the length of the waiting time for the next replenishment would determine whether the backlogging will be accepted or not. Hence, the backlogging rate is variable and depends on the waiting time for the next replenishment. The model determined the optimal time with positive inventory, cycle length and order quantity that minimise total variable cost. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions have been established. Some numerical examples have been given to illustrate the theoretical results of the model. Sensitivity analysis of some model parameters on optimal solutions has been carried out and suggestions toward minimising the total variable cost of the inventory system were also given.

***Keywords*:** Non-instantaneous deterioration, time-dependent quadratic demand rate, trade credit policy, time-dependent linear holding cost, time-dependent partial backlogging rate.

1. Introduction

In the conventional economic order quantity (EOQ) model developed by Harris [1] it is implicitly assumed that the demand rate is constant, fixed costs of inventory units, ordering costs and holding costs, and items have infinite shelf lives. However, in real practice, the demand rate of many items such as aircrafts, computers, fashionable goods, photographic films, televisions, computer chips and so on may always be in dynamic state as the age of inventory have negative impact on demand due to loss of quality, spoilage, loss of market potential and depletion. Many researchers have attempted to modify the assumptions of the basic EOQ model and consider some more realistic factors such as time-dependent demand rate, deterioration and shortages in order to make the model corresponds with reality. An attempt to modify the basic EOQ model for a case of time varying demand rate was carried out by Silver and Meal ([2], [3]). Many inventory models are developed on the assumption that the demand rate of items varies either linearly or exponentially to time. Silver [4], Donalson [5], Goswami and Chaudhuri [6], Chakrabarti and Chaudhuri [7], Chakrabarti *et al.* [8], Giri *et al.* [9], Kar *et al.* [10] and so on developed inventory models under various conditions and assumed a steady rise or fall in the demand rate per unit time which is rarely seen to occur for so many products.

Similarly, Hollier and Mak [11], Dash *et al.* [12], Ahmed and Musa [13] and so on developed inventory models under various conditions and assumed a very rapid change in the demand rate per unit time which is also rare because the demand rate of any product may not change with the higher rate of change as exponential.

A best alternative to linear/exponential demand rates would be to think of accelerated/retarded rise or fall in demand rate. Accelerated growth in the demand rate takes place in the case of newly produced items such as aircrafts, computers, machines and its spare parts, mobile phones, televisions and so on. Accelerated decline in the demand rate is found to occur in the case of obsolete aircrafts, cars,

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clothes, computers, machines and its spare parts, mobile phones, televisions and so on. Accelerated rise and fall in demand rate could be found seasonal products whose demand rise rapidly to a peak in the mid-season and then falls rapidly as the season wanes out. These different types of demand rates can be best represented by quadratic function of time. Khanra and Chaudhuri [14], Ghosh and Chaudhuri [15], Ghosh and Chaudhuri [16], Khanra *et al.* [17], Sarkar *et al.* [18], Mishra [19], Uthayakumar and Karuppasamy [20], Priya and Senbagam [21] and so on developed inventory models with time-dependent quadratic demand rate under various assumptions.

Most of the classical EOQ models assumed that inventory items have an infinite life span and the depletion of inventory is as a result of constant demand rate only. However, there is inventory depletion due to deterioration in some cases. Hence deterioration plays important role in many inventory systems and its effects cannot be ignored. Inventory model for fashionable products deteriorating at the end of prescribed storage period was first study by Whitin [22]. Later, Ghare and Schrader [23] presented a revised form of EOQ model for exponentially decaying items with constant rate of deterioration, where the consumption rate of deteriorating items assumed to be closely related to a negative exponential function of time. Afterwards, Covert and Philip [24] extended Ghare and Schrader's [23] model to develop an EOQ model for instantaneous deteriorating items, where the rate of deterioration follows two parameter Weibull distribution. Philip [25] extended Covert and Philip's [24] model to develop an inventory model for instantaneous deteriorating items, where deterioration rate follows three parameter Weibull distribution and shortages are not allowed to occur. Similarly, Mahapatra *et al.* [26], Chen [27], Mandal and Venkataraman [28] and so on study inventory models under various conditions with the assumption that the deterioration starts immediately items are received.

However, most of items such as aircrafts, computers, seasonal products, machines and its spare parts fashionable goods, android mobiles, televisions, photographic films, and so on have a span of maintaining quality or original condition and deterioration starts after that span. Wu *et al.* [29], Baraya and Sani ([30], [31], [32]), Dari and Sani ([33], [34], [35]), Babangida and Baraya ([36], [37], [38], [39]), Bello and Baraya ([40], [41]) and so on study on inventory models for non-instantaneous deteriorating items under various assumptions.

The traditional EOQ models assumed implicitly that the retailers should pay the purchasing cost as soon as items are received. However, in real practice, a manufacturer/supplier offers the retailer a delay period in paying for purchasing cost (known as trade credit period) and retailer can accumulate revenues by selling items and by earning interests. The concept of trade credit in the inventory literature was first introduced by Haley and Higgins [42]. Goyal [43] was the first to propose an EOQ model for non-decaying items with constant demand rate under permissible delay in payments and assumed that the unit purchasing cost and selling price per unit are the same. Later, Aggarwal and Jaggi [44] extended Goyal' [43] model to develop an inventory model for deteriorating items with constant demand rate under permissible delay in payments. The rate of deterioration is assumed to be constant and items deteriorate at instant of arrival. Then, Teng [45] developed an EOQ model for non-decaying items with constant demand rate under permissible delay in payments and assumed that the unit selling price was not equal to the unit purchasing cost and found out that it is more economical to order less quantity in order to take the benefit of trade credit more frequently. Shaikh *et al.* [46], Babangida and Baraya ([36], [37], [38], [39]) and so on studied inventory models under trade credit policy with various assumptions.

In most of the classical inventory models, holding cost is considered to be constant. However, in the real life situations, holding cost of many items may be in a dynamic state as there is change in time value of money and price index. The cost of storing deteriorating and perishable items when more storage facilities and services are needed may always be high. The holding cost for most of items held is a linear function of the length of time over which items are stored. Usually, the cost of holding items such fruits, vegetables, fish, meat, milks etc., in the stock is higher when better preserving facilities are used to maintain the freshness of items and to prevent spoilage, and consequently this lower deterioration rate. Moreover, sometimes holding cost increases due to inflation, bank interest, hiring charge and so forth. Thus, it is important to consider an inventory model with time varying holding cost. Babangida and Baraya [37], Baraya and Sani [31], Musa and Sani [47] and so on studied inventory models with time varying holding cost under various assumptions.

In the classical EOQ model, shortages are not allowed. However, sometimes customers demand cannot be fulfilled by the supplier from the current stocks, this situation is known as stock out or shortage condition. In real life situations, stock out is unavoidable due to various uncertainties. According to sharma [48], allowing shortages occur increase cycle length, spread the ordering cost over a long period of time and hence reducing the total variable cost. Several authors such as Deb and Chaudhuri [49], Goswami and Chaudhuri [6], Ghosh and Chaudhuri ([15], [16]), Babangida and Baraya ([38], Goswami and Chaudhuri [50], , Roy [51], Choudhury *et al.* [52] and so on studied inventory models with completely backlogged shortages under various assumptions.

However, when shortages occur, one cannot be certain that all customers are willing to wait for a backorder due to customers' impatient and dynamic nature of human beings. When shortages occur, some customers whose needs are not critical at that time may wait for the back-orders to be fulfilled, while others may opt to buy from other sellers. Consequently, the opportunity cost due to lost sales should be considered. Baraya and Sani [32] developed an EOQ model for delayed deteriorating items with inventory-level-dependent demand rate and constant partially backlogged shortages. Bello and Baraya [40] developed an inventory model for non-instantaneous deteriorating item with two-phase demand rates and constant partial backlogging rate.

For most items, such as fashionable goods, electronics, automobiles and its spare parts, photographic films, seasonal products and so on, the length of the waiting time for the next replenishment would determine whether the backlogging will be accepted or not. Therefore, the backlogging rate should be variable and depend on the waiting time for the next replenishment. That is, the longer the waiting time, the lower the backlogging rate will be and vice versa. Moreover, some related studies on inventory models, where the backlogging rate is variable and depend on the waiting time for the next replenishment, can be found in Geetha and Uthayakumar [53], Chang and Feng [54], Sarkar and Sarkar [55], Dutta and Kumar [56], Wu *et al.* [29], Wee [57], Chang and Dye [58], Yang *et al.* [59] and so on.

It can be observed from the above review that non-instantaneous deterioration, two phase demand rates (time dependent quadratic demand rate before deterioration sets in and constant demand rate after deterioration sets in), time-dependent partial backlogging, linear holding cost and trade credit policy are the most appealing and realistic features to consider in developing inventory policies for items

such as computers, machines and its spare parts, seasonal products, cars, clothes, fashionable goods, android mobiles, automobiles, photographic films and so on.
The purpose of this research is to develop an EOQ model that will determine the time with positive inventory, cycle length and order quantity that minimizes total variable cost. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions have been established. Some numerical examples have been given to illustrate the theoretical results of the model. Sensitivity analysis of some model parameters on optimal solutions have been carried out and suggestions toward minimizing the total variable cost of the inventory system were also given.

2. Notation and assumptions
The inventory system is developed based on the following notation and assumptions.
2.1 Notations

A	The fixed ordering cost per order.
C	The purchasing cost per unit per unit time (\$/unit/ year).
S	The selling price per unit per unit time (\$/unit/ year).
C_b	Shortage cost per unit per unit of time.
C_π	The unit cost of lost sales per unit.
I_c	The interest charged in stock by the supplier per unit cost per year (\$/unit/year) $(I_c \geq I_e)$.
I_e	The interest earned per unit cost per year (\$/unit/year).
M	The trade credit period (in year) for settling accounts.
θ	The deterioration rates function $(0 < \theta < 1)$.
t_d	The length of time in which the product exhibits no deterioration.
t_1	Length of time in which the inventory has no shortage.
T	The length of the replenishment cycle time (time unit).
Q_m	The maximum inventory level.
B_m	The backorder level during the shortage period.
Q	The order quantity during the cycle length, i.e., $Q = (Q_m + B_m)$.

2.2 Assumptions
This model was developed under the following assumptions.
1. The replenishment rate is infinite.
2. The lead time is zero.
3. A single non-instantaneous deteriorating item is modelled.
4. During the fixed period, t_d, there is no deterioration and at the end of this period, the items deteriorate at the rate θ.
5. There is no replacement or repair for deteriorated items during the period under consideration.
6. Demand rate before deterioration begins is time dependant quadratic and is given by $\alpha + \beta t + \gamma t^2$ where $\alpha \geq 0, \beta \neq 0, \gamma \neq 0$.
7. Demand rate after deterioration sets in is assumed to be constant and is given by λ.
8. Holding cost $C_1(t)$ per unit time is linear time-dependent and is assumed to be $C_1(t) = h_1 + h_2 t$; where $h_1 > 0$ and $h_2 > 0$.
9. During the trade credit period M $(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.
10. Shortages are allowed and partially backlogged during the stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment i.e. the longer the waiting time is, the smaller the backlogging rate will be. The backlogging rate for negative inventory is given by $B(t) = \frac{1}{1+\delta(T-t)}$, δ is backlogging parameter ($0 < \delta < 1$)and $(T - t)$ is waiting time $(t_1 \leq t \leq T)$, $1 - B(t)$ is the remaining fraction lost.

3. Formulation of the model
At the beginning of each replenishment cycle (i.e., at time $t = 0$), Q_m units of a single product from the manufacturer arrives. During the time interval $[0, t_d]$, the inventory level $V_1(t)$ is depleting gradually due to market demand only and it is assumed to be quadratic function of time t. At time interval $[t_d, t_1]$ the inventory level $V_2(t)$ is depleting due to combined effects of demand from the customers and deterioration and the demand at time is reduced to λ, a constant demand. At time $t = t_1$, the inventory level depletes to zero. Shortages occur at the time $t = t_1$ and are partially backlogged at the rate δ. The whole process of the inventory is repeated. The behaviour of the inventory system is described in figure below

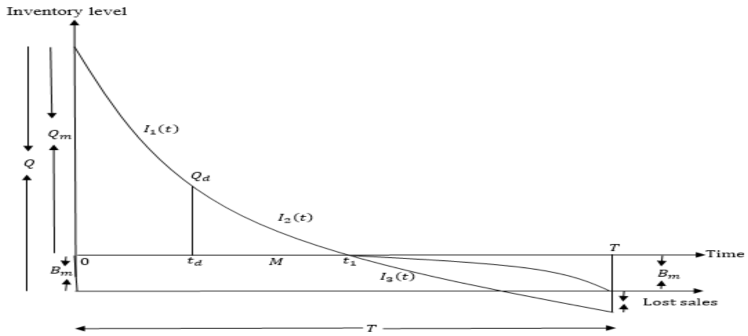

Figure 1 Graphical representation of the Inventory

Based on the description in Figure 1, during the time interval $[0, T]$, the change of inventory at any time t is represented by the following differential equations

$$\frac{dI_1(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \le t \le t_d \tag{1}$$

with boundary conditions $I_1(0) = Q_m$ and $I_1(t_d) = Q_d$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -\lambda, \qquad t_d \le t \le t_1 \tag{2}$$

with boundary conditions $I_2(t_1) = 0$ and $I_2(t_d) = Q_d$.

$$\frac{dI_3(t)}{dt} = -\frac{\lambda}{1 + \delta(T - t)}, \qquad t_1 \le t \le T \tag{3}$$

with condition $I_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are respectively given by

$$I_1(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \alpha(t_d - t) + \frac{\beta}{2}\left(t_d^2 - t^2\right) + \frac{\gamma}{3}\left(t_d^3 - t^3\right), \qquad 0 \le t \le t_d \tag{4}$$

$$I_2(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \qquad t_d \le t \le t_1 \tag{5}$$

and

$$I_3(t) = -\frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_1)] - ln[1 + \delta(T - t)]\right] \qquad t_1 \le t \le T \tag{6}$$

From Figure1, using the condition $I_1(0) = Q_m$ in equation (4), the maximum inventory level is given by

$$Q_m = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) \tag{7}$$

Moreover, the value of Q_d can be derived at $t = t_d$, and then from equation (5), it follows that

$$Q_d = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \tag{8}$$

The maximum backordered units B_m is obtained at $t = T$, and then from equation (6), it follows that

$$B_m = -I_3(T) = \frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_1)]\right] \tag{9}$$

Thus the order size during total time interval $[0,\ T]$ is

$$Q = Q_m + B_m = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) + \frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_1)]\right] \tag{10}$$

The total cost per unit time for a replenishment cycle (denoted by $TC(T)$ is given by

$$TC(t_{1,}T) = \begin{cases} TC_1(t_{1,}T) & 0 < M \le t_d \\ TC_2(t_{1,}T) & t_d < M \le t_1 \\ TC_3(t_{1,}T) & M > t_1 \end{cases} \tag{11}$$

where

For $0 < M \le t_d$

$TC_1(t_1, T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost+ interest payable – interest earned}

$$= \frac{1}{T}\left\{A + \left[\int_0^{t_d}(h_1 + h_2 t)I_1(t)dt + \int_{t_d}^{t_1}(h_1 + h_2 t)I_2(t)dt\right] + C\left[Q_d - \int_{t_d}^{t_1}\lambda\, dt\right] + \left[C_b\int_{t_1}^{T} -I_3(t)dt\right] + C_\pi\left[\lambda\int_{t_1}^{T}\left(1 - \frac{\lambda}{1 + \delta(T - t)}\right)dt\right]\right.$$
$$\left. + cI_c\left[\int_M^{t_d}I_1(t)dt + \int_{t_d}^{t_1}I_2(t)dt\right] - sI_e\left[\int_0^M(\alpha + \beta t + \gamma t^2)t dt\right]\right\}$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ A &+ h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] \\
&+ h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3} + \frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)} - \frac{\lambda t_1^2}{2\theta}\right] \\
&+ C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right] + \left(C_\pi\lambda + \frac{\lambda C_b}{\delta}\right)\left[(T-t_1) - \frac{ln\big(1+\delta(T-t_1)\big)}{\delta}\right] \\
&+ cI_c\left[\frac{\lambda(t_d-M)}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \frac{\alpha}{2}(t_d-M)^2 + \frac{\beta}{6}(2t_d+M)(t_d-M)^2 + \frac{\gamma}{12}\left(3t_d^2 + 2t_dM + M^2\right)(t_d-M)^2\right. \\
&\left. + \frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right)\right] - sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\} \qquad (12)
\end{aligned}
$$

For $t_d < M \le t_1$

$TC_2(t_1,T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost+ interest payable – interest earned}

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ A &+ \left[\int_0^{t_d}(h_1+h_2t)I_1(t)dt + \int_{t_d}^{t_1}(h_1+h_2t)I_2(t)dt\right] + C\left[Q_d - \int_{t_d}^{t_1}\lambda\, dt\right] + \left[C_b\int_{t_1}^{T} -I_3(t)dt\right] + C_\pi\left[\lambda\int_{t_1}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt\right] \\
&+ cI_c\left[\int_M^{t_1} I_2(t)dt\right] - sI_e\left[\int_0^{t_d}(\alpha+\beta t+\gamma t^2)tdt + \int_{t_d}^{M}\lambda tdt\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ A &+ h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] \\
&+ h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3} + \frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)} - \frac{\lambda t_1^2}{2\theta}\right] \\
&+ C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right] + \left(C_\pi\lambda + \frac{\lambda C_b}{\delta}\right)\left[(T-t_1) - \frac{ln\big(1+\delta(T-t_1)\big)}{\delta}\right] \\
&+ cI_c\left[\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-M)} - 1 - \theta(t_1-M)\right)\right] \\
&- sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right]\Bigg\} \qquad (13)
\end{aligned}
$$

For $M > t_1$

$TC_3(t_1,T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost – interest earned}

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ A &+ \left[\int_0^{t_d}(h_1+h_2t)I_1(t)dt + \int_{t_d}^{t_1}(h_1+h_2t)I_2(t)dt\right] + C\left[Q_d - \int_{t_d}^{t_1}\lambda\, dt\right] + \left[C_b\int_{t_1}^{T} -I_3(t)dt\right] + C_\pi\left[\lambda\int_{t_1}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt\right] \\
&+ cI_c\left[\int_M^{t_1} I_2(t)dt\right] - sI_e\left[\int_0^{t_d}(\alpha+\beta t+\gamma t^2)tdt + (M-t_1)\int_0^{t_d}(\alpha+\beta t+\gamma t^2)dt + \int_{t_d}^{t_1}\lambda tdt + (M-t_1)\int_{t_d}^{t_1}\lambda dt\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ A &+ h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] \\
&+ h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3} + \frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)} - \frac{\lambda t_1^2}{2\theta}\right] \\
&+ C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right] + \left(C_\pi\lambda + \frac{\lambda C_b}{\delta}\right)\left[(T-t_1) - \frac{ln\big(1+\delta(T-t_1)\big)}{\delta}\right] \\
&- sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + (M-t_1)\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) - \frac{\lambda}{2}(t_1-t_d)^2 + M\lambda(t_1-t_d)\right]\Bigg\} \qquad (14)
\end{aligned}
$$

Using the well-known approximations $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and $ln(1+x) = x - \frac{x^2}{2} + \cdots$ when $-1 < x \le 1$ in equations (12), (13) and (14) yields

$$
TC_1(t_1,T) = \frac{\lambda}{T}\left\{\frac{1}{2}W_1t_1^2 - X_1t_1 + Y_1 + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_1T\right\} \qquad (15)
$$

Where

$$
W_1 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_b + C_\pi\delta) + cI_c(\theta(t_d - M) + 1)\right],
$$

$X_1 = \left[h_1 t_d^2 \theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_c(M + (t_d - M)\theta t_d)\right]$ and

$$Y_1 = \frac{1}{\lambda}\Bigg[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2}$$
$$+ cI_c\left(\frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}\left(3t_d^2 + 2t_dM + M^2\right)(t_d - M)^2 + \lambda M t_d - \frac{\lambda t_d^2}{2}\right.$$
$$\left. + \frac{\lambda}{2}(t_d - M)\theta t_d^2\right) - sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg]$$

Similarly

$$TC_2(t_1, T) = \frac{\lambda}{T}\left\{\frac{1}{2}W_2t_1^2 - X_2t_1 + Y_2 + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_1T\right\} \tag{16}$$

Where

$W_2 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_b + C_\pi\delta) + cI_c\right]$,

$X_2 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_cM\right]$ and

$$Y_2 = \frac{1}{\lambda}\Bigg[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2} + cI_c\frac{\lambda}{2}M^2$$
$$- sI_e\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right)\Bigg].$$

and

$$TC_3(t_1, T) = \frac{\lambda}{T}\left\{\frac{1}{2}W_3t_1^2 - X_3t_1 + Y_3 + + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_1T\right\} \tag{17}$$

where

$W_3 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_b + C_\pi\delta) + sI_e\right]$,

$X_3 = \left[h_1t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + sI_e\left[(M + t_d) - \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda}\right]\right]$ and

$$Y_3 = \frac{1}{\lambda}\Bigg[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2}$$
$$- sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)M - \frac{\lambda}{2}(2M + t_d)t_d\right]\Bigg]$$

4. Optimal Decision

This section determines the optimal ordering policies that minimise the total variable cost per unit time. The necessary and sufficient conditions for the existence and uniqueness of optimal solutions have been established. The necessary conditions for the total variable cost per unit time $TC_i(t_1, T)$ to be minimum are $\frac{\partial TC_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_i(t_1,T)}{\partial T} = 0$ for $i = 1, 2, 3$. The value of (t_1, T) obtained from $\frac{\partial TC_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_i(t_1,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TC_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_i(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_i(t_1,T)}{\partial t_1 \partial T}\right)^2\right\} > 0$ is satisfied gives a minimum value for the total variable cost per unit time $TC_i(t_1, T)$.

Optimality condition for case 1: $0 < M \le t_d$

The necessary condition for the total variable cost $TC_1(t_1, T)$ in equation (15) to be the minimum are $\frac{\partial TC_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_1(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TC_1(t_1, T)}{\partial t_1} = \frac{\lambda}{T}\{W_1t_1 - X_1 - (C_b + C_\pi\delta)T\} = 0 \tag{18}$$

and

$$T = \frac{1}{(C_b + C_\pi\delta)}(W_1t_1 - X_1) \tag{19}$$

Note that

$$W_1t_1 - X_1 = \Bigg[h_1(t_d\theta(t_1 - t_d) + t_1) + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + C\theta(t_1 - t_d) + (C_b + C_\pi\delta)t_1$$
$$+ cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\Bigg] > 0$$

since $(t_d - M) \ge 0, (t_1 - t_d),\ (t_1 - M) > 0$

Similarly

$$\frac{\partial TC_1(t_1, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_1t_1^2 - X_1t_1 + Y_1 - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \tag{20}$$

Substituting T from equation (19) into equation (20) yields

$$W_1\big(W_1-(C_b+C_\pi\delta)\big)t_1^2-2X_1\big(W_1-(C_b+C_\pi\delta)\big)t_1-(2(C_b+C_\pi\delta)Y_1-X_1^2)=0 \quad (21)$$

Let $\Delta_1 = W_1\big(W_1-(C_b+C_\pi\delta)\big)t_d^2-2X_1\big(W_1-(C_b+C_\pi\delta)\big)t_d-(2(C_b+C_\pi\delta)Y_1-X_1^2)$, then the following result is obtained.

Lemma 4.1

(i) If $\Delta_1 \leq 0$, then the solution of $t_1 \in [t_d,\infty)$ (say t_{11}^*) which satisfies equation (21) not only exists but also is unique.

(ii) If $\Delta_1 > 0$, then the solution of $t_1 \in [t_d,\infty)$ which satisfies equation (21) does not exist.

Proof of (i):: From equation (21), a new function $F_1(t_1)$ is defined as follows

$$F_1(t_1)=W_1\big(W_1-(C_b+C_\pi\delta)\big)t_1^2-2X_1\big(W_1-(C_b+C_\pi\delta)\big)t_1-(2(C_b+C_\pi\delta)Y_1-X_1^2),\ t_1 \in [t_d,\infty) \quad (22)$$

Taking the first-order derivative of $F_1(t_1)$ with respect to $t_1 \in [t_d,\infty)$, it follows that

$$\frac{F_1(t_1)}{dt_1}=2(W_1t_1-X_1)\big(W_1-(C_b+C_\pi\delta)\big)>0$$

Because $(W_1t_1-X_1)>0$ and

$$\big(W_1-(C_b+C_\pi\delta)\big)=\left[h_1(t_d\theta+1)+h_2\left(\frac{t_d\theta}{2}+1\right)t_d+C\theta+cI_c(\theta(t_d-M)+1)\right]>0$$

Hence $F_1(t_1)$ is a strictly increasing of t_1 in the interval $[t_d,\infty)$.

Moreover, $\lim_{t_1\to\infty}F_1(t_1)=\infty$ and $F_1(t_d)=\Delta_1\leq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{11}^* \in [t_d,\infty)$ such that $F_1(t_{11}^*)=0$. Hence t_{11}^* is the unique solution of equation (21). Thus, the value of t_1 (denoted by t_{11}^*) can be found from equation (21) and is given by

$$t_{11}^*=\frac{X_1}{W_1}+\frac{1}{W_1}\sqrt{\frac{(2W_1Y_1-X_1^2)(C_b+C_\pi\delta)}{\big(W_1-(C_b+C_\pi\delta)\big)}} \quad (23)$$

Once t_{11}^* is obtained, then the value of T (denoted by T_1^*) can be found from equation (19) and is given by

$$T_1^*=\frac{1}{(C_b+C_\pi\delta)}(W_1t_{11}^*-X_1) \quad (24)$$

Equations (23) and (24) give the optimal values of t_{11}^* and T_1^* respectively for the cost function in equation (15) only if X_1 satisfies the inequality given in equation (25)

$$X_1^2<2W_1Y_1 \quad (25)$$

Proof of (ii): If $\Delta_1>0$, then from equation (22), $F_1(t_1)>0$. Since $F_1(t_1)$ is an increasing function of $t_1 \in [t_d,\infty)$, we have $F_1(t_1)>0$ for all $t_1 \in [t_d,\infty)$. Thus, a value of $t_1 \in [t_d,\infty)$ cannot be found such that $F_1(t_1)=0$. This completes the proof.

Theorem 4.1

(i) If $\Delta_1\leq 0$, then the total variable cost $TC_1(t_1,T)$ is convex and reaches its global minimum at the point (t_{11}^*,T_1^*), where (t_{11}^*,T_1^*) is the point which satisfies equations (21) and (18).

(ii) If $\Delta_1>0$, then the total variable cost $TC_1(t_1,T)$ has a minimum value at the point (t_{11}^*,T_1^*) where $t_{11}^*=t_d$ and $T_1^*=\frac{1}{(C_b+C_\pi\delta)}(W_1t_d-X_1)$

Proof of (i): When $\Delta_1\leq 0$, it is seen that t_{11}^* and T_1^* are the unique solutions of equations (21) and (18) respectively from Lemma l(i). Taking the second derivative of $TC_1(t_1,T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{11}^*,\ T_1^*)$, it follows that

$$\left(\frac{\partial^2TC_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right)\left(\frac{\partial^2TC_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right)-\left(\frac{\partial^2TC_1(t_1,\ T)}{\partial t_1\partial T}\bigg|_{(t_{11}^*,\ T_1^*)}\right)^2$$
$$=\frac{\lambda^2(C_b+C_\pi\delta)}{T_1^{*2}}\left[h_1(t_d\theta+1)+h_2\left(\frac{t_d\theta}{2}+1\right)t_d+C\theta+cI_c(\theta(t_d-M)+1)\right]>0 \quad (26)$$

It is therefore concluded from equation (26) and Lemma 4.1 that $TC_1(t_{11}^*,\ T_1^*)$ is convex and $(t_{11}^*,\ T_1^*)$ is the global minimum point of $TC_1(t_1,\ T)$. Hence the values of t_1 and T in equations (23) and (24) respectively are optimal.

Proof of (ii): When $\Delta_1>0$, $F_1(t_1)>0$ for all $t_1\in[t_d,\ \infty)$. Thus, $\frac{\partial TC_1(t_1,\ T)}{\partial T}=\frac{F_1(t_1)}{T^2}>0$ for all $t_1\in[t_d,\infty)$ which implies $TC_1(t_1,\ T)$ is an increasing function of T. Thus $TC_1(t_1,\ T)$ has a minimum value when T is minimum. Therefore, $TC_1(t_1,\ T)$ has a minimum value at the point $(t_{11}^*,\ T_1^*)$ where $t_{11}^*=t_d$ and $T_1^*=\frac{1}{(C_b+C_\pi\delta)}(W_1t_d-X_1)$. This completes the proof.

Optimality condition for case 2: $t_d<M\leq t_1$

The necessary condition for the total variable cost $TC_2(t_1,T)$ in equation (16) to be the minimum are $\frac{\partial TC_2(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_2(t_1,T)}{\partial T}=0$, which give

$$\frac{\partial TC_2(t_1,T)}{\partial t_1}=\frac{\lambda}{T}\{W_2t_1-X_2-(C_b+C_\pi\delta)T\}=0 \quad (27)$$

and

$$T=\frac{1}{(C_b+C_\pi\delta)}(W_2t_1-X_2) \quad (28)$$

Note that

$$W_2t_1 - X_2 = \left[h_1(t_d\theta(t_1-t_d)+t_1) + \frac{h_2t_d\theta}{2}(t_1-t_d)t_d + h_2\left(t_1-\frac{t_d}{2}\right)t_d + C\theta(t_1-t_d) + (C_b+C_\pi\delta)t_1 + cI_c(t_1-M)\right] > 0$$

since $(t_1-t_d) > 0, (t_1-M) \geq 0$

Similarly

$$\frac{\partial TC_2(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_2t_1^2 - X_2t_1 + Y_2 - \frac{T^2}{2}(C_b+C_\pi\delta)\right\} \quad (29)$$

substituting T from equation (28) into equation (29) yields

$$W_2\big(W_2-(C_b+C_\pi\delta)\big)t_1^2 - 2X_2\big(W_2-(C_b+C_\pi\delta)\big)t_1 - (2(C_b+C_\pi\delta)Y_2 - X_2^2) = 0 \quad (30)$$

Let $\Delta_2 = W_2\big(W_2-(C_b+C_\pi\delta)\big)M^2 - 2X_2\big(W_2-(C_b+C_\pi\delta)\big)M - (2(C_b+C_\pi\delta)Y_2 - X_2^2)$, then the following result is obtained.

Lemma 4.2

(i) If $\Delta_2 \leq 0$, then the solution of $t_1 \in [M,\infty)$ (say t_{12}^*) which satisfies Equation (30) not only exists but also is unique.

(ii) If $\Delta_2 > 0$, then the solution of $t_1 \in [M,\infty)$ which satisfies Equation (30) does not exist.

Proof: The process proof is similar to Lemma 4.1.

Thus, the value of t_1 (denoted by t_{12}^*) can be found from equation (30) and is given by

$$t_{12}^* = \frac{X_2}{W_2} + \frac{1}{W_2}\sqrt{\frac{(2W_2Y_2 - X_2^2)(C_b+C_\pi\delta)}{\big(W_2-(C_b+C_\pi\delta)\big)}} \quad (31)$$

Once t_{12}^* is obtained, then the value of T (denoted by T_2^*) can be found from equation (28) and is given by

$$T_2^* = \frac{1}{(C_b+C_\pi\delta)}(W_2t_{12}^* - X_2) \quad (32)$$

Equations (31) and (32) give the optimal values of t_{12}^* and T_2^* respectively for the cost function in equation (16) only if X_2 satisfies the inequality given in equation (33)

$$X_2^2 < 2W_2Y_2 \quad (33)$$

Theorem 4.2

(i) If $\Delta_2 \leq 0$, then the total variable cost $TC_2(t_1,T)$ is convex and reaches its global minimum at the point (t_{12}^*, T_2^*), where (t_{12}^*, T_2^*) is the point which satisfies Equations (30) and (27).

(ii) If $\Delta_2 > 0$, then the total variable cost $TC_2(t_1,T)$ has a minimum value at the point (t_{12}^*, T_2^*) where $t_{12}^* = M$ and $T_2^* = \frac{1}{(C_b+C_\pi\delta)}(W_2M - X_2)$

Proof: The process proof is similar to Theorem 4.1.

Optimality condition for case 3: $M > t_1$

The necessary condition for the total variable cost $TC_3(t_1,T)$ in equation (17) to be the minimum are $\frac{\partial TC_3(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_3(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TC_3(t_1,T)}{\partial t_1} = \frac{\lambda}{T}\{W_3t_1 - X_3 - (C_b+C_\pi\delta)T\} = 0 \quad (34)$$

and

$$T = \frac{1}{(C_b+C_\pi\delta)}(W_3t_1 - X_3) \quad (35)$$

Note that

$$W_3t_1 - X_3 = \Bigg[h_1(t_d\theta(t_1-t_d)+t_1) + \frac{h_2t_d\theta}{2}(t_1-t_d)t_d + h_2\left(t_1-\frac{t_d}{2}\right)t_d + C\theta(t_1-t_d) + (C_b+C_\pi\delta)t_1 \\ + sI_e\left[(t_1-t_d) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda}\right] - M\Bigg] > 0$$

since $(t_1-t_d) > 0$

Similarly

$$\frac{\partial TC_3(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_3t_1^2 - X_3t_1 + Y_3 - \frac{T^2}{2}(C_b+C_\pi\delta)\right\} \quad (36)$$

Substituting T from equation (35) into equation (36) yields

$$W_3\big(W_3-(C_b+C_\pi\delta)\big)t_1^2 - 2X_3\big(W_3-(C_b+C_\pi\delta)\big)t_1 - (2(C_b+C_\pi\delta)Y_3 - X_3^2) = 0 \quad (37)$$

Let $\Delta_{31} = W_3\big(W_3-(C_b+C_\pi\delta)\big)t_d^2 - 2X_3\big(W_3-(C_b+C_\pi\delta)\big)t_d - (2(C_b+C_\pi\delta)Y_3 - X_3^2)$

and

$\Delta_{32} = W_3\big(W_3-(C_b+C_\pi\delta)\big)M^2 - 2X_3\big(W_3-(C_b+C_\pi\delta)\big)M - (2(C_b+C_\pi\delta)Y_3 - X_3^2)$, then the following result is obtained.

Lemma 4.3

(i) If $\Delta_{31}\leq 0\leq\Delta_{32}$, then the solution of $t_1\in[t_d,M]$ (say t_{13}^*) which satisfies equation (37) not only exists but also is unique.

(ii) If $\Delta_{32}< 0$, then the solution of $t_1\in[t_d,M]$ which satisfies Equation (37) does not exist.

Proof: The process proof is similar to Lemma 4.1.
Thus, the value of t_1 (denoted by t_{13}^*) can be found from equation (37) is given by

$$t_{13}^*=\frac{X_3}{W_3}+\frac{1}{A_3}\sqrt{\frac{(2W_3Y_3-X_3^2)(C_b+C_\pi\delta)}{\big(W_3-(C_b+C_\pi\delta)\big)}} \tag{38}$$

Once t_{13}^* is obtained, then the value of T (denoted by T_3^*) can be found from equation (35) and is given by

$$T_3^*=\frac{1}{(C_b+C_\pi\delta)}(W_3t_{13}^*-X_3) \tag{39}$$

Equations (38) and (39) give the optimal values of t_{13}^* and T_3^* for the cost function in equation (17) only if X_3 satisfies the inequality given in equation (40)

$$X_3^2<2W_3Y_3 \tag{40}$$

Theorem 4.3

(i) If $\Delta_{31}\leq 0\leq\Delta_{32}$, then the total variable cost $TC_3(t_1,T)$ is convex and reaches its global minimum at the point (t_{13}^*,T_3^*), where (t_{13}^*,T_3^*) is the point which satisfies equations (37) and (34).

(ii) If $\Delta_{32}< 0$, then the total variable cost $TC_3(t_1,T)$ has a minimum value at the point (t_{13}^*,T_3^*) where $t_{13}^*=M$ and $T_3^*=\frac{1}{(C_b+C_\pi\delta)}(W_3M-X_3)$

(iii) If $\Delta_{31}> 0$, then the total variable cost $TC_3(t_1,T)$ has a minimum value at the point (t_{13}^*,T_3^*) where $t_{13}^*=t_d$ and $T_3^*=\frac{1}{(C_b+C_\pi\delta)}(W_3t_d-X_3)$

Proof: The process proof is similar to Theorem 4.1.
Thus, the EOQ corresponding to the optimal cycle length T^* will be computed as follows:
EOQ^* =Total demand before deterioration sets in+total demand after deterioration sets in
+total number of deteriorated items + total number of items backordered

$$=\int_0^{t_d}(\alpha+\beta t+\gamma t^2)dt+\int_{t_d}^{t_1^*}\lambda dt+\left[\frac{\lambda}{\theta}\left(e^{\theta(t_1^*-t_d)}-1\right)-\lambda(t_1^*-t_d)\right]+\frac{\lambda}{\delta}\big[ln[1+\delta(T^*-t_1^*)]\big]$$

$$=\alpha t_d+\beta\frac{t_d^2}{2}+\gamma\frac{t_d^3}{3}+\frac{\lambda}{\theta}\left(e^{\theta(t_1^*-t_d)}-1\right)+\frac{\lambda}{\delta}\big[ln[1+\delta(T^*-t_1^*)]\big] \tag{41}$$

5. Numerical Examples

This section provides some numerical examples to illustrate the theoretical results of model developed.

Example 5.1 ($\mathbf{0<M\leq t_d}$)

Consider an inventory system with the following input parameters: $A=\$350$/order, $C=\$45$/unit/year, $S=\$65$/unit/year, $h_1=\$15$/unit/year, $h_2=\$5$/unit/year, $C_b=\$20$/unit/year, $C_\pi=\$5$/unit/year, $\theta=0.05$ units/year, $\alpha=980$ units, $\beta=180$ units, $\gamma=15$ units, $\lambda=450$ units, $t_d=0.2136$ year (78 days), $M=0.0684$ year (25 days), $I_c=0.10, I_e=0.08$ and $\delta=0.8$. It is seen that $M\leq t_d$, $\Delta_1=-16.5278<0$, $X_1^2=3.78255$, $2W_1Y_1=102.8074$ and hence $X_1^2<2W_1Y_1$. Substituting the above values in equations $(23),(24),(15)$ and (41), the value of optimal time with positive inventory, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{11}^*=0.2625$ year (96 days), $T_1^*=0.5186$ year (189 days), $TC_1(T_1^*,\ t_{11}^*)=\$1837.8012$ per year and $EOQ_1^*=327.6931$ units per year.

Example 5.2 ($\mathbf{t_d<M\leq t_1}$)

The data are same as in Example 4.1 except that $M=0.2382$ year (87days). It is seen that $M>t_d$, $\Delta_2=-6.8850<0$ $X_2^2=7.3008$, $2W_2Y_2=86.3460$ and hence $X_2^2<2W_2Y_2$. Substituting the above values in equations $(31),(32),(16)$ and(41), the value of optimal time with positive inventory, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{12}^*=0.2596$ year (95 days), $T_2^*=0.4636$ year (169 days), $TC_2(T_2^*,\ t_{12}^*)=\$1419.0087$ per year and $EOQ_2^*=307.6548$ units per year.

Example 5.3 ($\mathbf{M>t_1}$)

The data are same as in Example 4.1 except that $t_d=0.1254$ (46 days) and $M=0.2378$ year (87 days). It is seen that $M>t_d$, $\Delta_{31}=-15.3534<0$, $\Delta_{32}=1.4087>0$, $X_3^2=3.0857$, $2W_3Y_3=55.0985$. Here $\Delta_{31}\leq 0\leq\Delta_{32}$ and $X_3^2<2W_3Y_3$. Substituting the above values in equations $(38),(39),(17)$ and (41), the value of optimal time with positive inventory, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{13}^*=0.1978$ year (72 days), $T_3^*=0.3745$ year (137 days), the optimal total variable cost $TC_3(T_3^*,\ t_{13}^*)=\$1253.4062$ per year and $EOQ_3^*=223.0945$ units per year. Therefore, $\boldsymbol{TC(T^*,t_1^*)=min\{TC_1(t_{11}^*,\ T_1^*),TC_2(t_{12}^*,\ T_2^*),TC_3(t_{13}^*,\ T_3^*)\}=TC_3(t_{13}^*,\ T_3^*)=₦1253.4062}$ per year.

Table 5.1: Comparison between model with constant partial backlogging rate and time-dependent partial backlogging rate and time-dependent linear holding cost.

MODELS	TOTAL VARIABLE COST		
	Case 1: $0 < M \leq t_d$	Case 2: $t_d < M \leq t_1$	Case 3: $M > t_1$
Model with constant partial backlogging rate	$TC_1(T_1^*,\ t_{11}^*) = ₦2293.5980$	$TC_2(T_2^*,\ t_{12}^*) = ₦1919.0162$	$TC_3(T_3^*,\ t_{13}^*) = ₦1722.3973$
Model with time-dependent partial backlogging rate	$TC_1(T_1^*,\ t_{11}^*) = ₦1837.8012$	$TC_2(T_2^*,\ t_{12}^*) = ₦1419.0087$	$TC_3(T_3^*,\ t_{13}^*) = ₦1253.4062$

It is observed from Table 5.1 above that model with time-dependent partial backlogging rate has the least total variable cost compared to model with constant partial backlogging rate. This is because the length of the Waiting time would determine whether backlogging will be accepted or not, hence, the backlogging rate is variable and depends on the waiting time for the next replenishment.

6. Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-20\%, -10\%$, $+10\%$ to 20% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these parameters on time with positive inventory, cycle length, total variable cost and the economic order quantity per cycle for example 5.1, 5.2 and 5.3 are summarized in Table 6.1, 6.2 and 6.3.

Table 6.1 Effect of changes of some parameters on decision variables for example 5.1

Parameters	% Change in parameter	% Change in t_{11}^*	% Change in T_1^*	% Change in EOQ_1^*	% Change in $TC_1(t_{11}^*, T_1^*)$
θ	-40	1.066	0.476	0.272	-0.219
	-20	0.528	0.236	0.135	-0.108
	$+20$	-0.518	-0.232	-0.133	0.107
	$+40$	-1.028	-0.459	-0.263	0.212
C	-40	11.764	3.997	2.485	-5.163
	-20	5.431	1.802	1.125	-2.476
	$+20$	-4.714	-1.495	-0.942	2.301
	$+40$	-8.850	-2.746	-1.736	4.453
S	-40	0.257	0.266	0.148	0.277
	-20	0.128	0.133	0.074	0.139
	$+20$	-0.129	-0.133	-0.074	-0.139
	$+40$	-0.257	-0.267	-0.148	-0.278
I_c	-40	10.371	3.372	2.111	-4.880
	-20	4.841	1.539	0.967	-2.356
	$+20$	-4.278	-1.300	-0.825	2.211
	$+40$	-8.089	-2.406	-1.533	4.296
I_e	-40	0.257	0.266	0.148	0.277
	-20	0.128	0.133	0.074	0.139
	$+20$	-0.129	-0.133	-0.074	-0.139
	$+40$	-0.257	-0.267	-0.148	-0.278
δ	-40	-1.479	1.691	1.440	-1.599
	-20	-0.723	0.820	0.689	-0.782
	$+20$	0.692	-0.773	-0.634	0.749
	$+40$	1.356	-1.504	-1.219	1.466
C_b	-40	-9.057	11.301	5.446	-9.795
	-20	-3.968	4.667	2.277	-4.292
	$+20$	3.191	-3.472	-1.722	3.451
	$+40$	5.817	-6.159	-3.073	6.290
C_π	-40	-1.479	1.691	0.830	-1.599
	-20	-0.723	0.820	0.403	-0.782
	$+20$	0.692	-0.773	-0.381	0.749
	$+40$	1.356	-1.504	-0.743	1.466

Table 6.2 Effect of changes of some parameters on decision variables for example 5.2

Parameters	% Change in parameter	% Change in t_{12}^*	% Change in T_2^*	% Change in EOQ_2^*	% Change in $TC_2(t_{12}^*, T_2^*)$
θ	-40	1.030	0.484	0.266	-0.259
	-20	0.510	0.240	0.132	-0.129
	$+20$	-0.501	-0.236	-0.129	0.127
	$+40$	-0.994	-0.467	-0.257	0.252
C	-40	9.421	4.520	2.590	-2.162
	-20	4.329	2.070	1.187	-1.008
	$+20$	-3.727	-1.774	-1.018	0.888
	$+40$	-6.972	-3.312	-1.900	1.676
S	-40	2.935	3.244	1.743	3.666
	-20	1.479	1.635	0.879	1.848
	$+20$	-1.504	-1.662	-0.894	-1.879
	$+40$	-3.033	-3.353	-1.804	-3.790
I_c	-40	8.093	3.887	2.227	-1.846
	-20	3.760	1.801	1.032	-0.868
	$+20$	-3.294	-1.572	-0.901	0.776
	$+40$	-6.206	-2.956	-1.695	1.474
I_e	-40	2.935	3.244	1.743	3.666
	-20	1.479	1.635	0.879	1.848
	$+20$	-1.504	-1.662	-0.894	-1.879
	$+40$	-3.033	-3.353	-1.804	-3.790
δ	-40	-1.182	1.671	1.284	-1.477
	-20	-0.577	0.810	0.615	-0.721
	$+20$	0.552	-0.764	-0.568	0.690
	$+40$	1.081	-1.486	-1.095	1.350
C_b	-40	-7.290	11.173	5.358	-9.107
	-20	-3.179	4.612	2.236	-3.972
	$+20$	2.540	-3.429	-1.687	3.173
	$+40$	4.619	-6.084	-3.008	5.770
C_π	-40	-1.182	1.671	0.814	-1.477
	-20	-0.577	0.810	0.395	-0.721
	$+20$	0.552	-0.764	-0.374	0.690
	$+40$	1.081	-1.486	-0.728	1.350

Table 6.3 Effect of changes of some parameters on decision variables for example 5.3

Parameters	% Change in parameter	% Change in t_{13}^*	% Change in T_3^*	% Change in EOQ_3^*	% Change in $TC_3(t_{13}^*, T_3^*)$
θ	-40	0.683	0.294	0.146	-0.042
	-20	0.338	0.146	0.072	-0.021
	$+20$	-0.331	-0.143	-0.071	0.020
	$+40$	-0.656	-0.282	-0.140	0.041
C	-40	0.667	0.287	0.144	-0.041
	-20	0.330	0.142	0.071	-0.020
	$+20$	-0.323	-0.139	-0.070	0.020
	$+40$	-0.641	-0.276	-0.138	0.040
S	-40	42.273	22.588	11.097	5.564
	-20	20.519	11.365	5.566	3.448
	$+20$	-19.737	-11.808	-5.752	-4.951
	$+40$	-39.222	-24.483	-11.897	-11.735
I_c	-40	0.000	0.000	0.000	0.000
	-20	0.000	0.000	0.000	0.000
	$+20$	0.000	0.000	0.000	0.000
	$+40$	0.000	0.000	0.000	0.000
I_e	-40	42.273	22.588	11.097	5.564
	-20	20.519	11.365	5.566	3.448
	$+20$	-19.737	-11.808	-5.752	-4.951
	$+40$	-39.222	-24.483	-11.897	-11.735
δ	-40	-2.488	1.605	1.185	-1.864
	-20	-1.218	0.778	0.568	-0.913
	$+20$	1.168	-0.733	-0.523	0.876
	$+40$	2.291	-1.426	-1.007	1.717
C_b	-40	-15.060	10.749	4.443	-11.286
	-20	-6.651	4.433	1.849	-4.985
	$+20$	5.407	-3.290	-1.391	4.052
	$+40$	9.896	-5.834	-2.478	7.416
C_π	-40	-2.488	1.605	0.673	-1.864
	-20	-1.218	0.778	0.327	-0.913
	$+20$	1.168	-0.733	-0.309	0.876
	$+40$	2.291	-1.426	-0.601	1.717

Based on the computed results shown on Tables 6.1, 6.2 and 6.3, the following managerial insights are obtained.

1. When the rate of deterioration (θ) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and economic order quantity (EOQ^*) decrease, while total variable cost $\left(TC(t_1^*, T^*)\right)$ increases and vice versa. When the number of deteriorated items increases, then the total variable cost will be high. Hence, the retailer shall order less quantity to avoid items being deteriorating when the deterioration rate increases. This decreases the inventory holding cost and hence reducing the total variable cost. The rate of deterioration can also be reduced by improving the equipments in the warehouse.
2. When the unit purchasing cost (C) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and economic order quantity (EOQ^*) decrease, while the total variable cost $\left(TC(t_1^*, T^*)\right)$ increases and vice versa. In a real market situation, the higher the cost of an item, the higher the total variable cost and vice versa. The retailer shall order less quantity when unit purchasing cost increases.
3. When the unit selling price (S) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$ decrease and vice versa. In a real market situation, the higher the price, the lower the quantity demanded and vice versa. This means that if the unit selling price per unit increases, the retailer shall order less quantity of items to take the benefits of the trade credit more frequently.
4. When the interest charge (I_c) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and economic order quantity (EOQ^*) decrease, while the total variable cost $\left(TC(t_1^*, T^*)\right)$ increases and vice versa. This means that when the interest charge increases, the retailer shall order fewer amounts of items to take the benefit of trade credit more frequently. As for $M > t_1$, the increase/decrease in interest charge (I_c) does not affect the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$. This is because the interest charge is zero when $M > t_1$.

5. When the interest earned (I_e) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$ decrease and vice versa. Hence, the retailer shall order fewer items to take the benefit of trade credit more frequently when interest earned increases.
6. When the backlogging parameter (δ) increases, the optimal time with positive inventory (t_1^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$ decrease, while the cycle length (T^*) and economic order quantity (EOQ^*) increase and vice versa.
7. When the shortage cost (C_b) increases, the optimal time with positive inventory (t_1^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$ decrease, while the cycle length (T^*) and economic order quantity (EOQ^*) increase and vice versa. This means that when the shortages cost increase, total variable cost increases and the number of back-ordered items reduce drastically which in turn decreases the total variable cost.
8. When the cost of lost sales (C_π) increases, the optimal time with positive inventory (t_1^*) and total variable cost $\left(TC(t_1^*, T^*)\right)$ decrease, while the cycle length (T^*) and economic order quantity (EOQ^*) increase and vice versa.

7. Conclusion

In this article, an EOQ model for non-instantaneous deteriorating item with two-phase demand rates, time-dependent linear holding cost and shortages under trade credit policy has been developed. Shortages are allowed and partially backlogged. The length of the Waiting time would determine whether backlogging will be accepted or not, hence, the backlogging rate is variable and depends on the waiting time for the next replenishment. The optimal time with positive inventory, cycle length and order quantity that minimise the total variable cost have been determined. Some numerical examples have been given to illustrate the theoretical results of the model. Sensitivity analysis of some model parameters has been carried out to see the effect of changes of these parameters on decision variables. The results show that the retailer reduces the total variable cost by ordering less to shorten the optimal time with positive inventory and cycle length when the rate of deterioration, unit purchasing cost, unit selling price, interest charge, shortage cost, backlogging parameter, cost of lost sales and interest earned increases respectively. The proposed model can be extended by taking more realistic assumptions such as two storage facilities, variable deterioration rate, quadratic holding cost, ramp type or trapezoidal type or probabilistic demand rates and so on.

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