Transactions of the Nigerian Association of Mathematical Physics
***Volume 17, (October – December, 2021 Issue*), pp83 – 88**
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A SEMI-ANALYTIC DECOMPOSITION METHOD WITH SHIFTED CHEBYSHEV POLYNOMIAL OF FIRST KIND FOR SOLVING SOME CLASS OF SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

A semi-analytic decomposition method for the solution of non-linear initial value problems based on a reliable decomposition algorithm with shifted Chebyshev orthogonal polynomial of first kind is presented to obtain an approximated solution of some class of second order nonlinear ordinary differential equations. The proposed method surmounts the use of special polynomials. Numerical tests are carried out on Duffing and Van der pol differential oscillatory equations. The results obtained in this paper demonstrate reliability, efficiency and accuracy of the new algorithm when compared to other known methods in the literature.

Keywords: Decomposition method, shifted Chebyshev polynomial, ordinary differential equation, Duffing's equations, Van der pol's equations and orthogonal polynomial.

INTRODUCTION

Most phenomena in our world are essentially best modeled by nonlinear equations. The advent of high performance digital computers has made it easier to solve nonlinear problems. However, it is seldom more difficult to get an analytic approximation than a numerical one of a given nonlinear problem. This paper considers the solution of the nonlinear differential equation of the forms

$$y''(x) = f(x, y), y(0) = \alpha_{1,} y'(0) = \alpha_2 \quad (1)$$

and the general form of the second order ordinary differential equation

$$''(x) = f(x, y, y'), y(0) = \alpha_{1,} y'(0) = \alpha_2 \quad (2)$$

with oscillatory solutions where $f : \Re \ x \ \Re^n \to \Re^n$ is smooth and satisfy Lipschitz condition.

Recently, efforts were geared towards various modification of Adomian decomposition method to enhance efficiency, accuracy, reliability and possible reduction of computational burden unlike the convectional approach of reducing equations (1) – (2) to an equivalent system of first order differential equations. Although, Runge-Kutta method, finite difference method, variational iteration method, polynomial spline and homotopy perturbation methods have all been developed to solve this kind of problems. Yet, all have well known deficiencies and drawbacks. For instance, perturbation techniques involve the use of perturbation parameters to transform a nonlinear problem into an infinite number of linear sub-problems and then approximate it by the sum of these several sub-problems. This approach is very cumbersome. A great deal of interest has been focus on the decomposition techniques and its applications in solving both linear and nonlinear function equations, stochastic and deterministic problems [1]

Iteration method reported by Daftardar-Gejji and Jafari in [2] is another form of Adomian decomposition methods. The difference arises in the surmount of special Adomian polynomials. Also, Wazwaz [3-4], Pourdvish [5], Osilagun and Taiwo [6] have carried out work in modifications for solving functional equations of form (1) – (2). This current paper also considers two other class of equations (1) – (2). These are the Duffing's equations and Van der pol's equations. Inspired and motivated by the ongoing research in this area, our paper aims to improve the efficiency and accuracy of the Adomian decomposition method (ADM) based on the simplification of its nonlinearity vis-à-vis the use of orthogonal polynomials. Our choice of shifted Chebyshev polynomial of first kind stems from the desire to ensure equi-distribution of the error in

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our derived approximant through the entire range of integration. This shifted Chebyshev polynomial of degree n, $P_n(x)$ is defined over the range [0,1] by

$$P_n(x) = \cos[n\cos^{-1}(2x-1) \quad (3)$$

is most appropriate for our purpose of study due to its mini-max approximation and equi-oscillation properties. Moreover, this orthogonal shifted Chebyshev polynomials are Jacobi orthogonal polynomials and eigenfunctions of singular Sturm-Liouville problem

$$[(1-x^2)P_n(x)]' + \frac{\alpha^2}{\sqrt{1-x^2}}P_n(x) = 0 \quad (4)$$

as proven in Kreyszigs [7] .

This paper outlines as follows section two deals with the description of the solution technique. Numerical experiment to illustrate the efficiency, accuracy and reliability of the developed recursive formula is elucidated in section three and discussion of results and its concluding remarks are entail in the last section of the paper.

Analysis of the Method.

In this section, we elucidate the succinctness of the proposed method on improved decomposition method based on non use of special Adomian polynomials in handling the nonlinearity of the functional equation vis-à-vis the use of shifted Chebyshev polynomials to improve efficiency and accuracy. For this reason, consider the general functional equation.

$$F[y(x)] = g(x) \quad (5)$$

where F represents a general nonlinear ordinary differential operator and $g(x)$ is the force terms or a given function. The linear term in $F[y(x)]$ is decomposed into $L[y(x)] + R[y(x)]$ where L is an easily invertible highest order derivative, R is the remainder of the linear operator whose order is less than order of L .

so equation (5) can therefore be equally expressed as

$$L[y(x)] + R[y(x)] + N[y(x)] = g(x) \quad (6)$$

where $N[y(x)]$ represents the nonlinear operator from a Banach space B into B.

On applying the inverse operation L^{-1} on both sides of equation (6) yields

$$y(x) = \phi(x) + L^{-1}g(x) - L^{-1}[R(y(x))] - L^{-1}[N(y(x))] \quad (7)$$

where $$\phi(x) = \sum_{k=0}^{n-1}\frac{x^k}{k!}y^k(0) \quad (8)$$

is obtained via the prescribed initial conditions.

The nonlinear operator $N[y(x)]$ is decomposed as

$$N(\sum_{n=0}^{\infty}y_n) = N[y_0(x)] + \sum_{n=0}^{\infty}\left(N\left(\sum_{k=0}^{n}y_k\right)\right) \quad (9)$$

rather than the use of special Adomian polynomial to handle its nonlinearity. According to the standard Adomian decomposition postulated by George [1], the solution $y(x)$ is defined by an infinite series

$$y(x) = \sum_{n=0}^{\infty}y_n(x) \quad (10)$$

We obtain the recurrence relation.

$$y_0 = L^{-1}[g(x)] + \phi(x)$$
$$y_1 = -L^{-1}[R[y_0(x)]] - L^{-1}[N(y_0)] \quad (11)$$
$$y_{n+1} = -L^{-1}[R(y_n)] - L^{-1}\left(N\left(\sum_{k=0}^{n}y_k\right)\right), \qquad n \ge 1$$

For smooth implementation of ADM, g(x) is always express in Taylor's series about the origin for a finite number of desirable terms which happens to be the best practice for ease of computational burden. In similar manner to earlier works of Hossein [8], Liu [9], Wei-chung and Cha'o-kuang [10] where $g(x)$ is expressed in terms of shifted chebyshev polynomial of the first kind. That is,

$$g(x) \approx \sum_{k=0}^{n} a_k p_k(2x-1), 0 \le x \le 1 \tag{12}$$

where $P_{n+1}(x) = 2(2x-1)P_n(x) - P_{n-1}(x), \quad n \ge 1$ (13)

is the general recursive relation that generate these polynomials. Thus, equations (3) and (13) give

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= 2x-1 \\ p_2(x) &= 8x^2 - 8x + 1 \\ p_3(x) &= 32x^3 - 48x^2 + 18x - 1 \\ p_4(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ &. \\ &. \\ &. \end{aligned} \tag{14}$$

and

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-1}^{1} \frac{g(\tfrac{1}{2}x + \tfrac{1}{2})P_0(x)}{\sqrt{1-x^2}}\,dx \\ a_k &= \frac{1}{\pi}\int_{-1}^{1} \frac{g(\tfrac{1}{2}x + \tfrac{1}{2})P_k(x)}{\sqrt{1-x^2}}\,dx, k = 1,2,... \end{aligned} \tag{15}$$

So, combining equations (11) and (12), yields

$$\begin{aligned} y_0 &= L^{-1}[\sum_{i=0}^{n} a_i p_i)] + \phi(x) \\ y_1 &= -L^{-1}[R(y_0)] - L^{-1}[N(y_0(x))] \\ y_{n+1} &= -L^{-1}[R(y_n)] - L^{-1}\left(N\left(\sum_{k=0}^{n} y_k\right)\right), \qquad n \ge 1 \end{aligned} \tag{16}$$

Alternatively, equation (16) in the spirit of Wazwaz and El-Sayed [11] can further be broken down to a simpler form for ease of huge computational burden as

$$\begin{aligned} y_0 &= L^{-1}[\alpha_0 p_0] + \phi(x) \\ y_1 &= L^{-1}[\alpha_1 p_1] - L^{-1}[R(y_0)] - L^{-1}[N(y_0)] \\ y_1 &= L^{-1}[\alpha_2 p_2] - L^{-1}[R(y_1)] - L^{-1}\left(N\left(\sum_{k=0}^{1} y_k\right)\right) \\ &. \\ &. \\ &. \\ {}_{n+1} &= L^{-1}[\alpha_n p_n] - L^{-1}[R(y_n)] - L^{-1}N\left(\sum_{k=0}^{n} u_k\right), \qquad n \ge 1 \end{aligned} \tag{17}$$

It's observed and noted that the approximate solution $y_n(x) = \sum_{k=0}^{n} y_k$ by the recursive algorithms (11), (16) and (17) exhibit similar behavior when compared to the shifted Chebyshev expansion of the exact solution $y(x)$ since Chebyshev orthogonal polynomial yields spectral accuracy. It allows the approximation of smooth function where truncation error approaches zero faster than any negative power of the number of basis function used in the approximation as that number tends to infinity.

3.0 Numerical Experiments

In this section, we present three problems that are solved using the proposed algorithms. The algorithms are performed by Maple 17 with 10 digits precision on all the experiments.

Problem 1: consider the second order nonlinear initial value problem

$$y''(x) + xy'(x) + x^2 y^3 = (2+6x^2)\, e^{x^2} + x^2 e^{3x^2}, \qquad 0 \le x \le 1$$

$$y(0) =, y'(0) = 0$$

whose exact solution is $y(x) = \exp(x^2)$

Numerical solution for this problem based on experiment with the scheme (16) and (17) are presented in Table 1 and Figure 1.

Problem 2:

Consider a class of equation (1) known as the Duffing's equation,

$$y'' + 3y + y^3 = \cos x \sin 2x$$

$$y(0) =, y'(0) = 1$$

The analytic solution of this problem is $y(x) = \sin x$

The numerical result obtained from the experimentation of the schemes (16) and (17) based on equations (11) – (13) and its absolute error are shown in Table 2 and Figure 2

Problem 3:

We also consider another class of equation (2) known as the Van der pol's equation

$$y''(t) + y'(t) + y(t) + y'(t) y^2(t) = 2\cos t - \cos^3 t, \qquad 0 \le t \le 1$$

$$y(0) =, y'(0) = 1$$

whose exact solution is $y(t) = \sin t$

The experimentation is carried out using the same scheme (16) and (17). The result obtained is presented in Table 3 and Figure 3 respectively.

4.0 Discussion of Result

Problem 1

Table 1: The comparison of the exact solution, ADM [12], and Proposed method

x	Exact	Hosseini [12]	$y-$ computed
0.0	1	1	1
0.1	1.010050167	1.009999104	1.009999138
0.2	1.040810774	1.040683938	1.040686673
0.3	1.094174284	1.094145054	1.094167463
0.4	1.173510871	1.173229489	1.173245107
0.5	1.284025417	1.281510938	1.283516338
0.6	1.433329415	1.421432057	1.434358142
0.7	1.632316622	1.588348757	1.601351057
0.8	1.896480879	1.759238545	1.844263537
0.9	2.247907980	1.876035239	2.152677109
1.0	1.718281828	1.824500000	2.382416730

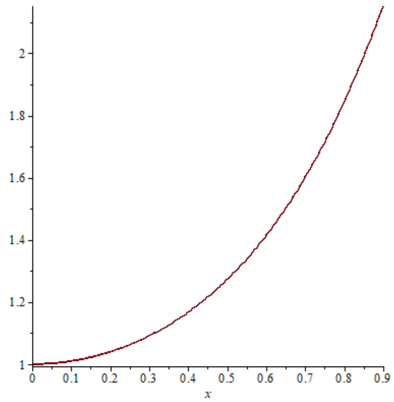

Figure 1: Graph of the exact and the proposed method

The result obtained above from the Table 1 and Figure 1 shows that the proposed method is more accurate than the result of Hosseini [12].

Problem 2

Table 2: Comparison of approximation method of solutions for Duffing's equation

x	Exact	Legendre	Error	Taylor	Error	$y(x)$ computed	Error
0.2	0.1986693308	0.1986693328	2.012×10^{-9}	0.1986693308	5.0×10^{-12}	0.1986693308	0
0.4	0.3894183423	0.3894183422	1.16×10^{-10}	0.3894183321	1.0224×10^{-8}	0.3894183423	0
0.6	0.5646424734	0.5646424754	1.951×10^{-9}	0.5646416032	8.70197×10^{-7}	0.5646424734	0
0.8	0.7173560909	0.7173565307	4.3981×10^{-7}	0.7173361590	1.9931859×10^{-5}	0.7173560909	0
1.0	0.8414709848	0.8414723506	1.36581×10^{-6}	0.8412387879	$2.32196948 \times 10^{-4}$	0.8414713876	4.028×10^{-7}

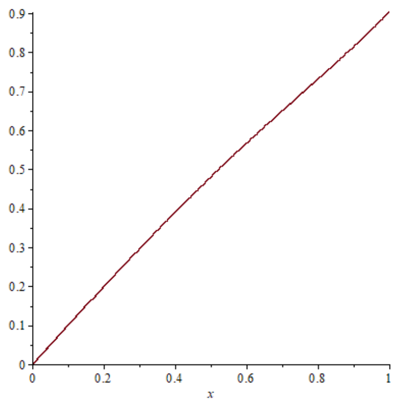

Figure. 2: Graph of the exact and the proposed method

The proposed scheme gives the best result as shown in the Table 2 and Figure 2 in comparison to the other mention methods.

Problem 3

Table 3: The comparison of the exact solution and proposed method

t	Exact $y(t) = \sin t$	$y(t) - computed$	Error
0.0	0	0	0
0.1	0.09983341665	0.09983341665	0.0000
0.2	0.1986693308	0.1986693308	0.0000
0.3	0.2955202067	0.2955203005	9.38×10^{-8}
0.4	0.3894183421	0.3894185344	1.923×10^{-7}
0.5	0.4794255386	0.4794259746	4.30×10^{-7}
0.6	0.5646424734	0.5646465764	4.103×10^{-6}
0.7	0.6442176872	0.6449776875	7.60×10^{-4}
0.8	0.7173560909	0.7173999908	1.110×10^{-5}
0.9	0.7833269096	0.8140339609	3.07×10^{-2}
1.0	0.8414709848	0.9034665488	6.16×10^{-2}

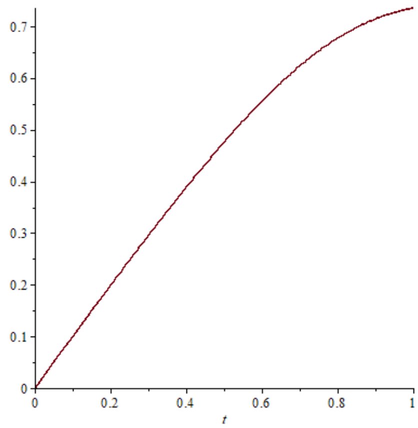

Figure 3: Graph of the exact and the proposed method

The result from the Table 3 and Figure 3 shows that the new scheme is more accurate and efficient when compared to other techniques mention in the literature.

5.0 Conclusion

A semi-analytic method for the solution of second order IVP has been presented. The techniques utilized the special features of shifted Chebyshev polynomial, no use of special polynomial of any kind, restrictive assumptions, linearization nor perturbation. The technique is further implemented in obtaining the approximate solution of both the Duffing's equations and Van der pol equations. The results obtained for the test problems as shown in the Tables and Figures reveal the efficiency, accuracy and reliability of the proposed scheme when compared to other methods mentioned in the literature.

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